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ON ONE-SIDEDLY GRAPH CLIQUISH FUNCTIONS

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Abstract: In the present paper we introduce a new notion of one-sidedly (right, left) graph cliquish functions from the real line to a metric space and study its relation with other types of generalized continuity. We also deal with some properties relating to that new notion of generalized continuity.

Keywords: graph continuity; graph quasi-continuity; graph cliquish functions; right-sidedly (left-sidedly) quasi-continuity; right sidedly (left-sidedly) cliquish functions.

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1. INTRODUCTION AND BASIC NOTATIONS

In what follows Y is a metric space with metric d. Through the paper $\mathbb{R}$ is the real line. Furthermore $\mathbb{Z},\mathbb{Q}$ stand for the set of integers and rational numbers respectively, ϕ denotes the empty set and $S(x,r)$ is the open sphere with centre x and radius r. For a subset $A \subseteq \mathbb{R}$, $cl(A), int(A)$ denote the closure and interior of A respectively. For a function $f: \mathbb{R} \to Y$, $G(f)$ denotes the graph of $f$and then the symbol $cl(G(f)$ denotes the closure of $G(f)$in the product topology $\mathbb{R} \times Y_d$(Y_d being the topology on Y induced by d).

The notion of graph continuity of real valued functions on the closed interval [0,1] was introduced by Z. Grande [4]. K. Sakalava [11] also dealt with that notion. A function $f: \mathbb{R} \to Y$ is

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said to be graph continuous [4] if there exists a continuous function $g:\mathbb{R}\to Y$ such that $G(g)\subseteq cl(G(f))$. A. Mikuka [10] defined graph quasi-continuity and other types of continuity and studied its relation with graph continuity and other types of continuity. A function $f:\mathbb{R}\to Y$ is said to be graph quasi continuous [10] if there exists a quasi continuous function $g:\mathbb{R}\to Y$ such that $G(g)\subseteq cl(G(f))$. In [7], [8] a notion of graph cliquish functions and its relations with other types of generalized continuous functions were investigated . A function $f:\mathbb{R}\to Y$ is said to be graph cliquish[7] if there exists a cliquish function $g:\mathbb{R}\to Y$such that $G(g)\subseteq cl(G(f))$.

Recall that a function $f:\mathbb{R}\to Y$ is said to be :

- Almost continuous (in the sense of Husain) at a point $x\in\mathbb{R}$ if for any neighbourhood V of $f(x)$, the set $int(cl(f^{-1}(V)))$ is a neighbourhood of x. [6]
- Quasi continuous at a point $x\in\mathbb{R}$ if for each open neighbourhood U of x and each neighbourhood V of $f(x)$ there exists a non-empty open set $G\subseteq U$ such that $f(G)\subseteq V$. [9]
- Cliquish at a point $x\in\mathbb{R}$ if for each $\varepsilon>0$ and each open neighbourhood U of x, there exists a non-empty open set $G\subseteq U$ such that $d(f(y),f(z))<\varepsilon$ whenever $y,z\in G$. [12]
- Right-sidedly (left-sidedly) quasi-continuous at a point $x\in\mathbb{R}$ if for each $\delta>0$ and each open neighbourhood V of $f(x)$, there is a non-empty open set $U\subseteq(x,x+\delta)$(resp. $U\subseteq(x-\delta,x)$) such that $f(U)\subseteq V$. [1]
- Right-sidedly (left-sidedly) cliquish at a point $x\in\mathbb{R}$ if for each $\delta>0$ and $\varepsilon>0$ there is a non-empty open set $U\subseteq(x,x+\delta)$(resp. $U\subseteq(x-\delta,x)$) such that $d(f(y),f(z))<\varepsilon$ whenever $y,z\in U$. [3]

f is called almost continuous (respectively quasi-continuous, cliquish, right(left)-sidedly quasi-continuous, right(left)-sidedly cliquish) if it is so at each point.

By $AE(f),A^{+}(f),A^{-}(f)$ we denote the sets of all points at which f is almost continuous, right sidedly, left-sidedly cliquish respectively.

Here we introduce the notion of one-sidedly graph cliquish functions as follows:

Definition 1.1: A function $f: \mathbb{R} \to Y$ is said to be right-sidedly (left-sidedly) graph cliquish if there exists a right –sidedly (respectively left-sidedly) cliquish function $g: \mathbb{R} \to Y$ such that $G(g) \subseteq cl(G(f))$.

2. ONE-SIDEDLY GRAPH CLIQUISH FUNCTIONS AND OTHER TYPES OF FUNCTIONS

Evidently every right-sidedly (left-sidedly) cliquish function is right-sidedly (respectively left-sidedly) graph cliquish. Also, every right-sidedly (left-sidedly) graph cliquish function with closed graph is right-sidedly (respectively left-sidedly) cliquish.

The following implications follow from the above definitions:

		One-sidedly quasi-continuity	⟹	One-sidedly cliquish
		⇓		⇓
Continuity	⇒	Quasi-continuity	⇒	Cliquish
⇓		⇓		⇓
Graph continuity	⇒	Graph quasi-continuity	⇒	Graph cliquish

And all of these are not invertible.

Example 2.1: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Here f is right-sidedly (left-sidedly) graph cliquish but f is not cliquish.

Example 2.2: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x < 1 \\ 2, & x \geq 1 \end{cases}$$

Here f is right-sidedly (left-sidedly) graph cliquish. Also f is right-sidedly, left-sidedly cliquish.

Example 2.3: Let X be the space of real numbers with the discrete metric and $f: X \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$

Here $cl\big(G(f)\big) = [Q \times \{1\}] \cup [(X \setminus Q) \times \{0\}]$

There is no one-sidedly cliquish function $g: X \to \mathbb{R}$ such that $G(g) \subseteq cl(G(f))$ and so, f is not one-sidedly graph cliquish.

3. Results on One-Sidedly Graph Cliquish Functions

The following results, lemmas are known:

Result 3.1: A function $f: \mathbb{R} \to Y$ is cliquish if and only if $A^+(f) \cap A^-(f)$ is dense in $\mathbb{R}$.[3] Using this result it easily follows that

Result 3.2: If a function $f: \mathbb{R} \to Y$ is right-sidedly (left-sidedly) cliquish then $A^-(f)$ (respectively $A^+(f)$) is dense in $\mathbb{R}$.

Result 3.3: If $f: \mathbb{R} \to Y$ is almost continuous at a point $x \in \mathbb{R}$ then there exists an open neighbourhood U of x such that $f^{-1}(V)$ is dense in U for any neighbourhood V of $f(x)$.

It easily follows from the definition of almost continuity.

Lemma 3.1: Let $A \subseteq W \subseteq \mathbb{R}$. If A is semi-open in $\mathbb{R}$ then A is semi-open in the subspace W. [6]

Lemma 3.2: If a set A is dense and semi-open in $\mathbb{R}$ and a set B is dense in $\mathbb{R}$ then $A \cap B$ is dense in $\mathbb{R}$. [10]

Now we can formulate the following theorems on one-sidedly graph cliquish functions.

Theorem 3.1: Let $f: \mathbb{R} \to Y$ be given. For a one-sidedly cliquish function $g: \mathbb{R} \to Y$ with $G(g) \subseteq cl(G(f))$ the set $A(f, g, \varepsilon) = \{x \in \mathbb{R}: \ d(f(x), g(x)) < \varepsilon\}$ is dense for any $\varepsilon > 0$.

Proof: Assume that $g: \mathbb{R} \to Y$ is right-sidedly cliquish.

Let $\varepsilon > 0$ and U be a non-empty open set in $\mathbb{R}$. By the Result 3.2, $A^-(g)$ is dense in $\mathbb{R}$.

Let $x_0 \in U \cap A^-(g)$. $x_0 \in U \Rightarrow \exists\ \delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq U$.

$x_0 \in A^-(g) \Rightarrow \exists$ a non-empty open set $U_1 \subseteq (x_0 - \delta, x_0)$ such that $d(g(x), g(y)) < \varepsilon/2$ whenever $x, y \in U_1$.

Let $x_1 \in U_1$. Then $(x_1, g(x_1)) \in cl(G(f))$.

So, $[U_1 \times S(g(x_1), \varepsilon/2\ )] \cap G(f) \neq \varphi$.

Choose $x_2 \in U_1$ such that $d(f(x_2), g(x_1)) < \varepsilon/2$.

Now, $d(f(x_2), g(x_2)) \leq d(f(x_2), g(x_1)) + d(g(x_1), g(x_2)) < \varepsilon$.

So, $x_2 \in A(f,g,\varepsilon)$. Hence, $A(f,g,\varepsilon)$ is dense in $\mathbb{R}$.

Remark 3.1: Let $f:\mathbb{R}\to Y$ be given and $g:\mathbb{R}\to Y$ be a one-sidedly cliquish function such that for any $\varepsilon > 0$, the set $A(f,g,\varepsilon)$ is dense in $\mathbb{R}$. Then it is not necessarily true that $G(g) \subseteq cl(G(f))$.

Example 3.1: Let Y be the space of real numbers with the discrete metric d. Let $f:\mathbb{R}\to Y$, $g:\mathbb{R}\to Y$ be defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{Z} \\ -1, & x \in \mathbb{Q}\setminus\mathbb{Z} \\ 1, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, x \in \mathbb{Z} \\ 1, x \in \mathbb{R}\setminus\mathbb{Z} \end{cases}$$

g is left-sidedly as well as right-sidedly cliquish.

Let $\varepsilon > 0$. Then $A(f,g,\varepsilon) = \{x \in \mathbb{R}: \ d(f(x), g(x)) < \varepsilon\}$

$$= \begin{cases} \mathbb{R}\setminus\mathbb{Q}, & 0 < \varepsilon \le 1 \\ \mathbb{R}, & \varepsilon > 1 \end{cases}$$

$A(f,g,\varepsilon)$ is dense for any $\varepsilon > 0$. But $G(g) \nsubseteq cl(G(f))$.

Theorem 3.2: Let $f:\mathbb{R}\to Y$ be given. For a one-sidedly cliquish function $g:\mathbb{R}\to Y$ with $G(g) \subseteq cl(G(f))$ the set $B(f,g,\varepsilon) = \{x \in \mathbb{R}: d(f(x), g(x)) \ge \varepsilon\}$ is nowhere dense for any $\varepsilon > 0$.

Proof: Let $\varepsilon > 0$ and U be a non-empty open set in $\mathbb{R}$. Suppose that $g:\mathbb{R}\to Y$ is left-sidedly cliquish. Then by the Result 3.2, $A^+(g)$ is dense in $\mathbb{R}$.

Let $x_0 \in U \cap A^+(g)$.

$x_0 \in U \Rightarrow \exists \delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq U$.

$x_0 \in A^+(g) \Rightarrow \exists$ a non empty open set $U_1 \subseteq (x_0, x_0 + \delta)$ such that $d(g(x), g(y)) < \varepsilon/3$ whenever $x, y \in U_1$.

By the Theorem 3.1, $A(f, g, \varepsilon/3)$ is dense in $\mathbb{R}$.

Let $x_1 \in U_1 \cap A(f, g, \varepsilon/3)$. Then $x_1 \in U_1$ and $d(f(x_1), g(x_1)) < \varepsilon/3$.

Now, $(x_1, g(x_1)) \in cl(G(f)$. So, $[U_1 \times S(f(x_1), \varepsilon/3)] \cap G(f) \neq \varphi$.

Choose $x_2 \in U_1$ such that $d(f(x_2), f(x_1)) < \varepsilon/3$.

Now, $d(f(x_2), g(x_2)) \le d(f(x_2), f(x_1)) + d(f(x_1), g(x_1)) + d(g(x_1), g(x_2)) < \varepsilon$.

So, $x_2 \in \mathbb{R}\setminus B(f,g,\varepsilon)$. Thus $B(f,g,\varepsilon)$ is nowhere dense.

Corollary 3.1: Let $f:\mathbb{R}\to Y$ be given. For a one-sidedly cliquish function $g:\mathbb{R}\to Y$ with $G(g)\subseteq cl(G(f))$ the set $A(f,g,\varepsilon)$ is semi-open for any $\varepsilon>0$.

It follows from the result [2] that the complement of a nowhere dense set is semi-open.

Theorem 3.3: Let $f:\mathbb{R}\to Y$ be given. For a one-sidedly cliquish function $g:\mathbb{R}\to Y$ with $G(g)\subseteq cl\big(G(f)\big)$, the set $\{x\in\mathbb{R}: f(x)\neq g(x)\}$ is of first category.

Proof: $\{x\in\mathbb{R}: f(x)\neq g(x)\}=\bigcup_{n=1}^{\infty} B(f,g,\frac{1}{n})$.

The set $B(f,g,\frac{1}{n})$ is nowhere dense by the Theorem 3.2 and so the proof is completed.

Corollary 3.2: Let $f:\mathbb{R}\to Y$ be given. For a one-sidedly cliquish function $g:\mathbb{R}\to Y$ with $G(g)\subseteq cl\big(G(f)\big)$, the set $\{x\in\mathbb{R}: f(x)=g(x)\}$ is dense in $\mathbb{R}$.

It follows from the fact that $\{x\in\mathbb{R}: f(x)=g(x)\}=\mathbb{R}\setminus\{x\in\mathbb{R}: f(x)\neq g(x)\}$ is residual in $\mathbb{R}$.

Theorem 3.4: Let $f:\mathbb{R}\to Y$ be given. For a right-sidedly (left-sidedly) cliquish function $g:\mathbb{R}\to Y$ if $B(f,g,\varepsilon)$ is nowhere dense for any $\varepsilon>0$ then f is right-sidedly (respectively left-sidedlt) cliquish.

Proof: Let $g:\mathbb{R}\to Y$ be left-sidedly cliquish.

Let $x_0\in\mathbb{R},\delta>0$ and $\varepsilon>0$. Then there is a non-empty open set $U\subseteq(x_0-\delta,x_0)$ such that $d(g(x),g(y)<\frac{\varepsilon}{3}$ whenever $x,y\in U$.

Since $B\left(f,g,\frac{\varepsilon}{3}\right)$ is nowhere dense, there is a non-empty open set $G\subseteq U$ such that $G\cap B\left(f,g,\frac{\varepsilon}{3}\right)=\varphi$.

Then $d(f(x),g(x)<\frac{\varepsilon}{3}$ for all $x\in G$. Let $x,y\in G$.

Then $d\big(f(x),f(y)\big)\leq d\big(f(x),g(x)\big)+d\big(g(x),g(y)\big)+d\big(g(y),f(y)\big)<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$.

So, f is left-sidedly cliquish.

Theorem 3.5: Let $f:\mathbb{R}\to Y$ be right-sidedly(left-sidedly) quasi-continuous and $g:\mathbb{R}\to Y$ be right-sidedly (respectively left-sidedly) cliquish such that $G(g)\subseteq cl\big(G(f)\big)$. Then $f(x)=g(x)$ for each $x\in AE(g)$.

Proof: Suppose that $f: \mathbb{R} \to Y$ be left-sidedly quasi-continuous and $g: \mathbb{R} \to Y$ be left-sidedly cliquish. If possible, let $f(x) \neq g(x)$ for some $x \in AE(g)$.

Suppose $r = d\big(f(x), g(x)\big)$. Then $r > 0$.

Since $x \in AE(g)$, there is an open neighbourhood U of x such that $g^{-1}\left(S\left(g(x), \frac{r}{4}\right)\right)$ is dense in U by the Result 3.3.

Using the Theorem 3.1, $A(f, g, {}^{r}/_{4})$ is dense in $\mathbb{R}$ and hence dense in the open subspace U of x. Also, $A(f, g, {}^{r}/_{4})$ is semi-open in U by the Corollary 3.1 and using the Lemma 3.1.

Hence by the Lemma 3.2 $A(f, g, {}^{r}/_{4}) \cap g^{-1}\left(S\left(g(x), \frac{r}{4}\right)\right)$ is dense in U.

$x \in U \Rightarrow \exists \delta > 0$ such that $(x - \delta, x) \subseteq U$.

Since f is left − sidedly quasi continuous at x, there is a non-empty open set $H \subseteq (x - \delta, x)$ such that $f(H) \subseteq S\left(f(x), \frac{r}{2}\right)$

Choose $x_1 \in H \cap A(f, g, {}^{r}/_{4}) \cap g^{-1}\left(S\left(g(x), \frac{r}{4}\right)\right)$.

Then $x_1 \in H$, $d\big(f(x_1), g(x_1)\big) < \frac{r}{4}$, $d\big(g(x_1), g(x)\big) < \frac{r}{4}$.

Now, $d\big(f(x_1), g(x)\big) \leq d\big(f(x_1), g(x_1)\big) + d\big(g(x_1), g(x)\big) < \frac{r}{2}$.

So, $f(x_1) \in S\left(g(x), \frac{r}{2}\right)$. Again, $f(x_1) \in f(H)$.

Thus we arrive at a contradiction as $S\left(g(x), \frac{r}{2}\right) \cap S\left(f(x), \frac{r}{2}\right) = \varphi$.

Remark 3.2: In the Theorem 3.5**,** the one-sidedly quasi-continuity cannot be replaced by the one-sidedly cliquishness of f even if g is continuous.

It follows from the following example.

Example 3.2: The functions $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ are defined as $f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$ and $g(x) = 1\ \forall x \in \mathbb{R}$.

f is both right-sidedly, left-sidedly cliquish but f is neither right-sidedly nor left-sidedly quasi continuous (f fails to be one-sidedly quasi continuous at 0). g is continuous and $G(g) \subseteq cl(G(f)$. Here $f(0) \neq g(0)$.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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