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D-LOCAL ANTIMAGIC VERTEX COLORING OF A GRAPH AND SOME GRAPH OPERATIONS

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Abstract. Let $G(V,E)$ be a simple, connected (p,q)-graph. A d-local antimagic labelling is a bijection $f : E(G) \to \{1,2,3,4,\dots q\}$ such that for any two adjacent vertices, v_1 and v_2, $w(v_1) \neq w(v_2)$ where $w(v_i) = \sum_{e\in E(v_i)} f(e) - deg(v_i)$,and $E(v_i)$ is the set of edges incident to v_i for $i = 1,2,\dots, p$. Any d-local antimagic labelling induces a proper vertex coloring of G where the vertex, v_i is assigned the color $w(v_i)$ for $i = 1,2,\dots p$ and this coloring is called d-local antimagic coloring of G. The minimum number of colors required to color the vertices in a d-local antimagic coloring of G is called the d-local antimagic chromatic number of G and it is denoted as $\chi_{dla}(G)$. In this paper, we study the d-local antimagic vertex coloring of paths, cycles, star graphs, complete bipartite graphs and some graph operations such as the subdivision of each edge of a graph by a vertex and determine the exact value of the parameter, d-local antimagic chromatic number for these graphs.

Keywords: antimagic labelling; local antimagic labelling; local antimagic chromatic number; d-local antimagic labelling; d-local antimagic chromatic number.

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1. INTRODUCTION

. By a graph, we mean a finite undirected graph without loops or parallel edges. A coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices have distinct colors. We can represent the colors by natural numbers so that the function $w : V(G) \to \mathbb{N}$ is a vertex coloring of a graph G and $w(v)$ denote the color of a vertex v. If any two adjacent

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vertices v_1 and v_2, $w(v_1) \neq w(v_2)$ then w is called a proper vertex coloring of G. Topics related to coloring of graphs has been studied by many researchers and Suresh Kumar [5] presented a survey of various graph colorings and related parameters.

. Hartsfield and Ringel [4] introduced the concept of antimagic labeling of a graph. Let $G = (V,E)$ be a graph and let $F : E \to \{1,2,\dots|E|\}$ be a bijection. For each vertex u of G, let the weight of the vertex, u is $w(u) = \sum_{e\in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u. If $w(u) \neq w(v)$ for any two distinct vertices u and v of G, then, f is called an antimagic labeling of G. A graph G is called antimagic if G has an antimagic labeling.

. Arumugam et al.[1] defined a local antimagic labelling as a bijection $f : E(G) \to \{1,2,3,\dots q\}$ such that for any two adjacent vertices u and v, $w(u) \neq w(v)$. Thus, any local antimagic labeling induces a proper vertex coloring of G, where the vertex v is assigned the color $w(v)$. We define this coloring as local antimagic coloring of G. The local antimagic chromatic number $\chi_{la}(G)$)is defined as the minimum number of colors taken over all local antimagic colorings of G. We use the following results about this parameter in the sequel

Theorem 1.1. [1] *For any tree T with l leaves, $\chi_{la}(T) \geq l+1$*

Theorem 1.2. [1] *For the path graph, P_n with $n \geq 3$, $\chi_{la}(P_n) = 3$.*

Theorem 1.3. [1] *For the cycle graph, C_n , $\chi_{la}(C_n) = 3$.*

Theorem 1.4. [1] *For the complete bipartite graph $G = K_{2,n}$*

$$\chi_{la}(K_{2,n}) = \begin{cases} 2 & \text{if } n \text{ is even, and } n \geq 4 \\ 3 & \text{if } n \text{ is odd Or } n=2 \end{cases}$$

. In this paper, we investigate a variation of local antimagic coloring, namely d-local antimagic vertex coloring, which takes the degrees of the vertices also into consideration. We also determine the d-local antimagic chromatic number of some special classes of graphs such as star graphs, path graphs, cycle graphs, complete bipartite graphs and some graph operations such as subdivision of each edge of a graph by one or two vertex each. For the terms and notations not mentioned here, reader may refer Harary [3].

2. Main Results

We begin with the definition of local antimagic coloring of a graph.

Definition 2.1. [1]

A local antimagic labelling is a bijection $f : E(G) \to \{1,2,.......|E(G)|\}$ such that for any two adjacent vertices v_1 and v_2, $w(v_1) \neq w(v_2)$ where $w(v_i) = \sum_{e\in E(v_i)} f(e)$ and $E(v_i)$ is the set of edges incident to v_i for $i = 1,2,\ldots p$. Thus, a local antimagic labelling induces a proper vertex coloring of G where the vertex,v_i is assigned the color $w(v_i)$. This coloring is called a local antimagic coloring of G. The minimum number of colors required to color the vertices in a local antimagic coloring of a graph, G is called the local antimagic chromatic number of G and it is denoted as $\chi_{la}(G)$. Taking the vertex degrees also into consideration, we define a new coloring and an associated graph parameter.

Definition 2.2.

A d-local antimagic labelling is a bijection $f : E(G) \to \{1,2,\ldots.|E(G)|\}$ such that for any two adjacent vertices v_1 and v_2, $w(v_1) \neq w(v_2)$ where $w(v_i) = \sum_{e\in E(v_i)} f(e) - deg(v_i)$ and $E(v_i)$ is the set of edges incident to v_i for $i = 1,2,\ldots,p$. Thus any d-local antimagic labelling induces a proper vertex coloring of G, where the vertex v_i is assigned the color $w(v_i)$. We define this coloring as d-local antimagic coloring of G. The minimum number of colors required to color the vertices in a d-local antimagic coloring of a graph, G is called the d-local antimagic chromatic number of G and it is denoted as $\chi_{dla}(G)$.

Lemma 2.3. For any graph G, $\chi_{dla}(G) \geq \chi_{la}(G)$

Proof: If G is local antimagic colourable, χ_{la} (G) is the minimum number of colors such that $w(v_1) \neq w(v_2)$ whenever v_1 and V_2 are adjacent where $w(w_1) = \sum_{e\in E(v_i)} f(e)$. If G is d-local antimagic colourable,$\chi_{dla}(G)$ is the minimum number of colors such that$w(v_1) \neq w(v_2)$ whenever v_1 and v_2 are adjacent where $w(v_i) = \sum_{e\in E(v_i)} f(e) - deg(v_i)$. It is enough to prove that if G is d-local antimagic colourable with k colors, then G is local antimagic colourable also with k colors. That is, for any adjacent vertices, v_1, v_2 of G. $\sum_{e\in E(v_1)} f(e) - deg(v_1) \neq \sum_{e\in E(v_2)} f(e) - deg(v_2)$ $\implies \sum_{e\in E(v_1)} f(e) \neq \sum_{e\in E(v_2)} f(e)$. Let v_1 and v_2 are adjacent vertices of G. We will consider two cases.

Case.1: deg(v_1) = deg (v_2), $w(v_1) \neq w(v_2) \implies \sum_{e\in E(v_1)} f(e) - deg(v_1) \neq \sum_{e\in E(v_2)} f(e)$, which implies $\sum_{e\in E(v_1)} f(e) \neq \sum_{e\in E(v_2)} f(e)$. So, G is local antimagic colourable and $\chi_{dla}(G) \geq \chi_{la}(G)$

Case.2. $\deg(v_1) \neq \deg\ (v_2)$ $w(v_1) \neq w(v_2)$ implies that $\sum_{e\in E(v_1)} f(e) - deg(v_1) \neq \sum_{e\in E(v_2)} f(e) - deg(v_2)$ which further implies that $\sum_{e\in E(v_1)}(f(e)-1) \neq \sum_{e\in E(v_2)}(f(e)-1)$. Since $f(e)$ is a bijection, $f'(e) = f(e)-1$ is a bijection from $E(G) \to \{0,1\ldots q-1\}$ and , $\sum_{e\in E(v_1)} f'(e) \neq \sum_{e\in E(v_2)} f'(e)$. Hence, G is local antimagic colourable and $\chi_{dla}(G) \geq \chi_{la}(G)$

Theorem 2.4. *For any graph G,$\chi_{dla}(G) \geq \chi_{la}(G) \geq \chi(G)$, where $\chi(G)$ is the usual vertex chromatic number of G*

Proof: The lower bound immediately follows from the fact that where $\chi(G)$ is the minimum number of colors in any vertex coloring of G and $\chi_{dla}(G)$ is the minimum number of colors in a vertex coloring of G induced by a local antimagic coloring of G. The upper bound follows from Lemma 2.3.

Theorem 2.5. $\chi_{dla}(K_{1,n}) = n+1$ *for* $n \geq 3$

Proof: Let v be the apex vertex and $v_1, v_2, \ldots v_n$ are the pendent vertices of $K_{1,n}$. Define a function $f : E(K_{1,n}) \to \{1,2,3,\ldots,n\}$ as follows: $f(e_i = vv_i) = i$, if $1 \leq i \leq n$. The range of f is $\{1,2,3,\ldots,n\}$ which is same as the co-domain. For any two distinct edges,$e_1, e_2 \in E(G)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$. Hence, f is a bijection.
$w(v_i) = \sum_{e\in E(v_i)} f(e) - deg(v_i) = i-1 for 1 \leq i \leq n$, and $w(v) = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$.
So, f is a d-local antimagic labeling of $K_{1,n}$ and f induces a proper d-local antimagic vertex coloring of G using $n+1$ colors. Hence,$\chi_{dla}(K_{1,n}) \leq n+1$. By Theorem.1.1, $\chi_{la}(K_{1,n}) \geq n+1$ and by Theorem 2.4, $\chi_{dla}(G) \geq \chi_{la}(G)$ so that $\chi_{dla}(G) \geq n+1$. Hence, $\chi_{dla}(K_{1,n}) = n+1$.

Theorem 2.6. $\chi_{dla}(P_n) = 4, n \geq 4$

Proof: Let the vertices $V(P_n) = \{v_1, v_2, v_3, \ldots v_n\}$. $E(P_n) = \{e_i = v_iv_{i+1},\ 1 \leq i \leq n\}$
Case.1 n is even
Define a function $f : E(P_n) \to \{1,2,3,...,n-1\}$ as follows: $f(e_i) = n-1$, $f(e_{n-1}) = 1$

$$f(e_i) = \begin{cases} n-i & i \text{ odd}, 1 \leq i \leq n-3 \\ \text{i} & i \text{ even } 2 \leq i \leq n-2 \end{cases}$$

The range of f is $\{n-1, n-3, \ldots 3, 1\} \cup \{2, 4, \ldots n-2\} = \{1,2,3,\ldots n-1\}$, which is same as the co-domain of f. Also, for any two distict edges $e_1, e_2 \in E(G)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$. Hence f is a bijection. Then the colors are

$$w(v_i) = \begin{cases} n-1 & \text{if } i \text{ is even, } i \neq n \\ 0 & \text{if } i = n \\ n-2 & i = 1 \\ n-3 & \text{if i odd, } i \neq 1 \end{cases}$$

Hence, f induces a proper d-local antimagic vertex coloring of G using 4 colors so that $\chi_{dla}(P_n) \leq 4$. By Theorem 1.2, $\chi_{la}(P_n) = 3$. By Theorem 2.4, $\chi_{dla}(G) \geq \chi_{la}(G)$.Hence, $3 \leq \chi_{dla}(P_n) \leq 4$.

If possible, assume that $\chi_{dla}(P_n) = 3$. There are only three cases to consider, since if we consider any other case then adjacent vertices will receive same color. In the above induced coloring, the initial vertex v_1 and terminal vertex v_n receive different colors.

Case.1.1 v_1 and v_n takes the same color. Then $n-2 = 0 \implies n = 2$ which is not possible, since $n \geq 4$.

Case.1.2 The initial vertex v_1 and the odd labelled vertices v_i receive the same color. Then $n-2 = n-3$, which is impossible.

Case.1.3 The terminal vertex, v_n and the even labelled vertices, v_j and takes the same color. Then $n-1 = 0 \implies n = 1$,which is not possible since $n \geq 4$. So, our assumption is wrong.$\chi_{dla}(P_n) \neq 3$. Hence, $\chi_{dla}(P_n) = 4$

Case.2: n is odd. Define a function $f; E(P_n) \to \{1,2,3\ldots n-1\}$ as follows: $f(e_{n-1}) = n-1$

$$f(e_i) = \begin{cases} n-i+\frac{(i-2)}{2} & i \text{ even, } 2 \leq i \leq n-3 \\ 1+\frac{(i-1)}{2} & i \text{ odd, } 1 \leq i \leq n-2 \end{cases}$$

Range of f is $\{n-1\} \cup \{n-2, n-3, \ldots \frac{n+1}{2}\} cup \{2,3,4,\ldots,\frac{n-1}{2}\} = \{1,2,\ldots.n-1\}$, which is the same as the co-domain. Also, for any two distinct edges, $e_1, e_2 \in E(G)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$.So f is a bijection. $w(v_1) = 0$

$$w(v_i) = \begin{cases} n-3 & \text{if } i \text{ even, } 2 \leq i \leq n-3 \\ n-2 & i \text{ odd } 1 \leq i \leq n \\ n-3+(\frac{n-1}{2}) & \text{if } i = n-1 \end{cases}$$

Thus, f induces a d-local antimagic coloring of P_n, using 4 colors, so $\chi_{dla}(P_n) \leq 4$. But $\chi_{la}(P_n) = 3$ by Theorem 1.2, $\chi_{dla}(G) \geq \chi_{la}(G)$ by Theorem 2.4. So, $3 \leq \chi_{dla}(P_n) \leq 4$. If possible, assume that $\chi_{dla}(P_n) = 3$. In the above induced coloring pattern the initial vertex v_1 and terminal vertex v_n receive different colors. Also v_{n-1} receives another color.There are four cases to consider. If we consider any other case, then the adjacent vertices receive same color.

Case.1. v_1 and v_{n-1} takes the same color. So, $n-3+\frac{n-1}{2} = 0 \implies 3n = 7$ which is not possible for the integer value of n.

Case.2. v_1 and the odd labelled vertices,v_i , have the same color, So $n-2 = 0 \implies n = 2$, which is not possible since $n \geq 4$.

Case.3. v_{n-1} and the even labelled vertices v_j have the same color So $n-3 = n-3+\frac{n-1}{2} \implies n = 1$, which is not possible since $n \geq 4$.

Case.4. v_1 and odd labeled vertices v_i have the same color. So $n-2 = 0 \implies n = 2$, which is not possible, since $n \geq 4$. So, our assumption is wrong.$\chi_{dla}(P_n) \neq 3$ By Theorem 2.4, $\chi_{dla}(G) \geq \chi_{la}(G)$. Hence, $\chi_{dla}(P_n) = 4$.

Theorem 2.7. *For the cycle C_n, $\chi_{dla}(C_n) = 3$*

Proof: .Proof. Let $V(C_n) = \{v_1, v_2, v_3, \ldots v_n, v_1\}$ and . $E(C_n) = \{e_i = v_i v_i + 1\ ,\ 1 \leq i \leq n-1$, $e_n = v_n v_1\}$

Define a function $f : E(C_n) \to \{1, 2, 3, \ldots, n\}$ as follows:

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ odd} \\ n - \frac{i-2}{2} & i \text{ is even} \end{cases}$$

When n is odd

Here the range of f is $\{1, 2, \ldots \frac{n+1}{2}\} \cup \{n, n-1, n-2 \ldots \frac{n+3}{2}\} = \{1, 2, 3, \ldots, n\}$, which is same as the co-domain.$e_1, e_2 \in E(G)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$.So f is a bijection.

When n is even

Here the range of $f$is $\{1, 2, \ldots, \frac{n}{2}\} \cup \{n, n-1, n-2, \ldots \frac{n}{2}+1\} = \{1, 2, 3, \ldots, n\}$, which is same as the co-domain.$e_1, e_2 \in E(G)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$. So f is a bijection.

$$w(v_i) = \begin{cases} n-1 & \text{if } i \text{ even} \\ n & i \text{ odd }, i \neq 1 \\ \lfloor \frac{n}{2} \rfloor & \text{if } i = 1 \end{cases}$$

So, f is a d- local antimagic labeling of C_n and f induces a proper vertex coloring using only 3 colors. Hence $\chi_{dla}(C_n) \leq 3$. By Theorem 1.3,$\chi_{la}(C_n) = 3$. By Theorem 2.4, $\chi_{dla}(G) \geq \chi_{la}(G) = 3$. Hence, $\chi_{dla}(C_n) = 3$.

Theorem 2.8. *For a complete Bipartite graph $K_{m,2}$, $\chi_{dla}(K_{m,2}) = 3$ where $m \geq 3$*

Proof: Let $V = \{v_1, v_2, v_3, \dots v_n\}$ and $U = \{u_1, u_2\}$ be the bipartition of $K_{m,2}$ Define a functiojn $f : E(K_{m,2}) \to \{1,2,3,...,2m\}$ as follows:
$f(v_iu_1) = i$, $1 \leq i \leq m$, $f(v_iu_2) = 2m-(i-1)$, $1 \leq i \leq m$
Here the range of f is $\{1,2,3,\dots,m\} \cup \{2m, 2m-1, 2m-2, \dots, m+1\} = \{1,2,3,\dots,2m\}$, which is same as the co-domain.$e_1, e_2 \in E(G)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$. So f is a bijection.

$$w(v_i) = 2m-1 \; \forall \; 1 \leq i \leq m,$$
$$w_(u_i) = \frac{m(m+1)}{2} - m,$$
$$w(u_2) = \frac{m(3m+1)}{2} - m$$

Then, f is a d- local antimagic labeling of $K_{m,2}$ and f induces a proper vertex coloring. The number of colors used are 3. Hence $\chi_{dla}(K_{m,2}) \leq 3$. By theorem 1.4 $\chi_{la}(K_{m,2}) = 2$ if n is even and $\chi_{la}(K_{m,2}) = 3$ if n is odd. By Theorem 2.4, $\chi_{dla}(G) \geq \chi_{la}(G) = 3$. So,$\chi_{dla}(K_{m,2}) = 3$.

Theorem 2.9. *In any star $K_{1,n}$, $n \geq 3$, if each edge is subdived by one vertex the d-local antimagic Chromatic number will be increased by one, and if each edge is subdivided by two vertices the d-local antimagic Chromatic number will be increased by two*

Proof. Let v as the apex vertex and $v_2, v_3, \dots v_n$ are the pendent vertices of the graph $K_{1,n}$. Let u_i be the vertices inserted on each vv_i. The resulting graph,say H has $2n+1$ vertices and $2n$ edges. Let these edges are $e_{2i-1} = v_iu_i, 1 \leq i \leq n$ and $e_{2i} = u_iv, 1 \leq i \leq n$. Define a function $f : E(H) \to \{1,2,3,\dots 2n\}$ as follows: $f(e_{2i-1}) = 1+(i-1), 1 \leq i \leq n, \;\; f(e_{2i}) = (2n+1)-i$, $1 \leq i \leq n$. Here the range of f is $\{1,2,3.\dots.n,\} \cup \{2n, 2n-1, \dots, n+1\} = \{1,2,3,\dots en\}$ which is the co-domain of f. Also, $e_1, e_2 \in E(G)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$. So f is a bijection.

$$w(v_i) = i-1, \text{ if } 1 \le i \le n,$$
$$w(u_i) = 2n-1 \quad \text{if } 1 \le i \le n,$$
$$w(v) = \tfrac{n}{2}(3n+1)-n$$

. All the v_i's receive n different colors ,all the u_i's receive the same color $2n-1$ and apex vertex v receives another color, so we use in total $n+2$ colors only.$\chi_{dla}(H) \le n+2$. Arumugam et al. [1] proved that $\chi_{la}(H) \ge n+1$. So, $n+1 \le \chi_{dla}(H) \le n+2$. Assume that $\chi_{la}(H) = n+1$. The apex vertex v is adjacent to u_i's. So color of v and u_i's are different. Again u_i's are adjacent to v_i's,$1 \le i \le n$. So the color of u_i's and v_i's are different,$1 \le i \le n$. The only possibility of obtaining the chromatic number as $n+1$ is that n the color of v and the color of v_i's are the same. So $\frac{n}{2}(3n+1)-n = n-1$ gives $3n^2-3n+2 = 0$. Solving we can obtaining an integer value for n. So our assumption is wrong.Hence $\chi_{dla}(H) \ne n+1$. So $\chi_{dla}(H) = n+2$.
Again consider the subdivision of each edge with two vertices. Let v as the apex vertex and $v_1, v_2, v_3, \ldots v_n$ are the pendent vertices of the graph $K_{1,n}$ .Let us subdivide vv_i with two vertices such that u_i be adjacent to each v_i and w_i. The resulting graph say H_1 has $3n+1$ vertices and $3n$ edges.

$$e_{3i-2} = v_iu_i, 1 \le i \le n,$$
$$e_{3i-1} = u_iw_i, 1 \le i \le n,$$
$$e_{3i} = w_iv, 1 \le i \le n,$$

Define a bijection $f : E(H_1) \to \{1,2,3,...,3n\}$ as follows:

$$f(e_{3i-2}) = 1+(i-1), 1 \le i \le n,$$
$$f(e_{3i-1}) = (2n+1)-i, 1 \le i \le n,$$
$$f(e_{3i}) = (2n+i), 1 \le i \le n$$

The range of f is $\{1,2,3,\ldots,n\} \cup \{2n, 2n-1, \ldots, n+1\} \cup \{2n+1, 2n+2, \ldots, 3n\}$,which is $\{1,2,3,\ldots,.3n\}$, the same as the co-domain. Also,$e_1, e_2 \in E(H_1)$ and $e_1 \ne e_2 \implies f(e_1) \ne f(e_2)$. Hence,$f$ is a bijection.

$$w(v) = \tfrac{n}{2}(5n+1)-n$$
$$w(v_i) = i-1 \text{ if } 1 \le i \le n,$$
$$w(u_i) = 2n-1 \text{ if } 1 \le i \le n,$$
$$w(w_i) = 4n-1, \ 1 \le i \le n.$$

. All the v_i's receive n different colors, all the u_1's receive the same color $2n-1$, all the w_i's receive the same color $4n-1$ and apex vertex v receives another color, so we use in 6 total $n+3$ colors. $\chi_{dla}(H_1) \leq n+3$.

In any star $k_{1,n}; n \geq 3$, if each edge is subdivided by one vertex each, then the d-local antimagic chromatic number will be $n+2$.If each edge of $K_{1,n}; n \geq 3$ is subdivided by two vertices each, then the d-local antimagic chromatic number will increased or remains the same as H. In the first part we proved that $\chi_{dla}(H) = n+2$. So, $n+2 \leq \chi_{dla}(H_1) \leq n+3$ Assume $\chi_{la}(H_1) = n+2$. Then the apex vertex v is adjacent to w_i's, u_i be adjacent to each v_i and w_i be adjacent to u_i. There are three possibilities for our claim . (If we consider any other case, then the adjacent vertices receive same color.)

Case.1. The apex vertex, v and the vertices u_i's have the same color. So $2n-1 = \frac{n}{2}(5n+1)-n$ $\implies 5n^2-5n+2 = 0$, which is not possible for integer values of n.

Case.2. The apex vertex,v and the vertices v_i's have the same color. So $3n-1 = \frac{n}{2}(5n+1)-n \implies 5n^2-7n+2 = 0$ which is not possible for the integer value of n.

Case.3. v_i's and the vertices w_i's have the same color $n-1 = 4n-1 \implies n = 0$, which is not possible . So our assumption is wrong. Hence, $\chi_{dla}(H_1) = n+3$.

Theorem 2.10. *In any path P_n , $n \geq 3$, the subdivision of each edge with a vertex, the d-local anti-magic chromatic number of the new graph P_n^* will remain the same as that of P_n.*

Proof: Let the vertices of P_n be $v_1, v_2, v_3, \dots v_n$. Let u_i be the vertices inserted on each v_i, v_{i+1},The resulting graph P_n^* has $2n-1$ vertices and $2n-2$ edges. $e_{2i-1} = v_i u_i$, $1 \leq i \leq n-1$ $\forall$ odd i, $e_{2i} = v_i v_i + 1$ $1 \leq i \leq n-1$ $\forall$ even i. Define a function $f : E(P_n^*) \to \{1,2,3,...,2n\}$ as follows: $f(e_i) = 1$, $f(e_{2n-2}) = 2n-2$

$$f(e_i) = \begin{cases} 1 + \frac{i-1}{2} & \text{if } i \text{ is odd, } 1 \leq i \leq 2n-3 \\ (2n-1) - i + (\frac{i-2}{2}) & \text{if } i \text{ is even, } 2 \leq i \leq 2n-4 \end{cases}$$

. Here the range of f is $\{1\} \cup \{2n-2\} \cup \{2,3,4,\dots,n-1\} \cup \{2n-3, 2n-4, \dots n\} = \{1,2,3,\dots 2n-2\}$, which is same as the co-domain. $e_1, e_2 \in E(G)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$. So f is a bijection. $w(v_1) = 0, w(v_i) = 2n-3$ if $2 \leq i \leq n$,$w(u_i) = 2n-4$,if $1 \leq i \leq n-2$, $w(u_{n-1}) = 3n-5$. All the v_i's take the same color $2n-3$ except v_1 ,all the u_i's take the same color $2n-4$, except u_{n-1}, so we use in total 4 colors. Colors of all adjacent vertices are distinct. So,f is a d- local antimagic labeling of P_n^*. Here we use only 4 colors, $\chi_{dla}(P_n^*) \leq 4$. By Theorem 1.2,$\chi_{la}(P_n) = 3$. By Theorem 2.4, $\chi_{dla}(G) \geq \chi_{dla}(G)$.So $3 \leq \chi_{dla}(P_n^*) \leq 4$.

. When we subdivide each edge with a vertex then the number of edges becomes twice the number of edges of the original graph. So after carried out the subdivision, number of edges of the new graph becomes even, whenever n is odd or even.

. Assume that $\chi_{dla}(P_n^*) = 3$. There are only three possibilities to consider since the adjacent vertices receive same color in all other cases.

Case.1. v_1 and v_i's takes the same color. So $2n-3=0 \implies 2n=3$, which is not possible for the integer value of n.

Case.2. When u_{n-1} and v_1 takes the same color. So, $3n-5=0 \implies 3n=5$, which is not possible for the integer value of n.

Case.3. the u_i's and u_{n-1} take the same color.$2n-4=3n-5, \implies n=1$, which isn not possible since $n \geq 3$. So our assumption is wrong so that $\chi_{dla}(P_n^*) = 4$.

Theorem 2.11. *Let G_n be the graph obtained from the cycle C_n , $n \geq 3$ by subdividing each edge with a new vertex each. Then the d-local antimagic chromatic number of G_n is the same as that of C_n.*

Proof: Let $v_1, v_2, v_3, \ldots v_n$ be the vertices of cycle C_n. Subdivide each edge v_iv_{i+1}, with a vertex u_i so that $G_n = (v_1u_1v_2u_2v_3 \ldots u_nv_1), 1 \leq i \leq n$. Let the edges of G_n be $e_{2i} = u_i, v_{i+1}, ; 1 \leq i \leq n-1, e_{2i-1} = v_iu_i, ; 1 \leq i \leq n$, $e_2n = u_nv_1$,

Define a function $f : E(G) \to \{1,2,3,\ldots 2n\}$ as follows:

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd, } 1 \leq i \leq 2n-1 \\ 2n - (\frac{i-2}{2}) & \text{if } i \text{ is even, } 2 \leq i \leq 2n \end{cases}$$

Here the range of f is $\{1,2,3,\ldots.n\} \cup \{2n, 2n-1, \ldots, n+1\} = \{1,2,3,\ldots 2n\}$ which is same as the co-domain.$e_1, e_2 \in E(G_n)$ and $e_1 \neq e_2 \implies f(e_1) \neq f(e_2)$. So f is a bijection. $w(v_1) = \lfloor \frac{2n}{2} \rfloor$, $w(v_i) = 2n$ if $2 \leq i \leq n$ $w(u_i) = 2n-1$, if $1 \leq i \leq n$. Let G_n be the graph obtained from the cycle C_n , $n \geq 3$ by subdividing each edge with a new vertex each and G_n itself be a cycle. Here we use only 3 colors and adjacent vertices have distinct colors so that it is a proper vertex coloring.

Hence,$\chi_{dla}(G_n) \leq 3$. By Theorem 1.3, $\chi_{la}(G_n) = 3$. By Theorem 2.4, $\chi_{dla}(G) \geq \chi_{la}(G)$. Hence, $\chi_{dla}(G_n) = 3$.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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