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CRANK-NICOLSON APPROXIMATION OF FRACTIONAL ORDER FOR TIME FRACTIONAL RADON DIFFUSION EQUATION IN SOIL MEDIUM

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Abstract. The basic aim of this paper is to study the analysis for the Crank-Nicolson finite difference approximation for time fractional radon diffusion equation (TFRDE) in soil medium. The equation expresses the concentration of radon as function of space and time in soil medium. We discuss the stability and convergence of the scheme. Graphically the numerical solution of the test problem is carried out with the help of mathematical software Mathematica.

Keywords: time fractional differential equation; finite difference; Crank-Nicolson method; Caputo fractional derivative; Mathematica; stability; convergence.

2010 AMS Subject Classification: 35R11.

1. Introduction

The radon 222_{Rn} is naturally occurring radioactive noble gas which has no colour, odor, taste and produced by natural radioactive decay of uranium and thorium. Radon naturally present in soil, water, air, charcoal and also in building materials. The diffusion of radon 222_{Rn} has been studying extensively for last few decades. Radon is found naturally in charcoal, rocks, soils, earth crust, natural gases, and water and also in the man-made materials like concrete cement and other building materials. When the human beings are in contact with radon then only

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the gas can be inhaled or exhaled and if the decay potential is very high, the radon daughters resulting from the decay chain insert into the lungs and it affects on the different mechanism in the lung activities.Also radiations are liberated which increase the risk of having lung cancer. Due to hazardous properties of radon, researchers have great interest to study the radon transport through various medium. The research has been studied the soil radon transport in $[9,12,13]$. Furthermore, the radon transport through various media studied in $[1,2,3,4,5,6,7,8,10,11,14]$. The basic objective of this paper is to study the radon diffusion in soil medium, by solving it by using finite difference method (FDM).Partial differential equations play a crucial role to describe generally all systems, that comes under the change. Solving PDE of nonlinear kind is not easy task by employing analytical methods, so, mathematicians prefers numerical computational schemes to deal with.Numerical methods are effective to solve the heat and mass transfer problem, problems in fluid dynamics and other partial differential equations when the exact analytical methods fails to solve such problems. The development of high speed digital computers significantly increased the use of numerical methods in various branches of science, engineering and technology. Recently, for the solution of partial differential equations the FDM and the finite element method (FEM) are widely used. Depending on the nature of physical problem to be solved each method has its own advantages.

1.1. Diffusion theory. The diffusion theory came from the famous German physiologist Adolf Fick $(1829-1901)$. He stated that the flux density J is proportional to the gradient of concentration. This gives,

$$J = -D\frac{\partial C}{\partial t},$$

where J is the radon flux density $[Bq/m^2 s]$, D is diffusion coefficient $[m^2 s^{-1}]$ in the medium, C is the concentration, and $\frac{\partial C}{\partial t}$ is gradient of radon concentration. Now the change in concentration to change in time and position is stated by the Fick's second law which is the extension of Fick's first law, that gives,

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2} - \lambda C(x,t),$$

where $\lambda = 2.1 \times 10^{-6} s^{-1}$ is the decay constant.

The time fractional radon diffusion equation with boundary conditions can be written as

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2} - \lambda C(x,t),\ 0 < x < L, t \geq 0 \tag{1}$$

with initial condition:

$$C(x,0) = 0, 0 < x < L \tag{2}$$

and boundary conditions:

$$C(0,t) = C_0\ \frac{\partial C(L,t)}{\partial t} = 0\ , t \geq 0 \tag{3}$$

2. Crank-Nicolson Approximation of Fractional Order for Time Fractional Radon Diffusion Equation

We consider the following time fractional radon diffusion equation (TFRDE),

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = D\frac{\partial^2 C(x,t)}{\partial x^2} - \lambda C(x,t) 0 < \alpha \leq 1\ (x,t) \in [0,L] \times [0,T]$$

with initial condition:

$$C(x,0) = 0, 0 < x < L \tag{4}$$

and boundary conditions:

$$C(0,t) = C_0, \frac{\partial C(L,t)}{\partial t} = 0, \qquad t \geq 0 \tag{5}$$

We introduce the fractional order Crank-Nicolson type finite difference scheme to the (TFRDE) 4 to 5 We define

$$t_k = k\tau; k = 0,1,2,\cdots,N \text{ and } x_i = ih; i = 0,1,2,\cdots,M$$

where

$$\tau = \tfrac{T}{N} \text{ and } h = \tfrac{L}{M}$$

Let (x_i,t_k),$i = 0,1,2,\cdots,M$ and $k = 0,1,2,\cdots,N$ be the exact solution of the TFRDE 4 to 5 at the mesh point (x_i,t_k). Let C_i^k be the numerical approximation of the point $C(ih,k\tau)$.

In the TFRDE 4 to 5, the time fractional derivative is approximated in the Caputo sense by the following scheme

$$\begin{aligned}
\frac{\partial^\alpha C(x_i,t_k)}{\partial t^\alpha} &\approx \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{C(x_i,t_{j+1})-C(x_i,t_j)}{\tau}\int\limits_{j\tau}^{(j+1)\tau}\frac{d\eta}{(t_{k+1}-\eta)^\alpha}+o(\tau)\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{C(x_i,t_{j+1})-C(x_i,t_j)}{\tau}\int\limits_{(k-j)\tau}^{(k-j+1)\tau}\frac{d\xi}{\xi^\alpha}+o(\tau)\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{C(x_i,t_{k+1-j})-C(x_i,t_{k-j})}{\tau}\int\limits_{j\tau}^{(j+1)\tau}\frac{d\xi}{\xi^\alpha}+o(\tau)\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{C(x_i,t_{k+1-j})-C(x_i,t_{k-j})}{\tau}\left[\frac{(j+1)^{1-\alpha}-j^{1-\alpha}}{1-\alpha}\right]\\
&\quad +o(\tau)\\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[C(x_i,t_{k+1})-C(x_i,t_k)]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}[C(x_i,t_{k+1-j})\\
&\quad -C(x_i,t_{k-j})][(j+1)^{1-\alpha}-j^{1-\alpha}]+o(\tau)\\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[C_i^{k+1}-c_i^k]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j[C_i^{k-j+1}-C_i^{k-j}]+o(\tau)
\end{aligned}$$

where $b_j=(j+1)^{1-\alpha}-j^{1-\alpha},\ j=0,1,2,...,N$.

Now, for $\frac{\partial^2 C}{\partial x^2}$, we adopt the second order central difference scheme in space for each interior grid point x_i, $0\le i\le M$. Thus

$$\begin{aligned}
\frac{\partial^2 C(x_i,t_k)}{\partial x^2} &= \frac{1}{2}[\delta_x^2 C_i^{k+1}+\delta_x^2 C_i^k]\\
&= \frac{1}{2}[\frac{C_{i-1}^{k+1}-2C_i^{k+1}+C_{i+1}^{k+1}}{h^2}+\frac{C_{i-1}^{k}-2C_i^{k}+C_{i+1}^{k}}{h^2}]
\end{aligned}$$

where δ_x is the central difference.

Using time fractional approximation the Crank-Nicolson type numerical approximation to equation 4 to 5 is given as follows:

$$\begin{aligned}\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[C_i^{k+1} &- C_i^k] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k} b_j[C_i^{k-j+1} - C_i^{k-j}] \\ &= D\frac{1}{2}[\frac{C_{i-1}^{k+1} - 2C_i^{k+1} + c_{i+1}^{k+1}}{h^2} + \frac{C_{i-1}^{k} - 2C_i^{k} + c_{i+1}^{k}}{h^2}] - \lambda C(x_i, t_k)\end{aligned}$$

where $b_j = ((j+1)^{1-\alpha}) - j^{1-\alpha}$

$$\begin{aligned}[C_i^{k+1} - C_i^k] &+ \sum_{j=1}^{k} b_j[C_i^{k-j+1} - C_i^{k-j}] \\ &= \frac{D\Gamma(2-\alpha)\tau^{\alpha}}{2h^2}[C_{i-1}^{k+1} - 2C_i^{k+1} + C_{i+1}^{k+1} + C_{i-1}^{k} - 2C_i^{k} + C_{i+1}^{k}] \\ &\quad -\lambda\Gamma(2-\alpha)\tau^{\alpha}C_i^k\end{aligned}$$

let $r = \frac{D\Gamma(2-\alpha)\tau^{\alpha}}{2h^2}$ *and* $\mu = \lambda\Gamma(2-\alpha)\tau^{\alpha}$

we get

$$\begin{aligned}[C_i^{k+1} - C_i^k] &+ \sum_{j=1}^{k} b_j[C_i^{k-j+1} - C_i^{k-j}] \\ &= r[C_{i-1}^{k+1} - 2C_i^{k+1} + C_{i+1}^{k+1} + C_{i-1}^{k} - 2C_i^{k} + C_{i+1}^{k}] - \mu C_i^k\end{aligned}$$

After simplification

$$\begin{aligned}-rC_{i-1}^{k+1} + (1+2r)C_i^{k+1} - rC_{i+1}^{k+1} &= rC_{i-1}^{k} + (1-2r-\mu)C_i^k + rC_{i+1}^k \\ &\quad - \sum_{j=1}^{k} b_j[C_i^{k-j+1} - C_i^{k-j}]\end{aligned} \tag{6}$$

where

$$r = \frac{D\Gamma(2-\alpha)\tau^{\alpha}}{2h^2} \;\; \mu = \lambda\Gamma(2-\alpha)\tau^{\alpha}$$

and

$$b_j = (j+1)^{1-\alpha} - j^{1-\alpha}.$$

From 6, we obtain

$$\begin{aligned} -rC_{i-1}^{k+1} &+ (1+2r)C_i^{k+1} - rC_{i+1}^{k+1} \\ &= rC_{i-1}^{k} + (1-2r-\mu)C_i^k + rC_{i+1}^k - [b_1(C_i^k - C_i^{k-1}) + b_2(C_i^{k-1} - C_i^{k-2}) \\ &\quad + b_3(C_i^{k-2} - C_i^{k-3}) + ... + b_{k-1}(C_i^2 - C_i^1) + b_k(C_i^1 - C_i^0)] \\ &= rC_{i-1}^{k} + (1-2r-\mu)C_i^k + rC_{i+1}^k - [b_1C_i^k - (b_1-b_2)C_i^{k-1} \\ &\quad -(b_2-b_3)C_i^{k-2} - .... - (b_{k-1}-b_k)C_i^1 - b_kC_i^0] \end{aligned}$$

$$\begin{aligned} -rC_{i-1}^{k+1} &+ (1+2r)C_i^{k+1} - rC_{i+1}^{k+1} = rC_{i-1}^{k} + (1-2r-\mu-b_1)C_i^k \\ &+ rC_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})C_i^{k-j} + b_kC_i^0 \end{aligned} \tag{7}$$

Further, the initial condition is approximated as

$C_i^0 = 0,\ i = 0,1,2,3,\cdots,M$

For the two boundary points x_0 and x_M, the corresponding discretization schemes are $C_0^k = C_0$ and $\frac{\partial C(L,t)}{\partial x} = 0,\ \frac{C_{M+1}^k - C_{M-1}^k}{2h} = 0 \Rightarrow C_{M+1}^k = C_{M-1}^k$

Therefore, the fractional approximated initial boundary value problem is, for $k = 0$, from 6 we have

$$-rC_{i-1}^{1} + (1+2r)C_i^1 - rC_{i+1}^1 = rC_{i-1}^0 + (1-2r-\mu)C_i^0 + rC_{i+1}^0 \tag{8}$$

for $k \geq 1$, from 7 we have

$$\begin{aligned} -rC_{i-1}^{k+1} + (1-2r)C_i^{k+1} - rC_{i+1}^{k+1} &= rC_{i-1}^{k} + (1-2r-\mu-b_1)C_i^k + rC_{i+1}^k \\ &\quad + \sum_{j=1}^{k-1}(b_j - b_{j+1})C_i^{k-j} + b_kC_i^0 \end{aligned} \tag{9}$$

with initial condition:

$$C_i^0 = 0; i = 0,1,2,3,\cdots,M, \tag{10}$$

and boundary conditions:

$$C_0^k = C_0, \frac{\partial C(L,t)}{\partial x} = 0; k = 0,1,2,3,\cdots,N, \tag{11}$$

where

$$r = \frac{D\Gamma(2-\alpha)\tau^{\alpha}}{2h^2}, \ \mu = \lambda\Gamma(2-\alpha)\tau^{\alpha}$$

and

$$b_j = (j+1)^{1-\alpha} - j^{1-\alpha}; j = 1,2,3,\cdots,k$$

The fractional approximated initial boundary value problem 8-11 can be written in the following matrix equation form,

$$AC' = BC^0 + S, \ \ k = 0, \tag{12}$$

$$AC^{k+1} = CC^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})C^{k-j} + b_k C^0 + S, \ \ k \geq 1 \tag{13}$$

where

$$A = \begin{pmatrix} 1+2r & -r & & & & \\ -r & 1+2r & -r & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & & & & & \\ \cdots & \cdots & \cdots & & -2r & 1+2r \end{pmatrix},$$

$$B = \begin{pmatrix} 1-2r-\mu & r & & & & \\ r & 1-2r-\mu & r & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & & & & & \\ \cdots & \cdots & \cdots & & 2r & 1-2r-\mu \end{pmatrix},$$

$$C = \begin{pmatrix} 1-2r-\mu-b_1 & r & & & & \\ r & 1-2r-\mu-b_1 & r & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & & & & & \\ \cdots & \cdots & \cdots & & 2r & 1-2r-\mu-b_1 \end{pmatrix},$$

$C^k = [C_1^k, C_2^k, C_3^k, ..., C_M^k]^T; C^0 = [C_1^0, C_2^0, C_3^0, ..., C_M^0]^T;$
$S = [rC_0, 0, 0, 0, ..., 0]^T; r = \frac{D\Gamma(2-\alpha)\tau^\alpha}{2h^2}; \mu = \lambda\Gamma(2-\alpha)\tau^\alpha$ and
$b_j = (j+1)^{1-\alpha} - j^{1-\alpha},\ j = 1, 2, 3, \cdots, k.$

3. STABILITY

Lemma: If $\lambda_j(A),\ j = 1, 2, 3, \cdots, M-1$ represents the eigenvalues of matrix A,then
$(i)\ \lambda_j(A) \geq 1$
$(ii)\ \| A^{-1} \|_2 \leq 1$
where $\| A^{-1} \|_2$ second matrix norm of matrix A.
$iii\ \| B \|_2 < 1$
$iv\ \| C \|_2 < 1$

Proof:-i The Gerschgorin's theorem states that each eigenvalues λ of a square matrix A is in at least one of the following disk

$$| \lambda - a_{jj} | \leq \sum_{l=1; l\neq j}^{M-1} a_{lj},\ j = 1, 2, \cdots, M-1 \tag{14}$$

Therefore each eigenvalue λ of matrix A satisfy at least one of the following inequalities

$$| \lambda | \leq | \lambda - a_{jj} | + \sum_{l=1; l\neq j}^{M-1} | a_{lj} | \leq \sum_{l=1}^{M-1} | a_{lj} | \tag{15}$$

$$| \lambda | \geq | a_{jj} | - | \lambda - a_{jj} | \geq | a_{jj} | - \sum_{l=1; l\neq j}^{M-1} | a_{lj} | \tag{16}$$

Now we use the equation 16 to matrix A, then each eigenvalue λ of matrix A satisfy at least one of the following inequalities.

$$| \lambda_j(A) | \geq 1 + 2r - r = 1 + r > 1 \quad r > 0$$

$$| \lambda_j(A) | \geq 1 - r + 1 + 2r - r \geq 1$$

we get

$$| \lambda_j(A) | \geq 1$$

(ii) we have

$$\| A \|_2 = \max_{l \leq j \leq M-1} | \lambda_j(A) | \geq 1 \tag{17}$$

hence

$$\| A \|_2 \geq 1$$

we get

$$\| A^{-1} \|_2 \leq \frac{1}{|\lambda_j(A)|} \leq 1$$

(iii)

$$\| A \|_2 \leq r + 1 - 2r - \mu + r < 1$$

(iv)

$$\| C \|_2 \leq | r + 1 - b_1 - 2r + r - \mu | \leq (1 - \mu) - b_1 \leq 1 - b_1 < 1$$

Theorem: The solution of the finite difference scheme 8-11 for TFRDE 4-5 is unconditionally stable.

Proof: To prove that the above finite difference scheme is unconditionally stable, that is to show that

$$\| C^k \|_2 \leq \| C^0 \|_2 , \quad k \geq 1$$

From equation 12 we have

$$AC^1 = BC^0 \ \ , k = 0$$

$$C^1 = A^{-1} BC^0$$

$$\begin{aligned} \| C^1 \|_2 &= \| A^{-1} BC^0 \|_2 \\ &\leq \| A^{-1} \|_2 \| B \|_2 \| B^0 \|_2 \\ &\leq \| C^0 \|_2 \end{aligned}$$

we obtain

$$\| C^1 \|_2 \leq \| C^0 \|_2$$

Thus,the result is true for $n = 1$.

Assume that the result is true for $n = k$,

$$\| C^k \|_2 \leq \| C^0 \|_2$$

So now to prove, the result is true for $n = k + 1$;

from equation 12, we have

$$AC^{k+1} = CC^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})C^{k-j} + b_kC^0$$

$$C^{k+1} = A^{-1}CC^k + A^{-1}\sum_{j=1}^{k-1}(b_j - b_{j+1})C^{k-j} + b_kA^{-1}C^0$$

This yields

$$\begin{aligned}
\| C^{k+1} \|_2 &\leq \| A^{-1} \|_2 \| C \|_2 \| C^k \|_2 + \| A^{-1} \|_2 \sum_{j=1}^{k-1}(b_j - b_{j+1}) \| C^{k-j} \|_2 \\
&\quad + b_k \| A^{-1} \|_2 \| C^0 \|_2 \\
&\leq | (1-b_1) \| C^0 \|_2 + \sum_{j=1}^{k-1}(b_j - b_{j+1}) \| C^0 \|_2 + b_k \| C^0 \|_2 \\
&\leq \left(1 - b_1 + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k\right) \| C^0 \|_2 \\
&\leq (1 - b_1 + (b_1 - b_2) + (b_2 - b_3) \\
&\quad + \cdots + (b_{k-1} - b_k) + b_k) \| C^0 \|_2
\end{aligned}$$

we get

$$\| C^{k+1} \|_2 \leq \| C^0 \|_2$$

Hence by induction principle, the result is true for all k.

Finally, for all k

$$\| C^k \|_2 \leq \| C^0 \|_2$$

this shows that, the scheme is unconditionally stable.

4. CONVERGENCE

We introduce the another vector for,

$$\overline{C}_l^k = [C(x_0,t_k), \cdots, C(x_i,t_k), \cdots, C(x_{M-1},t_k)]^T$$

which represents the exact solution at the time level t_K whose size is M.

By above discretization scheme,

$$A\bar{C}^1 = B\bar{C}^0 + \tau^1 \quad k = 0 \tag{18}$$

$$A\bar{C}^{k+1} = B\bar{C}^{k} + \sum_{j=1}^{k-1}(b_j - b_{j+1})\bar{C}^{k-j} + b_k\bar{C}^{0} + \tau^{k+1} \quad k \geq 1 \tag{19}$$

where

$$A = \begin{pmatrix} 1+2r & -r & & & & \\ -r & 1+2r & -r & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & & & & & \\ \cdots & \cdots & \cdots & & -2r & 1+2r \end{pmatrix};$$

$$B = \begin{pmatrix} 1-2r-\mu & r & & & & \\ r & 1-2r-\mu & r & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & & & & & \\ \cdots & \cdots & \cdots & & 2r & 1-2r-\mu \end{pmatrix};$$

$$C = \begin{pmatrix} 1-2r-\mu-b_1 & r & & & & \\ r & 1-2r-\mu-b_1 & r & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & & & & & \\ \cdots & \cdots & \cdots & & 2r & 1-2r-\mu-b_1 \end{pmatrix};$$

$$r = \frac{D\Gamma(2-\alpha)\tau^{\alpha}}{2h^2}\ ,\ \mu = \lambda\Gamma(2-\alpha)\tau^{\alpha}\ ,$$

$$b_j = (j+1)^{1-\alpha} - j^{1-\alpha},\ j = 0,1,2,3,\cdots,N$$

and τ^k is the vector of the truncation error at the time level t^k.

Theorem: The finite difference scheme 8-11 for TFRDE 4-5 is unconditionally convergent. that is

$$\| E^k \|_2 \leq \| E^0 \|_2 \quad as(h,\tau) \to (0,0).$$

Proof: We subtract equation 12 from 18 and 13 from 19 respectively, to obtain

$$A(\bar{C}^1 - C^1) = B(\bar{C}^0 - C^0) + \tau^1 \tag{20}$$

$$A(\bar{C}^{k+1}-C^{k+1}) = C(barC^k - C^k) + \sum_{j=1}^{k-1}(b_j - b_{j+1})[\bar{C}^{k-j} - C^{k-j}]$$

$$+b_k[\bar{C}^0 - C^0] + \tau^{k+1} \tag{21}$$

Also set, error vector

$$E^k = (\bar{C}^k - C^k)$$

where $E^k = (e_1^k, e_2^k, ..., e_{M-1}^k)^T; C^k = (C_1^k, C_2^k, ..., C_{M-1}^k)^T$ By using equation 20, we have,

$$AE^1 = BE^0 + \tau^1$$

$$E^1 = A^{-1}BE^0 + A^{-1}\tau^1$$

which yields

$$\begin{aligned} \| E^k \|_2 &\leq \| A^{-1} \|_2 \| B \|_2 \| E^0 \|_2 + \| A^{-1} \|_2 \| \tau^1 \|_2 \\ &\leq \| E^0 \|_2 + O(\tau^{2-\alpha}, h^2) \end{aligned}$$

Assume that the result is true for k,

$$||E^k||_2 \leq ||E^0||_2 + O(\tau^{2-\alpha}, h^2)$$

Now to prove that the result is true for $k+1$, again from equation 21, we have

$$AE^{k+1} = CE^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})E^{k-j} + b_kE^0 + \tau^{k+1}$$

$$E^{k+1} = A^{-1}CE^k + A^{-1}\sum_{j=1}^{k-1}(b_j - b_{j+1})E^{k-j} + b_kA^{-1}E^0 + A^{-1}\tau^{k+1} \tag{22}$$

$$\begin{aligned} \| E^{k+1} \|_2 &\leq \| A^{-1} \|_2 \| C \|_2 \| E^k \|_2 + \| A^{-1} \|_2 \sum_{j=1}^{k-1}(b_j - b_{j+1}) \| E^{k-j} \|_2 \\ &\quad +b_k \| A^{-1} \|_2 \| E^0 \|_2 + \| A^{-1} \|_2 \| \tau^{k+1} \|_2 \\ &\leq | (1-b_1) | \| E^0 \|_2 + \sum_{j=1}^{k-1}(b_j - b_{j+1}) \| E^0 \|_2 \\ &\quad +b_k \| E^0 \|_2 \\ &\leq (1 - b_1 + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k) \| E^0 \|_2 + O(\tau^{2-\alpha}, h^2) \\ &\leq \| E^0 \|_2 + O(\tau^{2-\alpha}, h^2) \end{aligned}$$

Thus, the result is true for $k+1$ and hence by mathematical induction, it is true for all k.

Hence

$$\| E^k \|_2 \leq \| E^0 \|_2 \ , as\,(h,\tau) \to (0,0)$$

which proves that, the scheme is unconditionally convergent.

5. Numerical Solution

We have solved the time fractional radon diffusion equation 4-5 by Crank-Nicolson finite difference approximation. The approximated solution for time fractional radon diffusion equation 4-5 in soil medium with initial and boundary conditions is obtained. The solution is simulated by using mathematical software Mathematica. The numerical solutions obtained at $t = 0\cdot 05$ by considering the parameters $L = 1\cdot 7278cm, \lambda = 2\cdot 1\times 10^{-6}s^{-1}, \tau = 0\cdot 05, k = 4m^2/kg, p = 0\cdot 5g/cm^3, C_0 = 200Bq/m^3$, radon diffusion coefficient for soil $D = 4\cdot 1\times 10^{-7}m^2/s$, at $\alpha = 0\cdot 9, \alpha = 0\cdot 8$ is simulated in the following figure,

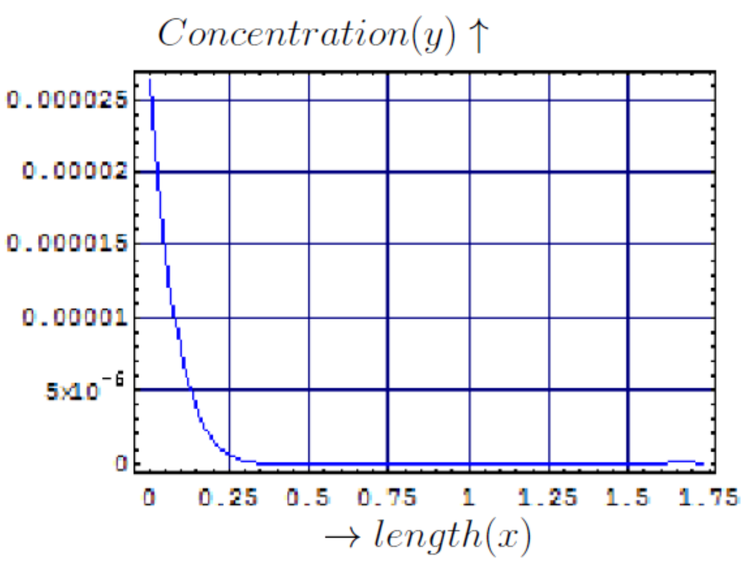

FIGURE 1. The approximate solution of radon diffusion equation $\alpha = 0.9$

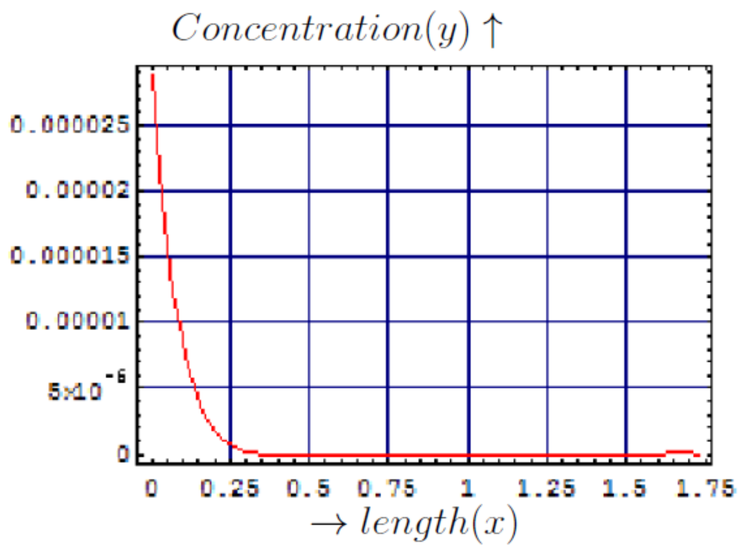

FIGURE 2. The approximate solution of radon diffusion equation $\alpha = 0.8$

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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