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# ADOMIAN DECOMPOSITION METHOD APPLIED TO CONTINUOUS-TIME BILINEAR STOCHASTIC PROCESSES WITH TIME-VARYING COEFFICIENTS

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**Abstract.** In the present paper, we apply the Adomian decomposition method for bilinear stochastic processes with time-varying coefficients in both time and frequency domain. More precisely, we derived an analytical approximate solution and we prove its convergence to the exact solution in time domain, furthermore we give an analytical approximate solution in frequency domain, i.e, we derived analytical approximate transfer functions which converge to the exact transfer functions.

**Keywords:** Adomian decomposition method; continuous-time bilinear processes; approximate solution; spectral representation; transfer functions.

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## 1. Introduction

The Adomian decomposition method $(ADM)$ is an approach proposed by George Adomian (1923-1996) [3, 4, 5] which provides an analytic approximation to linear and/or nonlinear problems, boundary value problems, algebraic equations and differential equations [2, 1, 18, 20]. The general principle is based on the so-called 'reversion method' which consists of decomposing the solution of a nonlinear functional equation in a series of functions elegantly computed

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[12, 15]. The method can be applied directly for different type of (stochastic) differential equations providing a solution with fast speed of convergence [7, 9, 10, 14, 16]. So, the aim of this Note is to apply the *ADM* (as an alternative of Itô solution) for solving a continuous-time bilinear (*COBL*) equation defined as a stochastic differential equations (*SDE*) with time varying coefficients in time and frequency domain.

## 2. Bilinear Stochastic Differential Equation

In this Note, we consider the first order continuous-time bilinear process ($COBL(1,1)$) driven by a stochastic differential equation of the form

$$dX(t) = (\alpha(t)X(t)+\mu(t))dt + (\gamma(t)X(t)+\beta(t))\,dw(t), t \geq 0, X(0) = \eta, \tag{1}$$

where $(w(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$ defined on some basic probability space $(\Omega, \mathscr{A}, P)$ and $\alpha(t)$, $\mu(t)$ and $\gamma(t)$ are complex measurable functions such that $\forall t \geq 0$, $\alpha(t) \neq 0$, and $\gamma(t) \neq 0$. To ensure the existence and uniqueness of the solution process $(X(t))_{t\geq 0}$ of equation (1), we assume in addition that, $\forall T > 0$, $\int_0^T |\alpha(t)|\,dt < \infty$ , $\int_0^T |\mu(t)|\,dt < \infty$, $\int_0^T |\gamma(t)|^2\ dt < \infty$ and that the initial state $\eta$ is a random variable defined on $(\Omega, \mathscr{A}, P)$, independent of $w$ with $E\{\eta\} = m(0)$ and $Var\{\eta\} = K(0)$. The existence of the Itô solution process $(X(t))_{t\geq 0}$ of equation (1) in time domain is however ensured by general results on *SDE* (see e.g., [6])

$$X(t) = \Phi(t)\left\{X(0) + \int_0^t \Phi^{-1}(s)\,(\mu(s) - \gamma(s)\,\beta\,(s))\,ds + \int_0^t \Phi^{-1}(s)\beta\,(s)\,dW(s)\right\} \tag{2}$$

where $\Phi(t) = \exp\left\{\int_0^t \left(\alpha(s) - \frac{1}{2}\gamma^2(s)\right) ds + \int_0^t \gamma(s)\,dW(s)\right\}$ its mean function is $\Psi(t) = \exp\left\{\int_0^t \alpha(s)ds\right\}$, $t \geq 0$. If the coefficients $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are constant and satisfying the condition $\gamma \neq 0, \alpha\beta \neq \mu\gamma, 2\alpha + \gamma^2 < 0$, then the process $(X(t))_{t\in\mathbb{R}}$ is a second-order stationary with

$$E\{X(t)\} = -\frac{\mu}{\alpha}, K(0) = Var\{X(t)\} = \frac{(\alpha\beta - \mu\gamma)^2}{\alpha^2\,|2\alpha+\gamma^2|} \text{ and } Cov(X(t), X(t+h)) = K(0)e^{\alpha|h|}, h \in \mathbb{R}.$$

## 3. Decomposition Method Applied to $COBL(1,1)$ in Time-Domain

Without loss of generality we assume that $\beta(t)=0$, for all $t$, since this assumption can be fulfilled by the transformation $Y(t)=X(t)+\dfrac{\beta(t)}{\gamma(t)}$ with some additional assumptions on the differentiability of the functions $\gamma(t)$ and $\beta(t)$. The Adomian's method suggest that the solution process $X(t)$ of SDE $(1)$ is sought in the form

$$X(t)=\sum_{j=0}^{\infty}X_j(t) \tag{3}$$

Substituting $(3)$ into $(1)$ we obtain

$$\sum_{j=0}^{\infty}X_j(t)=\eta+L^{-1}\mu(t)+L^{-1}\left(\alpha(t)+\gamma(t)w^{(1)}(t)\right)\sum_{j=0}^{\infty}X_j(t), \tag{4}$$

in wherein $w^{(1)}(t)=\dfrac{dw(t)}{dt}$ and $L$ is the operator defined by $L^{-1}[.]=\int_0^t(.)ds$. So we have by identification

$$X_0(t)=\eta+\int_0^t\mu(s)ds \tag{5}$$

and $\forall n\geq 1$,

$$X_n(t)=\int_0^t\left(\alpha(s)+\gamma(s)w^{(1)}(s)\right)X_{n-1}(s)ds, \tag{6}$$

More precisely, we have

$$\begin{aligned}
X_1(t)&=\int_0^t(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))X_0(s_1)ds_1\\
&=\eta\int_0^t(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))ds_1+\int_0^t\int_0^{s_1}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))\mu(s_2)ds_2ds_1\\
X_2(t)&=\int_0^t(\alpha(s_2)+\gamma(s_2)w^{(1)}(s_2))X_1(s_2)ds_2\\
&=\eta\int_0^t\int_0^{s_1}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))(\alpha(s_2)+\gamma(s_2)w^{(1)}(s_2))ds_2ds_1\\
&+\int_0^t\int_0^{s_1}\int_0^{s_2}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))(\alpha(s_2)+\gamma(s_2)w^{(1)}(s_2))\mu(s_3)ds_3ds_2ds_1\\
&\vdots\\
X_n(t)&=\eta\int_0^t\int_0^{s_1}\ldots\int_0^{s_{n-1}}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))\ldots(\alpha(s_n)+\gamma(s_n)w^{(1)}(s_n))ds_nds_{n-1}\ldots ds_2ds_1\\
&+\int_0^t\int_0^{s_1}\ldots\int_0^{s_{n-1}}\int_0^{s_n}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))\ldots(\alpha(s_n)+\gamma(s_n)w^{(1)}(s_n))\mu(s_{n+1})ds_{n+1}ds_nds_{n-1}\ldots ds_2ds_1
\end{aligned}$$

Finally, we approximate the solution by the truncated series

$$\forall N \geq 1, \Phi_N(t) = \sum_{n=0}^{N-1} X_n(t). \tag{7}$$

In order to show the convergence of the series (7) to the exact solution processes we introduce the concept of the so-called stochastic exponential associated with a centred Gaussian random variable $Z$ and defined by $\mathscr{E}\{Z\} = \exp\left\{Z - \frac{1}{2}E\left\{Z^2\right\}\right\}$. First, we have the following lemma

**Lemma 3.1.** *If $f$ is a real function defined on the interval $[t_0, T] \subset \mathbb{R}$. Then we have for all $t \in [t_0, T]$ and $\forall n \geq 2$,*

$$\int_{t_0}^{t}\int_{t_0}^{s_1} ... \int_{t_0}^{s_{n-1}} f(s_1)....f(s_{n-1})f(s_n)ds_n ds_{n-1}...ds_1 = \frac{\left[\int_{t_0}^{t} f(s)ds\right]^n}{n!}, \tag{8}$$

*where $t_0 \leq s_n \leq s_{n-1} \leq .... \leq s_2 \leq s_1 \leq t$.*

*Proof.* By induction, we have for $n = 2$,

$$\int_{t_0}^{t}\int_{t_0}^{s_1} f(s_1)f(s_2)ds_2 ds_1 = \int_{t_0}^{t} F(s_1)f(s_1)ds_1 = \frac{[F(t)]^2}{2},$$

where $F(t) = \int_{t_0}^{t} f(s)ds$. Now assume that the equation (8) holds for $n-1$, it means

$$\int_{t_0}^{s_1} ... \int_{t_0}^{s_{n-1}} f(s_2)....f(s_{n-1})f(s_n)ds_n ds_{n-1}...ds_2 = \frac{[F(s_1)]^{n-1}}{(n-1)!},$$

which implies

$$\int_{t_0}^{t}\int_{t_0}^{s_1} ... \int_{t_0}^{s_{n-1}} f(s_1)....f(s_{n-1})f(s_n)ds_n ds_{n-1}...ds_1 = \int_{t_0}^{t} \frac{[F(s_1)]^{n-1}}{(n-1)!} f(s_1)ds_1 = \frac{[F(t)]^n}{n!} = \frac{\left[\int_{t_0}^{t} f(s)ds\right]^n}{n!}.$$

□

**Theorem 3.2** (Theorem 1, El-Kalla [13] )**.** *If $q(t)$ is integrable and $r(t)$ is derivable functions, then*

$$e^{-r(t)} \int e^{r(t)} q(t)dt = \sum_{k=0}^{\infty} (-1)^k \int \frac{dr(t)}{dt} \underbrace{\cdots}_{k-fold} \int \frac{dr(t)}{dt} \int q(t)dtdt \underbrace{\cdots}_{k-fold} dt.$$

We are now in a position to prove the convergence of the approximate solution $\Phi_N(t)$ to the exact solution process

**Theorem 3.3.** *Let $\Phi_N(t)$ is the approximate solution (7) given by Adomian decomposition method of the COBL(1,1) driven by the equation (1), then almost surely, we have*

$$\lim_{N\to+\infty}\Phi_N(t)=X(t) \tag{9}$$

*Proof.* Applying Lemma 3.1 on the first term of $X_n(t)$, we obtain

$$\eta\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))...(\alpha(s_n)+\gamma(s_n)w^{(1)}(s_n))ds_nds_{n-1}...ds_2ds_1$$
$$=\eta\frac{\left(\int_0^t(\alpha(s)+\gamma(s)w^{(1)}(s))ds\right)^n}{n!},$$

and Applying Theorem 3.2 on the infinite sum of second term of $X_n(t)$, we obtain

$$\sum_{n=0}^{\infty}\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}}\int_0^{s_n}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))...(\alpha(s_n)+\gamma(s_n)w^{(1)}(s_n))\mu(s_{n+1})ds_{n+1}ds_nds_{n-1}...ds_2ds_1$$
$$=\mathscr{E}\left\{\int_0^t(\alpha(s)+\gamma(s)w^{(1)}(s))ds\right\}\int_0^t\mathscr{E}\left\{-\int_0^s(\alpha(u)+\gamma(u)w^{(1)}(u))du\right\}\mu(s)ds.$$

Then we can obtain

$$\lim_{N\to+\infty}\Phi_N(t)$$
$$=\eta\sum_{n=0}^{\infty}\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))...(\alpha(s_n)+\gamma(s_n)w^{(1)}(s_n))ds_nds_{n-1}...ds_2ds_1$$
$$+\sum_{n=0}^{\infty}\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}}\int_0^{s_n}(\alpha(s_1)+\gamma(s_1)w^{(1)}(s_1))...(\alpha(s_n)+\gamma(s_n)w^{(1)}(s_n))\mu(s_{n+1})ds_{n+1}ds_nds_{n-1}...ds_2ds_1$$
$$=\eta\sum_{n=0}^{\infty}\frac{\left(\int_0^t(\alpha(s)+\gamma(s)w^{(1)}(s))\right)^n}{n!}$$
$$+\mathscr{E}\left\{\int_0^t(\alpha(s)+\gamma(s)w^{(1)}(s))ds\right\}\int_0^t\mathscr{E}\left\{-\int_0^s(\alpha(u)+\gamma(u)w^{(1)}(u))du\right\}\mu(s)ds$$
$$=\mathscr{E}\left\{\int_0^t(\alpha(s)+\gamma(s)w^{(1)}(s))ds\right\}\left(\eta+\int_0^t\mathscr{E}\left\{-\int_0^s(\alpha(u)+\gamma(u)w^{(1)}(u))du\right\}\mu(s)ds\right),$$

where $\mathscr{E}\left\{\int_0^t(\alpha(s)+\gamma(s)w^{(1)}(s))ds\right\}$ denotes the stochastic exponential of $\int_0^t(\alpha(s)+\gamma(s)w^{(1)}(s))ds$, so we get

$$\mathscr{E}\left\{\int_0^t(\alpha(s)+\gamma(s)w^{(1)}(s))ds\right\}=\exp\left\{\int_0^t\left(\alpha(s)-\frac{\gamma^2(s)}{2}\right)ds+\int_0^t\gamma(s)w^{(1)}(s)ds\right\},$$

which implies

$$\lim_{N\to+\infty}\Phi_N(t)=\exp\left\{\int_0^t\left(\alpha(s)-\frac{\gamma^2(s)}{2}\right)ds+\int_0^t\gamma(s)w^{(1)}(s)ds\right\}$$
$$\times\left(\eta+\int_0^t\exp\left\{-\int_0^s\left(\alpha(u)-\frac{\gamma^2(u)}{2}\right)du-\int_0^s\gamma(u)w^{(1)}(u)du\right\}\mu(s)ds\right),$$

which it is the exact Itô solution process $(11)$ of SDE $(1)$ and the proof is complete. □

## 4. Decomposition Method in Frequency Domain

In frequency domain, the process $(X(t))_{t\in\mathbb{R}}$ admits the so-called Wiener-It$\hat{o}$ orthogonal representation

$$X(t)=f(t,0)+\sum_{r\geq1}\frac{1}{r!}\int_{\mathbb{R}^r}e^{it\underline{\lambda}_{(r)}}f(t,\lambda_{(r)})dZ(\lambda_{(r)}), \tag{10}$$

where $\underline{\lambda}_{(r)}=\sum_{i=1}^{r}\lambda_i$ and the integrals are multiple Wiener-Itô stochastic integrals with respect to the stochastic measure $Z(d\lambda)$, $f(t,0)=E\{X(t)\}$, $Z(d\lambda_{(r)})=\prod_{i=1}^{r}Z(d\lambda_i)$, (see Major [19] and Dobrushin [11]). and $f(t,\lambda_{(r)})$ are referred as the $r-th$ evolutionary transfer functions of $(X(t))_{t\in\mathbb{R}}$, uniquely determined and fulfill the condition $\sum_{r\geq0}\frac{1}{r!}\int_{\mathbb{R}^r}\left|f(t,\lambda_{(r)})\right|^2dF(\lambda_{(r)})<\infty$. As a property of the representation $(10)$ is that for any $f(t,\lambda_{(n)})$ and $f(s,\lambda_{(m)})$, we have

$$E\left\{\int_{\mathbb{R}^n}f(t,\lambda_{(n)})dZ(\lambda_{(n)})\int_{\mathbb{R}^m}f(t,\lambda_{(m)})dZ(\lambda_{(m)})\right\}=\delta_n^m n!\int_{\mathbb{R}^n}\widetilde{f}(t,\lambda_{(n)})\overline{\widetilde{f}(s,\lambda_{(n)})}dF(\lambda_{(n)}), \tag{11}$$

where $\delta_n^m$ is the delta function and $\widetilde{f}\left(t,\lambda_{(r)}\right)$ is the symmetrized version of $f$ by the vector $\lambda_{(r)}$ which is the average of those values of $f$ taken by all possible permutations of entries of $\lambda_{(r)}$. We need also the diagram formula given as follow

$$\int_{\mathbb{R}}f(\lambda)Z(d\lambda)\int_{\mathbb{R}^n}g\left(\lambda_{(n)}\right)Z(d\lambda_{(n)})=\int_{\mathbb{R}^{n+1}}g\left(\lambda_{(n)}\right)f\left(\lambda_{n+1}\right)Z(d\lambda_{(n+1)})$$
$$+\sum_{k=1}^{n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}g\left(\lambda_{(n)}\right)\overline{f\left(\lambda_k\right)}dF\left(\lambda_k\right)Z(d\lambda_{(n\backslash k)})$$

where $Z(d\lambda_{(n\backslash k)})=Z\left(d\lambda_1\right)...Z\left(d\lambda_{k-1}\right).Z\left(d\lambda_{k+1}\right)...Z\left(d\lambda_n\right)$.

**Theorem 4.1.** *Assume that the process $(X(t))_{t\geq 0}$ generated by the SDE (1) has a regular second-order solution. Then the evolutionary symmetrized transfer functions $f(t,\lambda_{(r)}),(t,r)\in \mathbb{R}\times\mathbb{N}$ of this solution are given by,*

$$f^{(1)}(t,\lambda_{(r)})=\begin{cases}\alpha(t)f(t,0)+\mu(t),\ \text{if } r=0\\ \left(\alpha(t)-i\underline{\lambda}_{(r)}\right)f(t,\lambda_{(r)})+r\left(\gamma(t)f(t,\lambda_{(r-1)})+\delta_{\{r=1\}}\beta(t)\right),\text{if } r\geq 1\end{cases}\tag{12}$$

*Proof.* See Bibi and Merahi [8]. □

**Remark 4.2.** *The existence and uniqueness of the evolutionary symmetrized transfer functions $f(t,\lambda_{(r)}),(t,r)\in\mathbb{R}\times\mathbb{N}$ of this solution is ensured by general results on linear ordinary differential equations (see, e.g.,* [17]*, ch. 1) so,*

$$f(t,\lambda_{(r)})=\begin{cases}\varphi_t(0)\left(f(0,0)+\int\limits_0^t\varphi_s^{-1}(0)\mu(s)ds\right)\ \text{if } r=0\\ \varphi_t\left(\underline{\lambda}_{(r)}\right)\left(f(0,\lambda_{(r)})+r\int\limits_0^t\varphi_s^{-1}\left(\underline{\lambda}_{(r)}\right)\left(\gamma(s)f(s,\lambda_{(r-1)})+\delta_{\{r=1\}}\beta(s)\right)ds\right)\ \text{if } r\geq 1\end{cases}\tag{13}$$

*where* $\varphi_t\left(\underline{\lambda}_{(r)}\right)=\exp\left\{\int\limits_0^t\left(\alpha(s)-i\underline{\lambda}_{(r)}\right)ds\right\}$.

Applying ADM method to obtain approximate transfer functions $\left(F_N(t,\lambda_{(r)})\right)_{N\geq 1}$, $(t,r)\in\mathbb{R}\times\mathbb{N}$ where

$$F_N(t,\lambda_{(r)})=\sum_{n=0}^{N-1}f_n(t,\lambda_{(r)})\to f(t,\lambda_{(r)})\text{ as } N\to\infty$$

.

## 4.1. First Technique.

We substitute the series $f(t,\lambda_{(r)})=\sum_{n=0}^{\infty}f_n(t,\lambda_{(r)})$ in the equations (12) we obtain

(1) For $r=0$, we have

$$Lf(t,0)=\alpha(t)f(t,0)+\mu(t),$$

which implies

$$\sum_{n=0}^{\infty}f_n(t,0)=f(0,0)+L^{-1}\left(\alpha(t)\sum_{n=0}^{\infty}f_n(t,0)+\mu(t)\right),$$

it follows, $f_0(t,0)=f(0,0)+\int_0^t\mu(s)ds$ and $\forall n\geq 1$,

$$f_n(t,0)=\int_0^t\alpha(s)f_{n-1}(s,0)ds.$$

After computing the terms $(f_n(t,0))_{n\geq 1}$ have the form

$$f_1(t,0) = f(0,0)\int_0^t \alpha(s_1)ds_1 + \int_0^t\int_0^{s_1}\alpha(s_1)\mu(s_2)ds_2ds_1,$$

$$f_2(t,0) = f(0,0)\int_0^t\int_0^{s_1}\alpha(s_1)\alpha(s_2)ds_2ds_1 + \int_0^t\int_0^{s_1}\int_0^{s_2}\alpha(s_1)\alpha(s_2)\mu(s_3)ds_3ds_2ds_1,$$

$$\vdots$$

$$f_n(t,0) = f(0,0)\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}}\alpha(s_1)...\alpha(s_n)ds_nds_{n-1}...ds_2ds_1$$
$$+\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}}\int_0^{s_n}\alpha(s_1)...\alpha(s_n)\mu(s_{n+1})ds_{n+1}ds_nds_{n-1}...ds_2ds_1.$$

(2) For $r=1$, we have

$$Lf(t,\lambda) = (\alpha(t)-i\lambda)f(t,\lambda) + (\gamma(t)f(t,0)+\beta(t)),$$

which implies

$$\sum_{n=0}^{\infty} f_n(t,\lambda) = f(0,\lambda) + L^{-1}\left((\alpha(t)-i\lambda)\sum_{n=0}^{\infty} f_n(t,\lambda) + \gamma(t)f(t,0)+\beta(t)\right),$$

it follows, $f_0(t,\lambda) = f(0,\lambda)+\int_0^t(\gamma(s)f(s,0)+\beta(s))\,ds$ and $\forall n\geq 1$,

$$f_n(t,\lambda) = \int_0^t (\alpha(s)-i\lambda)\,f_{n-1}(s,\lambda)ds.$$

After computing the terms $(f_n(t,\lambda))_{n\geq 1}$ have the form

$$f_1(t,\lambda) = f(0,\lambda)\int_0^t(\alpha(s_1)-i\lambda)\,ds_1 + \int_0^t\int_0^{s_1}(\alpha(s_1)-i\lambda)\,(\gamma(s_2)f(s_2,0)+\beta(s_2))\,ds_2ds_1,$$

$$f_2(t,\lambda) = f(0,\lambda)\int_0^t\int_0^{s_1}(\alpha(s_1)-i\lambda)\,(\alpha(s_2)-i\lambda)\,ds_2ds_1$$
$$+\int_0^t\int_0^{s_1}\int_0^{s_2}(\alpha(s_1)-i\lambda)\,(\alpha(s_2)-i\lambda)\,(\gamma(s_3)f(s_3,0)+\beta(s_3))\,ds_3ds_2ds_1,$$

$$\vdots$$

$$f_n(t,\lambda) = f(0,\lambda)\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}}(\alpha(s_1)-i\lambda)\ldots(\alpha(s_n)-i\lambda)\,ds_nds_{n-1}...ds_2ds_1$$
$$+\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}}\int_0^{s_n}(\alpha(s_1)-i\lambda)\ldots(\alpha(s_n)-i\lambda)\,(\gamma(s_{n+1})f(s_{n+1},0)+\beta(s_{n+1}))\,ds_{n+1}ds_nds_{n-1}...ds_2ds_1.$$

(3) For $r\geq 2$, we have

$$Lf(t,\lambda_{(r)}) = \left(\alpha(t)-i\underline{\lambda}_{(r)}\right)f(t,\lambda_{(r)}) + r\gamma(t)f(t,\lambda_{(r-1)}),$$

which implies

$$\sum_{n=0}^{\infty} f_n(t,\lambda_{(r)}) = f(0,\lambda_{(r)}) + L^{-1}\left(\left(\alpha(t) - i\underline{\lambda}_{(r)}\right)\sum_{n=0}^{\infty} f_n(t,\lambda_{(r)}) + r\gamma(t)f(t,\lambda_{(r-1)})\right),$$

it follows, $f_0(t,\lambda_{(r)}) = f(0,\lambda_{(r)}) + r\int_0^t \gamma(s)f(s,\lambda_{(r-1)})ds$ and $\forall n \geq 1$,

$$f_n(t,\lambda_{(r)}) = \int_0^t \left(\alpha(s) - i\underline{\lambda}_{(r)}\right) f_{n-1}(s,\lambda_{(r)})ds.$$

After computing the terms $\left(f_n(t,\lambda_{(r)})\right)_{n\geq 1}$ have the form

$$f_1(t,\lambda_{(r)}) = f(0,\lambda_{(r)})\int_0^t \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)ds_1 + r\int_0^t\int_0^{s_1}\left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\gamma(s_2)f(s_2,\lambda_{(r-1)})ds_2ds_1,$$

$$f_2(t,\lambda_{(r)}) = f(0,\lambda_{(r)})\int_0^t\int_0^{s_1}\left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\left(\alpha(s_2) - i\underline{\lambda}_{(r)}\right)ds_2ds_1$$

$$+ r\int_0^t\int_0^{s_1}\int_0^{s_2}\left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\left(\alpha(s_2) - i\underline{\lambda}_{(r)}\right)\gamma(s_3)f(s_3,\lambda_{(r-1)})ds_3ds_2ds_1,$$

$$\vdots$$

$$f_n(t,\lambda_{(r)}) = f(0,\lambda_{(r)})\int_0^t\int_0^{s_1}\ldots\int_0^{s_{n-1}}\left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\ldots\left(\alpha(s_n) - i\underline{\lambda}_{(r)}\right)ds_nds_{n-1}\ldots ds_2ds_1$$

$$+ r\int_0^t\int_0^{s_1}\ldots\int_0^{s_{n-1}}\int_0^{s_n}\left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\ldots\left(\alpha(s_n) - i\underline{\lambda}_{(r)}\right)\gamma(s_{n+1})f(s_{n+1},\lambda_{(r-1)})ds_{n+1}ds_nds_{n-1}\ldots ds_2ds_1.$$

**Theorem 4.3.** *Assume that the process $(X(t))_{t\geq 0}$ generated by the SDE (1) has a regular second-order solution. We consider the process $(\mathscr{F}_N(t))_{t\geq 0}$ which has the following spectral representation*

$$\mathscr{F}_N(t) = \widetilde{F}_N(t,0) + \sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}}\widetilde{F}_N(t,\lambda_{(r)})dZ(\lambda_{(r)}), \tag{14}$$

*where $\widetilde{F}_N(t,\lambda_{(r)}) = \sum_{n=0}^{N-1}\widetilde{f}_n(t,\lambda_{(r)})$, $\forall N \geq 1$. Then almost surely, we have*

$$\lim_{N\to+\infty}\mathscr{F}_N(t) = X(t).$$

*Proof.* First applying the Lemma 3.1, Theorem 3.2, we obtain for all $r \geq 0$,

$$\lim_{N\to+\infty} F_N(t,\lambda_{(r)}) = \lim_{N\to+\infty}\sum_{n=0}^{N-1} f_n(t,\lambda_{(r)}) = f(t,\lambda_{(r)}),$$

by substitution in (14) we get

$$\lim_{N\to+\infty}\mathscr{F}_N(t)=\widetilde{f}(t,0)+\sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r}e^{it\underline{\lambda}_{(r)}}\widetilde{f}(t,\lambda_{(r)})dZ(\lambda_{(r)})=X(t),$$

where the transfer functions $f(t,\lambda_{(r)}),(t,r)\in\mathbb{R}\times\mathbb{N}$ are given by (13) in theorem 4.1. $\square$

**Remark 4.4.** *Theorem 4.3 means that, $\forall n\geq 0$ there exists a process $(\chi_n(t))_{t\in\mathbb{R}}$ admits the spectral representation*

$$\chi_n(t)=\widetilde{f}_n(t,0)+\sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r}e^{it\underline{\lambda}_{(r)}}\widetilde{f}_n(t,\lambda_{(r)})dZ(\lambda_{(r)}),\forall n\geq 0,$$

*such that*

$$X(t)=\sum_{n=0}^{\infty}\chi_n(t).$$

**4.2. Second Technique.** In this subsection, we present a different technique to decomposition the transfer functions $f(t,\lambda_{(r)}$ by the substitution of the series $f(t,\lambda_{(r)})=\sum_{n=0}^{\infty}g_n(t,\lambda_{(r)})$ and $f(t,\lambda_{(r-1)})=\sum_{n=0}^{\infty}g_n(t,\lambda_{(r-1)})$, for all $r\geq 1$ in the differential equations (13), so

For $r=0$, we have

$$f(t,0)=\sum_{n=0}^{\infty}g_n(t,0),$$

where $g_0(t,0)=f(0,0)+\int_0^t\mu(s)ds=f_0(t,0)$, and

$$g_n(t,0)=\int_0^t\alpha(s)g_{n-1}(s,0)ds,\ \forall n\geq 1,$$

this implies

$$g_n(t,0)=f_n(t,0),\ \forall n\geq 0.$$

Now for all $r\geq 1$, we substitute the series $f(t,\lambda_{(r)})=\sum_{n=0}^{\infty}g_n(t,\lambda_{(r)})$ and $f(t,\lambda_{(r-1)})=\sum_{n=0}^{\infty}g_n(t,\lambda_{(r-1)})$ in the differential equations

$$f^{(1)}(t,\lambda_{(r)})=\left(\alpha(t)-i\underline{\lambda}_{(r)}\right)f(t,\lambda_{(r)})+r\left(\gamma(t)f(t,\lambda_{(r-1)})+\delta_{\{r=1\},}\beta(t)\right)$$

we obtain

(1) For $r=1$, we have

$$Lf(t,\lambda)=(\alpha(t)-i\lambda)f(t,\lambda)+(\gamma(t)f(t,0)+\beta(t)),$$

which implies

$$\sum_{n=0}^{\infty} g_n(t,\lambda) = f(0,\lambda) + \int_0^t \beta(s)ds + \int_0^t \sum_{n=0}^{\infty} \left[(\alpha(s) - i\lambda)\, g_n(s,\lambda) + \gamma(s) g_n(s,0)\right] ds,$$

it follows, $g_0(t,\lambda) = f(0,\lambda) + \int_0^t \beta(s)ds$ and

$$g_n(t,\lambda) = \int_0^t (\alpha(s) - i\lambda)\, g_{n-1}(s,\lambda)ds + \int_0^t \gamma(s) g_{n-1}(s,0)ds, \ \forall n \geq 1.$$

After computing the terms $(g_n(t,\lambda))_{n\geq 1}$ have the form

$$\begin{aligned}
&g_1(t,\lambda) = f(0,\lambda)\int_0^t (\alpha(s_1) - i\lambda)\, ds_1 + \int_0^t \int_0^{s_1} (\alpha(s_1) - i\lambda)\, \beta(s_2) ds_2 ds_1 + \int_0^t \gamma(s_1) g_0(s_1,0) ds_1,\\
&g_2(t,\lambda) = f(0,\lambda)\int_0^t \int_0^{s_1} (\alpha(s_1) - i\lambda)\, (\alpha(s_2) - i\lambda)\, ds_2 ds_1\\
&+ \int_0^t \int_0^{s_1} \int_0^{s_2} (\alpha(s_1) - i\lambda)\, (\alpha(s_2) - i\lambda)\, \beta(s_3) ds_3 ds_2 ds_1\\
&+ \int_0^t \int_0^{s_1} (\alpha(s_1) - i\lambda)\, \gamma(s_2) g_0(s_2,0) ds_2 ds_1 + \int_0^t \gamma(s_1) g_1(s_1,0) ds_1,\\
&\vdots\\
&g_n(t,\lambda) = f(0,\lambda)\int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} (\alpha(s_1) - i\lambda) \ldots (\alpha(s_n) - i\lambda)\, ds_n ds_{n-1} \ldots ds_2 ds_1\\
&+ \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} \int_0^{s_n} (\alpha(s_1) - i\lambda) \ldots (\alpha(s_n) - i\lambda)\, \beta(s_{n+1}) ds_{n+1} ds_n ds_{n-1} \ldots ds_2 ds_1\\
&+ \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} (\alpha(s_1) - i\lambda) \ldots (\alpha(s_{n-1}) - i\lambda)\, \gamma(s_n) g_0(s_n,0) ds_n ds_{n-1} \ldots ds_2 ds_1\\
&+ \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-2}} (\alpha(s_1) - i\lambda) \ldots (\alpha(s_{n-2}) - i\lambda)\, \gamma(s_{n-1}) g_1(s_{n-1},0) ds_{n-1} ds_{n-2} \ldots ds_2 ds_1 + \ldots\\
&\ldots + \int_0^t \int_0^{s_1} \int_0^{s_2} (\alpha(s_1) - i\lambda)\, (\alpha(s_2) - i\lambda)\, \gamma(s_3) g_{n-3}(s_3,0) ds_3 ds_2 ds_1\\
&+ \int_0^t \int_0^{s_1} (\alpha(s_1) - i\lambda)\, \gamma(s_2) g_{n-2}(s_2,0) ds_2 ds_1 + \int_0^t \gamma(s_1) g_{n-1}(s_1,0) ds_1.
\end{aligned}$$

(2) For $r \geq 2$, we have

$$Lf(t,\lambda_{(r)}) = \left(\alpha(t) - i\underline{\lambda}_{(r)}\right) f(t,\lambda_{(r)}) + r\gamma(t) f(t,\lambda_{(r-1)}),$$

which implies

$$\sum_{n=0}^{\infty} g_n(t,\lambda_{(r)}) = f(0,\lambda_{(r)}) + \sum_{n=0}^{\infty} \int_0^t \left[\left(\alpha(s) - i\underline{\lambda}_{(r)}\right) g_n(s,\lambda_{(r)}) + r\gamma(s) g_n(s,\lambda_{(r-1)})\right] ds,$$

it follows, $g_0(t,\lambda_{(r)}) = f(0,\lambda_{(r)})$ and $\forall n \geq 1$,

$$g_n(t,\lambda_{(r)}) = \int_0^t \left(\alpha(s) - i\underline{\lambda}_{(r)}\right) g_{n-1}(s,\lambda_{(r)})ds + r\int_0^t \gamma(s)g_{n-1}(s,\lambda_{(r-1)})ds.$$

After computing the terms $\left(g_n(t,\lambda_{(r)})\right)_{n\geq 1}$ have the form

$$g_1(t,\lambda_{(r)}) = f(0,\lambda_{(r)})\int_0^t \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right) ds_1 + r\int_0^t \gamma(s_1)g_0(s_1,\lambda_{(r-1)})ds_1,$$

$$g_2(t,\lambda_{(r)}) = f(0,\lambda_{(r)})\int_0^t \int_0^{s_1} \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\left(\alpha(s_2) - i\underline{\lambda}_{(r)}\right) ds_2 ds_1$$

$$+ r\int_0^t \int_0^{s_1} \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\gamma(s_2)g_0(s_2,\lambda_{(r-1)})ds_2 ds_1 + r\int_0^t \gamma(s_1)g_1(s_1,\lambda_{(r-1)})ds_1,$$

$$\vdots$$

$$g_n(t,\lambda_{(r)}) = f(0,\lambda_{(r)})\int_0^t \int_0^{s_1} \dots \int_0^{s_{n-1}} \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\dots\left(\alpha(s_n) - i\underline{\lambda}_{(r)}\right) ds_n ds_{n-1}\dots ds_2 ds_1$$

$$+ r\int_0^t \int_0^{s_1} \dots \int_0^{s_{n-1}} \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\dots\left(\alpha(s_{n-1}) - i\underline{\lambda}_{(r)}\right)\gamma(s_n)g_0(s_n,\lambda_{(r-1)})ds_n ds_{n-1}\dots ds_2 ds_1$$

$$+ r\int_0^t \int_0^{s_1} \dots \int_0^{s_{n-2}} \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\dots\left(\alpha(s_{n-2}) - i\underline{\lambda}_{(r)}\right)\gamma(s_{n-1})g_1(s_{n-1},\lambda_{(r-1)})ds_{n-1} ds_{n-2}\dots ds_2 ds_1 + \dots$$

$$\dots + r\int_0^t \int_0^{s_1} \int_0^{s_2} \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\left(\alpha(s_2) - i\underline{\lambda}_{(r)}\right)\gamma(s_3)g_{n-3}(s_3,0)ds_3 ds_2 ds_1$$

$$+ r\int_0^t \int_0^{s_1} \left(\alpha(s_1) - i\underline{\lambda}_{(r)}\right)\gamma(s_2)g_{n-2}(s_2,\lambda_{(r-1)})ds_2 ds_1 + r\int_0^t \gamma(s_1)g_{n-1}(s_1,\lambda_{(r-1)})ds_1.$$

**Theorem 4.5.** *Under the condition of theorem 4.3, $\forall n \geq 0$ there exists a process $(\mathscr{X}_n(t))_{t\in\mathbb{R}}$ admits the spectral representation*

$$\mathscr{X}_n(t) = \widetilde{g}_n(t,0) + \sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}}\widetilde{g}_n(t,\lambda_{(r)})dZ(\lambda_{(r)}), \forall n \geq 0,$$

*such that*

$$X(t) = \sum_{n=0}^{\infty}\mathscr{X}_n(t).$$

*Proof.* The result follows immediately, because

$$\sum_{n=0}^{\infty} g_n(t,\lambda_{(r)}) = \sum_{n=0}^{\infty} f_n(t,\lambda_{(r)}) = f(t,\lambda_{(r)}),$$

then

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathscr{X}_n(t) &= \sum_{n=0}^{\infty}\widetilde{g}_n(t,0)+\sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}}\sum_{n=0}^{\infty}\widetilde{g}_n(t,\lambda_{(r)})dZ(\lambda_{(r)}) \\
&= \sum_{n=0}^{\infty}\widetilde{f}_n(t,0)+\sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}}\sum_{n=0}^{\infty}\widetilde{f}_n(t,\lambda_{(r)})dZ(\lambda_{(r)}) \\
&= \widetilde{f}(t,0)+\sum_{r\geq 1}\frac{1}{r!}\int_{\mathbb{R}^r} e^{it\underline{\lambda}_{(r)}}\widetilde{f}(t,\lambda_{(r)})dZ(\lambda_{(r)}) \\
&= X(t).
\end{aligned}$$

□

## 5. Conclusion

In this paper we propose the ADM for solving bilinear SDE driven by Brownian motion wich behaves as a nonlinear stochastic differential equation. This method is applied in time and frequency domain where we have proved the convergence of the approximate processes to the exact solution.

## Conflict of Interests

The author declares that there is no conflict of interests.

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