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DD'S NEW FORMULA OF THE DENDRIMER'S TREE USING THE VERTICES' PAIRS NUMBER

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Abstract. The topological indices are widely used for describing the chemical structure of molecules, the chemical reaction networks, the World Wide Web, financial markets, ecosystems, social networks, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds, and making other chemical applications. In [2] a new formula to calculate the degree distance index using the $d_G^u(k)$ is proved. In is paper we are going to calculate the degree distance index of the Dendrimer's tree via $d_G^u(k)$ (vertices' pairs number of G that are at distance k from u).

Keywords: degree distance index; complex networks; Dendremer's tree; molecular descriptors; topological indices; Wiener index; the vertices' pairs number $d_G^u(k)$.

2010 AMS Subject Classification: 05C05, 05C07, 05C09, 05C12, 90C35, 94C15.

1. Introduction

In 1947 the graph invariants' history was begun when the physical chemist H. Wiener predicted the boiling points of paraffin using the Wiener index [1] and hundred of topological indices were defined; by chemists and mathematicians after; for modeling the physical properties of alcanes. In 1971, Gutman & Trinajstić defined the Zagreb indices that are degree-based

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topological indices [3] and were firstly proposed to be branching's measures of the carbon-atom skeleton [4]. These indices and Randic index are the most used topological indices in chemical and mathematical literature so far. For more detailed discussions of the most well-known topological indices, we refer the interested reader to [5, 6, 7, 8, 9, 10, 11] and the references therein.

The topological indices can be generally divided into three classes, some of them are : distance-based measures[12], which are based on distance (the shortest paths between pairs of vertices); like, the Wiener index that is a well-known topological index, which equals the sum of distances between all pairs of vertices of a molecular graph. The second classe is degree-based measures [17], that are computed by using vertex degree; like the first Zagreb index of G defined as $M(G) = \sum_{i=1}^{n}(deg(v_i))^2$ where $deg(w)$ is the degree of vertex w. The third classe is the measures based on both degrees and distances; like, the degree distance index of G, $DD(G)$, defined as $\sum_{\{u,v\}\subseteq V}(deg(u)+deg(v))d(u,v)$ where $d(u,v)$ denotes the distance between u and v [13, 14, 15, 16, 17, 18, 19, 20, 21]. The properties of the graph degree distance will be compared with properties of the Wiener index [22, 23, 24, 25, 26, 27, 28].

The rest of this paper is organized as follows : The next section presentes a short describtion of the Dendremer molecluars. In Section 3, we introduce the basic concepts, definitions and techniques employed in this work. In the fourth section we describe the proof of the degree distance index formula of the Dendremer's tree. We summarize our findings by the section 5 (Conclusion).

2. The Properties of Dendrimers

The term "Dendrimer" comes from the Greek terms dendron and meros which mean tree or branch and part respectively. The Dendrimers would therefore be trees or parts of trees. Indeed, they constitute a class of multi-branched, globular molecules, which have been the subject of considerable attention in many fields such as functional nanomaterials, catalysis or even biotechnology and nanomedicine, as evidenced by different families of Dendrimers synthesized over the past thirty years [29].

2.1. The structure. Their structure is perfectly defined by three distinct structural domains (Figure 1 (A)) [30]:

- A central core, which defines the initial dendritic architecture and, at the same time, serves as an anchoring site for the branches. In some cases, it can also be a specific function carrier that makes it fluorescent, photo-isomerizable or transforms it into a catalytic site.
- Dendrons, made up of a succession of units with at least one junction point and organized according to a radial geometry to form a series of concentric layers called generations (G).
- A multivalent surface formed by potential reactive sites in the form of terminal active groups at the periphery, often containing atoms with free electron pairs (O, N ..) or acidic functional groups. The terminal groups play a key role in the physicochemical properties of the Dendrimer.

It should be mentioned that the junction between the core and the dendrons is made through chemical groups called points of divergence.

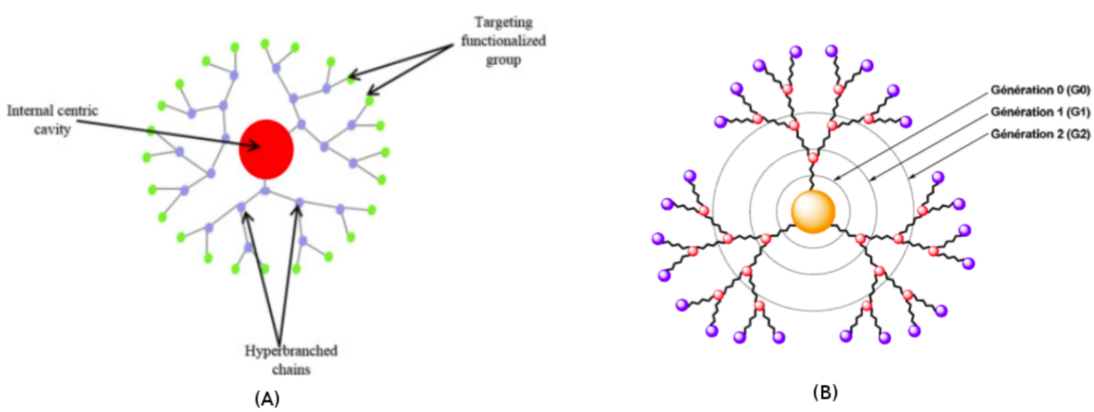

FIGURE 1. (A) Dendrimer general structure and (B) Structure of a generation 2 dendrimer

These latters are also found at the ends of the branches, allowing the iterative addition of a second series of branches or the grafting of terminal groups. The resulting tree structure is determined by the number of successive branches going from the core to the periphery of the molecule, called generations of the Dendrimer (Figure 1 (B)). where is a central core, is a sequence of repeating units and is a terminal surface groups. G0, 1, 2 : Generation 0, 1, 2 [31]

2.2. Intrinsic properties. Dendrimers are distinguished by their unique properties make them particularly attractive in nanomedicine and its personalized therapies. The unique properties of Dendrimers including a large and controllable size, a very high degree of branching, a great potential for solubilization in water, above all a great interaction capital, with other chemical or biological entities, a very high solubilization potential[32], a low viscosity[33], multivalent surface and high loading capacity[34], bio distribution and toxicity [35, 36] and illustrations for phosphorus Dendrimers in nanomedicine[37, 38].

3. Background and Notation

A graph is an object $G = (V,E)$ that consists of a nonempty set V and 2-element subsets of V namely E. The elements of V are called vertices and the elements of E are called edges. The distance $d(u,v)$ between two vertices u and v is the length of the shortest path connecting these two vertices. The diameter $D(G)$ of G denotes the maximum of the shortest path between all vertices of G. We denote by $deg(v)$ the degree of the vertex v which is the number of edges incident to v[2].

In this paper, we will focus on the parameter $d_G(k)$, which is defined as the vertices' pairs number of G that are at distance k, and the Wiener index of a vertex $v \in V(G)$, which equals to the sum of distances between a chosen vertex v and the other vertices in G, $w(v,G) = \sum_{u \in V(G)} d(u,v)$. Furthermore, we will concentrate on the degree distance index that is defined as $DD(G) = \sum_{\{u,v\} \subseteq V(G)} (deg(u) + deg(v))d(u,v) = \sum_{u \in V(G)} deg(u)w(u,G)$. The reader can see [39, 40, 41, 42, 43, 44] for more details and results on the calculation of the Wiener and degree distance indices.

Definition 1. [2] *Let G be a simple connected graph of diameter $D(G)$. The degree distance of the vertex v $dd(v,G)$ is defined as follows:*

$$dd(v,G) = \sum_{k=1}^{D(G)} deg(v)(k-2)d_G^v(k). \tag{1}$$

Theorem 1. [2] *Let G be a simple connected undirected graph, with n vertices, m edges and $D(G) \geq 2$. Then :*

$$DD(G) = 4m(n-1) + \sum_{v \in V(G)} dd(v,G). \tag{2}$$

4. Main Results

For proofing our main results we need some notations and previous results.

Definition 2. *Let's T_Δ^l be a Dendrimer's tree of l levels and its vertices' maximum degree is Δ. We note by :*

(1) n its order (number of vertices) and m its size (number of edges),

(2) u_i : the vertice that exist at the level $l-i$ (with $0 \leq i \leq l$),

(3) P_{l-i} : the number of vertices at the level $l-i$,

(4) $d_{T_\Delta^l}^{u_i}(k)$: the vertices' pairs number of T_Δ^l that are at distance k from u_i,

Lemma 1. *Let's T_Δ^l be a Dendrimer's tree described above. Then :*

(1) $P_{l-i} = \begin{cases} \Delta(\Delta-1)^{(l-i)-1}, & \text{if } 0 \leq k \leq l-1 \\ 1, & \text{if } i = l \end{cases}.$

(2) $n = \frac{\Delta}{\Delta-2}((\Delta-1)^l - 1) + 1.$

(3) $m = \frac{\Delta}{\Delta-2}((\Delta-1)^l - 1).$

(4) $d_{T_\Delta^l}^{u_i}(k) = \begin{cases} \Delta(\Delta-1)^{k-1}, & \text{if } k \leq i \\ (\Delta-1)^{\frac{k+i}{2}}, & \text{if } i \text{ and } k \text{ have the same parity} \\ (\Delta-1)^{\frac{k+i-1}{2}}, & \text{if } k \text{ and } i \text{ don't have the same parity} \end{cases}.$

(5) $deg(u_0) = 1$ *and* $deg(u_{2i}) = deg(u_{2i+1}) = \Delta$, *with* $1 \leq i \leq l$.

(6) $4m(n-1) = 4(n-1)^2$.

Proof. (1) By calculation.

(2)

$$\begin{aligned} n &= \sum_{i=0}^{l-1} P_{(l-1)} + 1 \\ &= \frac{\Delta}{\Delta-2}((\Delta-1)^l - 1) + 1 \end{aligned}$$

For the rest 3., 4., 5. and 6. are by calculation. □

Lemma 2. *Let's Δ, l, i, j, k and r are integers. Then :*

(1) $\sum_{k=1}^{l} k(\Delta-1)^k = \frac{\Delta-1}{(\Delta-2)^2} + \frac{l(\Delta-1)^{(l+1)}}{(\Delta-2)} - \frac{(\Delta-1)^{(l+1)}}{(\Delta-2)^2}$

(2) $\sum_{k=1}^{l} (\Delta-1)^k = \frac{(\Delta-1)^{(l+1)}-(\Delta-1)}{\Delta-2}$

(3) $\sum_{k=2j+2}^{2l-2j-1} k = \sum_{r=j+1}^{l-j-1} 2r + \sum_{r=j+1}^{l-j-1} (2r+1)$

(4) $\sum_{r=j+1}^{l-j-1} r(\Delta-1)^r = \frac{(\Delta-1)^{(j+1)}}{(\Delta-2)^2}(1+((\Delta-2)(l-2j-1)-1)(\Delta-1)^{(l-2j-1)}) + \frac{j(\Delta-1)^{(j+1)}}{\Delta-2}((\Delta-1)^{(l-2j-1)}-1)$

(5) $\sum_{r=j+1}^{l-j-1} (\Delta-1)^r = \frac{(\Delta-1)^{(j+1)}}{\Delta-2}((\Delta-1)^{(l-2j-1)}-1)$

Proof. Evident □

Theorem 2. *Let's T_Δ^l be a Dendrimer's tree as previously described. if l is odd, then:*

$$DD(T_\Delta^l) = \frac{\Delta(\Delta-1)^l}{(\Delta-2)^3}(\Delta^2+8\Delta-4) + \frac{\Delta(\Delta-1)^{(2l-1)}}{(\Delta-2)^3}(4\Delta^3 l - \Delta^3 - 12\Delta^2 l - 5\Delta^2 + 8\Delta l + 10\Delta - 4) - \frac{2\Delta^2}{(\Delta-2)^3}$$

Proof. We using the theorem 1 and the lemma 2 :

$$\begin{aligned}
DD(T_\Delta^l) &= 4m(n-1) + \sum_{u_i \in V(T_\Delta^l)} dd(u_i, T_\Delta^l) \\
&= 4m(n-1) + \sum_{u_i \in V(T_\Delta^l)} \sum_{k=1}^{2D(T_\Delta^l)-i} deg(u_i)(k-2) d_{T_\Delta^l}^{u_i}(k) \\
&= 4m(n-1) + \sum_{i=0}^{l} P_{l-i} \times dd(u_i, T_\Delta^l) \\
&= 4m(n-1) + \sum_{j=0}^{\frac{l-1}{2}} P_{(l-2j)} \times dd(u_{2j}, T_\Delta^l) + \sum_{j=0}^{\frac{l-1}{2}} P_{(l-(2j+1))} \times dd(u_{2j+1}, T_\Delta^l) \\
&= 4m(n-1) + P_l \times dd(u_0, T_\Delta^l) + \sum_{j=0}^{\frac{l-1}{2}} P_{(l-2j)} \times dd(u_{2j}, T_\Delta^l) \\
&+ P_{(l-(2\times\frac{l-1}{2}+1))} \times dd(u_{(2\times\frac{l-1}{2}+1)}, T_\Delta^l) + \sum_{j=0}^{\frac{l-2}{2}-1} P_{(l-(2j+1))} \times dd(u_{2j+1}, T_\Delta^l) \\
&= 4m(n-1) + P_l \times dd(u_0, T_\Delta^l) + \sum_{j=1}^{\frac{l-1}{2}} P_{(l-2j)} \times dd(u_{2j}, T_\Delta^l) + P_0 \times dd(u_l, T_\Delta^l) \\
&+ \sum_{j=0}^{\frac{l-3}{2}} P_{(l-(2j+1))} \times dd(u_{2j+1}, T_\Delta^l)
\end{aligned}$$

□

Theorem 3. *Let's T_Δ^l be a Dendrimer's tree as previously described. if l is even, then :*

$$DD(T_\Delta^l) = \frac{\Delta(\Delta-1)^{(2l-1)}}{(\Delta-2)^3}(4\Delta^3 l - \Delta^3 - 12\Delta^2 l - 5\Delta^2 + 8\Delta l + 10\Delta - 4) + \frac{\Delta(\Delta-1)^l}{(\Delta-2)^3}(\Delta^2 + 8\Delta - 4) - \frac{2\Delta^2}{(\Delta-2)^3}$$

Proof. We using the theorem 1 and the lemma 2:

$$\begin{aligned}
DD(T_\Delta^l) &= 4m(n-1) + \sum_{u_i \in V(T_\Delta^l)} dd(u_i, T_\Delta^l) \\
&= 4m(n-1) + \sum_{u_i \in V(T_\Delta^l)} \sum_{k=1}^{2D(T_\Delta^l)-i} deg(u_i)(k-2) d_{T_\Delta^l}^{u_i}(k) \\
&= 4m(n-1) + \sum_{i=0}^{l} P_{l-i} \times dd(u_i, T_\Delta^l) \\
&= 4m(n-1) + \sum_{j=0}^{\frac{l}{2}} P_{(l-2j)} \times dd(u_{2j}, T_\Delta^l) + \sum_{j=0}^{\frac{l-2}{2}} P_{(l-(2j+1))} \times dd(u_{2j+1}, T_\Delta^l) \\
&= 4m(n-1) + P_l \times dd(u_0, T_\Delta^l) + \sum_{j=1}^{\frac{l-2}{2}} P_{(l-2j)} \times dd(u_{2j}, T_\Delta^l) + P_0 \times dd(u_l, T_\Delta^l) \\
&+ \sum_{j=0}^{\frac{l-2}{2}} P_{(l-(2j+1))} \times dd(u_{2j+1}, T_\Delta^l)
\end{aligned}$$

□

For completing the theorem's 1 and 2 proofs, we have to calculate :

$P_0 \times dd(u_l, T_\Delta^l)$, $\quad P_l \times dd(u_0, T_\Delta^l)$, $\quad P_{(l-2j)} \times dd(u_{2j}, T_\Delta^l)$ and $\quad P_{(l-(2j+1))} \times dd(u_{2j+1}, T_\Delta^l)$.

Lemma 3. *Let's T_Δ^l be a Dendrimer's tree as described above. Then :*

$$dd(u_0, T_\Delta^l) = \frac{(\Delta-1)(3\Delta-2)}{(\Delta-2)^2} + \frac{(\Delta-1)^l}{(\Delta-2)^2}(2\Delta^2(l-1) - 4\Delta l + \Delta + 2) - 1$$

Proof. We using the lemmas 1 and 2:

$$\begin{aligned}
dd(u_0, T_\Delta^l) &= \sum_{k=1}^{2l} deg(u_0)(k-2) d_{T_\Delta^l}^{u_0}(k) \\
&= deg(u_0)\left(\sum_{i=0}^{l-1}((2i+1)-2) d_{T_\Delta^l}^{u_0}(2i+1) + \sum_{i=1}^{l}(2i-2) d_{T_\Delta^l}^{u_0}(2i)\right)
\end{aligned}$$

$$= \sum_{i=0}^{l-1}(2i-1)((\Delta-1)^{\frac{((2i+1)+0-1)}{2}}) + \sum_{i=1}^{l}(2i-2)((\Delta-1)^{\frac{2i+0}{2}})$$

$$= -1(\Delta-1)^0 + \sum_{i=1}^{l-1}(2i-1)(\Delta-1)^i + \sum_{i=1}^{l}(2i-2)(\Delta-1)^i$$

$$= 2\sum_{i=1}^{l-1} i(\Delta-1)^i + 2\sum_{i=1}^{l-1} i(\Delta-1)^i + 2l(\Delta-1)^l - \sum_{i=1}^{l-1}(\Delta-1)^i - 2\sum_{i=1}^{l-1}(\Delta-1)^i - 2(\Delta-1)^l - 1$$

$$= 4(\frac{\Delta-1}{(\Delta-2)^2}(1+((\Delta-1-1)(l-1)-1)(\Delta-1)^{(l-1)})) - 3(\frac{(\Delta-1)^l-(\Delta-1)}{(\Delta-1)-1}) + 2(l-1)(\Delta-1)^l - 1$$

$$= 4(\frac{\Delta-1}{(\Delta-2)^2} + \frac{(\Delta-1)^l}{(\Delta-2)^2}((\Delta-2)(l-1)-1)) - \frac{3((\Delta-1)^l-(\Delta-1))}{\Delta-2} + 2(l-1)(\Delta-1)^l - 1$$

$$= \frac{(\Delta-1)(3\Delta-2)}{(\Delta-2)^2} + \frac{(\Delta-1)^l}{(\Delta-2)^2}(2\Delta^2(l-1) - 4\Delta l + \Delta + 2) - 1$$

□

Lemma 4. *Let's T_Δ^l be a Dendrimer's tree as described above. Then :*

$$dd(u_l, T_\Delta^l) = \frac{\Delta^2}{(\Delta-2)^2}(2\Delta - 3 - (\Delta-1)^l + (\Delta-1)^l(l-2)(\Delta-2))$$

Proof. We using the lemmas 1 and 2:

$$dd(u_l, T_\Delta^l) = \sum_{k=1}^{l} deg(u_l)(k-2)d_{T_\Delta^l}^{u_l}(k)$$

$$= \Delta\sum_{k=1}^{l}(k-2)(\Delta(\Delta-1)^{k-1})$$

$$= \Delta^2(\frac{1}{\Delta-1}\sum_{k=1}^{l} k(\Delta-1)^k - \frac{2}{\Delta-1}\sum_{k=1}^{l}(\Delta-1)^k)$$

$$= \frac{\Delta^2}{\Delta-1}(\frac{\Delta-1}{(\Delta-2)^2} + \frac{l(\Delta-1)^{(l+1)}}{(\Delta-2)} - \frac{(\Delta-1)^{(l+1)}}{(\Delta-2)^2} - \frac{2(\Delta-1)^{(l+1)}}{\Delta-2} + \frac{2(\Delta-1)}{\Delta-2})$$

$$= \frac{\Delta^2}{\Delta-1}(\frac{\Delta-1}{(\Delta-2)^2}(1-(\Delta-1)^l + (\Delta-1)^l(l-2)(\Delta-2) + 2(\Delta-2)))$$

$$= \frac{\Delta^2}{(\Delta-2)^2}(2\Delta - 3 - (\Delta-1)^l + (\Delta-1)^l(l-2)(\Delta-2))$$

□

Lemma 5. *Let's T_Δ^l be a Dendrimer's tree as previously described. Then :*

$$dd(u_{2j},T_\Delta^l) = \frac{2\Delta(\Delta-1)}{(\Delta-2)^2}(\Delta-1)^{2j} - \frac{\Delta^2(\Delta-1)^l}{\Delta-2}2j$$
$$+\frac{\Delta}{(\Delta-2)^2}((\Delta-1)^l(2(\Delta-2)\Delta l+(\Delta+2)-2\Delta^2)+2\Delta^2-3\Delta)$$

Proof. We using the lemmas 1 and 2 :

$$dd(u_{2j},T_\Delta^l) = \sum_{k=1}^{2l-2j} deg(u_{2j})(k-2)d_{T_\Delta^l}^{u_{2j}}(k)$$
$$= \Delta(\sum_{k=1}^{2j}(k-2)d_{T_\Delta^l}^{u_{2j}}(k) + \sum_{k=2j+1}^{2l-2j}(k-2)d_{T_\Delta^l}^{u_{2j}}(k))$$

For simplifying the proof of $dd(u_{2j},T_\Delta^l)$ we can note by :

$$dd_1(u_{2j},T_\Delta^l) = \Delta\sum_{k=1}^{2j}(k-2)d_{T_\Delta^l}^{u_{2j}}(k)$$

$$dd_2(u_{2j},T_\Delta^l) = \Delta\sum_{k=2j+1}^{2l-2j}(k-2)d_{T_\Delta^l}^{u_{2j}}(k)$$

with:

$$dd_1(u_{2j},T_\Delta^l) = \Delta\sum_{k=1}^{2j}(k-2)d_{T_\Delta^l}^{u_{2j}}(k)$$
$$= \Delta\sum_{k=1}^{2j}(k-2)\Delta(\Delta-1)^{(k-1)}$$
$$= \frac{\Delta^2}{\Delta-1}(\sum_{k=1}^{2j}k(\Delta-1)^k - 2\sum_{k=1}^{2j}(\Delta-1)^k)$$
$$= \frac{\Delta^2}{\Delta-1}(\frac{\Delta-1}{(\Delta-1-1)^2}(1+((\Delta-1-1)2j-1)(\Delta-1)^{2j}) - 2\frac{(\Delta-1)^{2j+1}-(\Delta-1)}{\Delta-1-1})$$
$$= (\frac{\Delta}{\Delta-2})^2(1+2j(\Delta-1)^{2j}(\Delta-2)-(\Delta-1)^{2j}+2(\Delta-2)-2(\Delta-2)(\Delta-1)^{2j})$$
$$= \frac{\Delta^2}{\Delta-2}(\Delta-1)^{2j}2j + \frac{\Delta^2(3-2\Delta)}{(\Delta-2)^2}(\Delta-1)^{2j} + \frac{\Delta^2(2\Delta-3)}{(\Delta-2)^2}$$

And:

$$\begin{aligned} dd_2(u_{2j}, T_\Delta^l) &= \Delta \sum_{k=2j+1}^{2l-2j} (k-2) d_{T_\Delta^l}^{u_{2j}}(k) \\ &= \Delta \Big(\sum_{r=j+1}^{\frac{2l-2j}{2}} (2r-2) d_{T_\Delta^l}^{u_{2j}}(2r) + \sum_{r=j}^{\frac{2l-2j-2}{2}} (2r+1-2) d_{T_\Delta^l}^{u_{2j}}(2r+1) \Big) \\ &= \Delta \Big(\sum_{r=j+1}^{l-j} (2r-2)(\Delta-1)^{j+r} + \sum_{r=j}^{l-j-1} (2r-1)(\Delta-1)^{j+r} \Big) \end{aligned}$$

For simplifying the proof of $dd_2(u_{2j}, T_\Delta^l)$ we can note by :

$$dd_2(u_{2jP}, T_\Delta^l) = \sum_{r=j+1}^{l-j} (2r-2)(\Delta-1)^{j+r}$$

$$dd_2(u_{2jI}, T_\Delta^l) = \sum_{r=j}^{l-j-1} (2r-1)(\Delta-1)^{j+r}$$

With:

$$\begin{aligned} dd_2(u_{2jP}, T_\Delta^l) &= \sum_{r=j+1}^{l-j} (2r-2)(\Delta-1)^{j+r} \\ &= 2(\Delta-1)^j \sum_{r=j+1}^{l-j} (r-1)(\Delta-1)^r \\ &= 2(\Delta-1)^j \Big(\sum_{r=j+1}^{l-j} r(\Delta-1)^r - \sum_{r=j+1}^{l-j} (\Delta-1)^r \Big) \\ &= 2(\Delta-1)^j \Big(\sum_{r=1}^{l-2j} (r+j)(\Delta-1)^{(r+j)} - \sum_{r=1}^{l-2j} (\Delta-1)^{r+j} \Big) \\ &= 2(\Delta-1)^{2j} \Big(\sum_{r=1}^{l-2j} r(\Delta-1)^r + j \sum_{r=1}^{l-2j} (\Delta-1)^r - \sum_{r=1}^{l-2j} (\Delta-1)^r \Big) \\ &= 2(\Delta-1)^{2j} \Big(\sum_{r=1}^{l-2j} r(\Delta-1)^r + (j-1) \sum_{r=1}^{l-2j} (\Delta-1)^r \Big) \\ &= 2(\Delta-1)^{2j} \Big(\frac{\Delta-1}{(\Delta-2)^2} (1 + ((\Delta-2)(l-2j)-1)(\Delta-1)^{(l-2j)}) \\ &+ (j-1) \Big(\frac{(\Delta-1)^{(l-2j+1)} - (\Delta-1)}{\Delta-2} \Big) \Big) \end{aligned}$$

$$= -(\frac{\Delta-1}{\Delta-2})(\Delta-1)^{2j}2j + (\frac{\sqrt{2}(\Delta-1)}{\Delta-2})^2(\Delta-1)^{2j} - \frac{2(\Delta-1)^{(l+1)}}{\Delta-2}j$$
$$+ \frac{2(\Delta-1)^{(l+1)}((l-1)(\Delta-2)-1)}{(\Delta-2)^2}$$

And

$$dd_2(u_{2jI}, T_\Delta^l) = \sum_{r=j}^{l-j-1}(2r-1)(\Delta-1)^{j+r}$$
$$= (\Delta-1)^j(2\sum_{r=j}^{l-j-1} r(\Delta-1)^r - \sum_{r=j}^{l-j-1}(\Delta-1)^r)$$
$$= (\Delta-1)^j(2(j(\Delta-1)^j + \sum_{r=1}^{l-2j-1}(j+r)(\Delta-1)^{(j+r)}) - (\Delta-1)^j - \sum_{r=1}^{l-2j-1}(\Delta-1)^{(r+j)})$$
$$= (\Delta-1)^{2j}(2\sum_{r=1}^{l-2j-1} r(\Delta-1)^r + (2j-1)\sum_{r=1}^{l-2j-1}(\Delta-1)^r + (2j-1))$$
$$= (\Delta-1)^{2j}(\frac{2(\Delta-1)}{(\Delta-2)^2}(1+((\Delta-2)(l-2j-1)-1)(\Delta-1)^{(l-2j-1)})$$
$$+ \frac{2j-1}{\Delta-2}((\Delta-1)^{(l-2j)} - (\Delta-1)) + (2j-1))$$
$$= (\Delta-1)^{2j}(\frac{(\Delta-1)^{(l-2j)}}{(\Delta-2)^2}(2(\Delta-2)(l-2j-1) - 2 + (2j-1)(\Delta-2))$$
$$- \frac{\Delta-1}{(\Delta-2)^2}(-2+(\Delta-2)(2j-1)) + (2j-1))$$
$$= -\frac{(\Delta-1)^{2j}}{\Delta-2}2j + \frac{(3\Delta-4)(\Delta-1)^{2j}}{(\Delta-2)^2} - \frac{(\Delta-1)^l}{\Delta-2}2j + \frac{(\Delta-1)^l}{(\Delta-2)^2}(2(\Delta-2)(l-1)-\Delta)$$

Then:

$$dd_2(u_{2j}, T_\Delta^l) = \Delta(dd_2(u_{2jP}) + dd_2(u_{2jI}, T_\Delta^l))$$
$$= \Delta(-(\frac{\Delta-1}{\Delta-2})(\Delta-1)^{2j}2j + (\frac{\sqrt{2}(\Delta-1)}{\Delta-2})^2(\Delta-1)^{2j} - \frac{2(\Delta-1)^{(l+1)}}{\Delta-2}j$$
$$+ \frac{2(\Delta-1)^{(l+1)}((l-1)(\Delta-2)-1)}{(\Delta-2)^2}$$
$$- \frac{(\Delta-1)^{2j}}{\Delta-2}2j + \frac{(3\Delta-4)(\Delta-1)^{2j}}{(\Delta-2)^2} - \frac{(\Delta-1)^l}{\Delta-2}2j + \frac{(\Delta-1)^l}{(\Delta-2)^2}(2(\Delta-2)(l-1)-\Delta))$$
$$= \Delta(-\frac{\Delta}{\Delta-2}(\Delta-1)^{2j}2j + \frac{2\Delta^2-\Delta-2}{(\Delta-2)^2}(\Delta-1)^{2j} - \frac{\Delta(\Delta-1)^l}{\Delta-2}2j$$
$$+ \frac{(\Delta-1)^l}{(\Delta-2)^2}(2\Delta(\Delta-2)(l-1) - (3\Delta-2)))$$

Go back now to $dd(u_{2j},T_{\Delta}^{l})$, then:

$$\begin{aligned}
dd(u_{2j},T_{\Delta}^{l}) &= dd_1(u_{2j},T_{\Delta}^{l})+dd_2(u_{2j},T_{\Delta}^{l})\\
&= \frac{\Delta^2}{\Delta-2}(\Delta-1)^{2j}2j+\frac{\Delta^2(3-2\Delta)}{(\Delta-2)^2}(\Delta-1)^{2j}+\frac{\Delta^2(2\Delta-3)}{(\Delta-2)^2}+\Delta(-\frac{\Delta}{\Delta-2}(\Delta-1)^{2j}2j\\
&+\frac{2\Delta^2-\Delta-2}{(\Delta-2)^2}(\Delta-1)^{2j}-\frac{\Delta(\Delta-1)^l}{\Delta-2}2j+\frac{(\Delta-1)^l}{(\Delta-2)^2}(2\Delta(\Delta-2)(l-1)-(3\Delta-2)))\\
&= (\Delta-1)^{2j}\frac{1}{(\Delta-1)^2}(\Delta^2(3-2\Delta)+\Delta(2\Delta^2-\Delta-2))+2j(\frac{-\Delta^2(\Delta-1)^l}{\Delta-2})+\frac{\Delta^2(2\Delta-3)}{(\Delta-2)^2}\\
&+\frac{\Delta(\Delta-1)^l}{(\Delta-2)^2}(2\Delta(\Delta-2)(l-1)-(3\Delta-2))\\
&= \frac{2\Delta(\Delta-1)}{(\Delta-2)^2}(\Delta-1)^{2j}-\frac{\Delta^2(\Delta-1)^l}{\Delta-2}2j+\frac{\Delta}{(\Delta-2)^2}((\Delta-1)^l(2(\Delta-2)\Delta l\\
&+(\Delta+2)-2\Delta^2)+2\Delta^2-3\Delta)
\end{aligned}$$

□

Lemma 6. *Let's T_{Δ}^{l} be a Dendrimer's tree as previously described. Then:*

$$dd(u_{2j+1},T_{\Delta}^{l})=\frac{2\Delta(\Delta-1)^2}{(\Delta-2)^2}(\Delta-1)^{2j}-\frac{2\Delta^2(\Delta-1)^l}{\Delta-2}j+\frac{\Delta^2(2\Delta-3)}{(\Delta-2)^2}+\frac{\Delta(\Delta-1)^l}{(\Delta-2)^2}((\Delta-2)(2\Delta l-3\Delta-1)-2\Delta)$$

Proof. We using the lemmas 1 and 2:

$$\begin{aligned}
dd(u_{2j+1},T_{\Delta}^{l}) &= \sum_{k=1}^{2l-(2j+1)} deg(u_{2j+1})(k-2)d_{T_{\Delta}^{l}}^{u_{2j+1}}(k)\\
&= deg(u_{2j+1})(\sum_{k=1}^{2j+1}(k-2)d_{T_{\Delta}^{l}}^{u_{2j+1}}(k)+\sum_{k=(2j+1)+1}^{2l-(2j+1)}(k-2)d_{T_{\Delta}^{l}}^{u_{2j+1}}(k))
\end{aligned}$$

For simplifying the proof of $dd(u_{2j+1},T_{\Delta}^{l})$ we can note by :

$$dd_1(u_{2j+1},T_{\Delta}^{l})=deg(u_{2j+1})\sum_{k=1}^{2j+1}(k-2)d_{T_{\Delta}^{l}}^{u_{2j+1}}(k)$$

$$dd_2(u_{2j+1},T_{\Delta}^{l})=deg(u_{2j+1})\sum_{k=2j+2}^{2l-2j-1}(k-2)d_{T_{\Delta}^{l}}^{u_{2j+1}}(k)$$

With

$$
\begin{aligned}
dd_1(u_{2j+1},T_\Delta^l) &= deg(u_{2j+1})\sum_{k=1}^{2j+1}(k-2)d_{T_\Delta^l}^{u_{2j+1}}(k)\\
&= \Delta\sum_{k=1}^{2j+1}(k-2)\Delta(\Delta-1)^{(k-1)}\\
&= \frac{\Delta^2}{\Delta-1}(\sum_{k=1}^{2j+1}k(\Delta-1)^k-2\sum_{k=1}^{2j+1}(\Delta-1)^k)\\
&= \frac{\Delta^2}{\Delta-1}(\frac{2(\Delta-1)^2}{\Delta-2}j(\Delta-1)^{2j}+\frac{(\Delta-1)^2(\Delta-3)}{(\Delta-2)^2}(\Delta-1)^{2j}\\
&+\frac{\Delta-1}{(\Delta-2)^2}-\frac{2(\Delta-1)^2}{\Delta-2}(\Delta-1)^{2j}+\frac{2(\Delta-1)}{\Delta-2})\\
&= \frac{\Delta^2}{(\Delta-2)^2}(2j(\Delta-2)(\Delta-1)(\Delta-1)^{2j}-(\Delta-1)^2(\Delta-1)^{2j}+(2\Delta-3))
\end{aligned}
$$

And

$$
\begin{aligned}
dd_2(u_{2j+1},T_\Delta^l) &= deg(u_{2j+1})\sum_{k=2j+2}^{2l-2j-1}(k-2)d_{T_\Delta^l}^{u_{2j+1}}(k)\\
&= \Delta(\sum_{r=j+1}^{\frac{2l-2j-1-1}{2}}(2r-2)d_G^{u_{2j+1}}(2r)+\sum_{r=j+1}^{\frac{2l-2j-1-1}{2}}(2r+1-2)d_G^{u_{2j+1}}(2r+1))\\
&= \Delta\sum_{r=j+1}^{l-j-1}(2r-2)(\Delta-1)^{(j+r)}+\Delta\sum_{r=j+1}^{l-j-1}(2r-1)(\Delta-1)^{(j+r+1)}\\
&= \Delta(\Delta-1)^j(\sum_{r=j+1}^{l-j-1}r(\Delta-1)^r-\sum_{r=j+1}^{l-j-1}(\Delta-1)^r)+\Delta(\Delta-1)^{(j+1)}(2\sum_{r=j+1}^{l-j-1}r(\Delta-1)^r\\
&-\sum_{r=j+1}^{l-j-1}(\Delta-1)^r)\\
&= 2\Delta(\Delta-1)^j(1+\Delta-1)\sum_{r=j+1}^{l-j-1}r(\Delta-1)^r-\Delta(\Delta-1)^j(2+\Delta-1)\sum_{r=j+1}^{l-j-1}(\Delta-1)^r\\
&= 2\Delta^2(\Delta-1)^j\sum_{r=j+1}^{l-j-1}r(\Delta-1)^r-\Delta(\Delta-1)^j(\Delta+1)\sum_{r=j+1}^{l-j-1}(\Delta-1)^r\\
&= 2\Delta^2(\Delta-1)^j(\frac{(\Delta-1)^{j+1}}{(\Delta-2)^2}(1+((\Delta-2)(l-2j-1)-1)(\Delta-1)^{(l-2j-1)})\\
&+\frac{j(\Delta-1)^{(j+1)}}{\Delta-2}((\Delta-1)^{(l-2j-1)}-1))\Delta(\Delta-1)^j(\Delta+1)(\frac{(\Delta-1)^{(j+1)}}{\Delta-2}((\Delta-1)^{(l-2j-1)}-1))\\
&= \frac{\Delta(\Delta-1)^2(\Delta+2)}{(\Delta-2)^2}(\Delta-1)^{2j}-\frac{2\Delta^2(\Delta-1)}{\Delta-2}(\Delta-1)^{2j}j-\frac{2\Delta^2(\Delta-1)^l}{\Delta-2}j\\
&+\frac{\Delta(\Delta-1)^l}{(\Delta-2)^2}(-2\Delta+(\Delta-2)(2\Delta l-3\Delta-1))
\end{aligned}
$$

Go back now to $dd(u_{2j+1},T_{\Delta}^{l})$, then:

$$\begin{aligned}
dd(u_{2j+1},T_{\Delta}^{l}) &= dd_1(u_{2j+1},T_{\Delta}^{l})+dd_2(u_{2j+1},T_{\Delta}^{l})\\
&=(-\frac{\Delta^2(\Delta-1)}{\Delta-2}+\frac{\Delta^2(\Delta-1)}{\Delta-2})(\Delta-1)^{2j}2j+(\frac{\Delta(\Delta-1)^2(\Delta+2)}{(\Delta-2)^2}-\frac{\Delta^2(\Delta-1)^2}{(\Delta-2)^2})(\Delta-1)^{2j}\\
&-\frac{2\Delta^2(\Delta-1)^l}{\Delta-2}j+\frac{\Delta}{(\Delta-2)^2}(\Delta(2\Delta-3)+(\Delta-1)^l((\Delta-2)(2\Delta l-3\Delta-1)-2\Delta))\\
&=\frac{2\Delta(\Delta-1)^2}{(\Delta-2)^2}(\Delta-1)^{2j}-\frac{2\Delta^2(\Delta-1)^l}{\Delta-2}j+\frac{\Delta^2(2\Delta-3)}{(\Delta-2)^2}\\
&+\frac{\Delta(\Delta-1)^l}{(\Delta-2)^2}((\Delta-2)(2\Delta l-3\Delta-1)-2\Delta)
\end{aligned}$$

□

Lemma 7. *Let's T_{Δ}^{l} be a Dendrimer's tree as previously described. Then:*

$$\begin{aligned}
\sum_{j=1}^{\frac{l-2}{2}}P_{(l-2j)}\times dd(u_{2j},T_{\Delta}^{l}) &= \frac{\Delta^2(\Delta-1)^{l-1}}{(\Delta-2)^3}(2\Delta^3 l+5\Delta^2 l-7\Delta^2-8\Delta l+12\Delta+4l-8)\\
&+\frac{\Delta^2(\Delta-1)^{(2l-1)}}{(\Delta-2)^3}(2\Delta l-4\Delta-4l+5)-\frac{\Delta^2(\Delta-1(2\Delta-3))}{(\Delta-2)^3}
\end{aligned}$$

Proof. We using the lemmas 1, 2 and 5:

$$\begin{aligned}
\sum_{j=1}^{\frac{l-2}{2}}P_{(l-2j)}dd(u_{2j},T_{\Delta}^{l}) &= \sum_{j=1}^{\frac{l-2}{2}}(\Delta(\Delta-1)^{(l-2j-1)})(\frac{2\Delta(\Delta-1)}{(\Delta-2)^2}(\Delta-1)^{2j}-\frac{\Delta^2(\Delta-1)^l}{\Delta-2}2j\\
&+\frac{\Delta}{(\Delta-2)^2}((\Delta-1)^l)(\Delta+2+2(\Delta-2)\Delta l-2\Delta^2)+2\Delta^2-3\Delta)\\
&=\frac{2\Delta^2(\Delta-1)^{(l-1+1)}}{(\Delta-2)^2}\sum_{j=1}^{\frac{l-2}{2}}(\Delta-1)^{(-2j+2j)}-\frac{2\Delta^3(\Delta-1)^{(l-1+l)}}{\Delta-2}\sum_{j=1}^{\frac{l-2}{2}}((\Delta-1)^{-2})^j\\
&+\frac{\Delta^2(\Delta-1)^{(l-1)}}{(\Delta-2)^2}((\Delta-1)^l(\Delta+2+2\Delta l(\Delta-2)-2\Delta^2)+2\Delta^2-3\Delta)\sum_{j=1}^{\frac{l-2}{2}}((\Delta-1)^{-2})^j\\
&=\frac{2\Delta^2(\Delta-1)^l}{(\Delta-2)^2}\frac{l-2}{2}-\frac{2\Delta^3(\Delta-1)^{2l-1}}{\Delta-2}\frac{(\Delta-1)^{-2}}{((\Delta-1)^{-2}-1)^2}(1+(((\Delta-1)^{-2}-1)\frac{l-2}{2}-1)\\
&((\Delta-1)^{-2})^{\frac{l-2}{2}})+\frac{\Delta^2(\Delta-1)^{(l-1)}}{(\Delta-2)^2}+\frac{\Delta^2(\Delta-1)^{l-1}}{(\Delta-2)^2}\times\frac{((\Delta-1)^{-2})^{\frac{l-2}{2}+1}-(\Delta-1)^{-2}}{(\Delta-1)^{-2}-1}\\
&((\Delta-1)^l(\Delta+2+2\Delta l(\Delta-2)-2\Delta^2)+2\Delta^2-3\Delta)
\end{aligned}$$

$$= \frac{\Delta(\Delta-1)^{(l-1)}}{(\Delta-2)^3}(\Delta(\Delta-1)(l-2)(\Delta-2)+2(\Delta-1)^2+\Delta l(\Delta-1)^2(\Delta-2)$$

$$+2\Delta^2(\Delta-1)^2-\Delta^2(\Delta-2)-2\Delta^2 l-2\Delta^2 l(\Delta-2)^2+4\Delta l-2)-\frac{\Delta^2(\Delta-1)(2\Delta-3)}{(\Delta-2)^3}$$

$$+\frac{\Delta(\Delta-1)^{(2l-1)}}{(\Delta-2)^3}(2\Delta l(\Delta-2)+\Delta-2(\Delta-1)(\Delta+1)-2(\Delta-1)^2)$$

$$= \frac{\Delta^2(\Delta-1)^{l-1}}{(\Delta-2)^3}(2\Delta^3 l+5\Delta^2 l-7\Delta^2-8\Delta l+12\Delta+4l-8)$$

$$+\frac{\Delta^2(\Delta-1)^{(2l-1)}}{(\Delta-2)^3}(2\Delta l-4\Delta-4l+5)-\frac{\Delta^2(\Delta-1(2\Delta-3))}{(\Delta-2)^3}$$

□

Lemma 8. *Let's T^l_Δ be a Dendrimer's tree as previously described. Then :*

$$\sum_{j=0}^{\frac{l-2}{2}} P_{l-(2j+1)} \times dd(u_{2j+1}, T^l_\Delta) = \frac{\Delta^2(\Delta-1)^l}{(\Delta-2)^3}(5\Delta-6)+\frac{\Delta^2(\Delta-1)^{2l}}{(\Delta-2)^3}(2l(\Delta-2)-3(\Delta-1))-\frac{\Delta^2(2\Delta-3)}{(\Delta-2)^3}$$

Proof. We using the lemmas 1, 2 and 6:

$$\sum_{j=0}^{\frac{l-2}{2}} P_{l-(2j+1)} \times dd(u_{2j+1}, T^l_\Delta) = \sum_{j=0}^{\frac{l-2}{2}} \Delta(\Delta-1)^{(l-(2j+1)-1)}(\frac{2\Delta(\Delta-1)^2}{(\Delta-2)^2}(\Delta-1)^{2j}-\frac{2\Delta^2(\Delta-1)^l}{\Delta-2}j$$

$$+\frac{\Delta^2(2\Delta-3)}{(\Delta-2)^2}+\frac{\Delta(\Delta-1)^l}{(\Delta-2)^2}\times((\Delta-2)(2\Delta l-3\Delta-1)-2\Delta))$$

$$= \frac{2\Delta^2(\Delta-1)^l}{(\Delta-2)^2}\sum_{j=0}^{\frac{l-2}{2}}1-\frac{2\Delta^3(\Delta-1)^{(2l-2)}}{\Delta-2}\sum_{j=0}^{\frac{l-2}{2}}((\Delta-1)^{-2})^j j$$

$$+\sum_{j=0}^{\frac{l-2}{2}}((\Delta-1)^{-2})^j\frac{\Delta^3(2\Delta-3)(\Delta-1)^{(l-2)}}{(\Delta-2)^2}$$

$$+\frac{\Delta^2(\Delta-1)^{(2l-2)}}{(\Delta-2)^2}\times((\Delta-2)(2\Delta l-3\Delta-1)-2\Delta)\sum_{j=0}^{\frac{l-2}{2}}((\Delta-1)^{-2})^j$$

$$= \frac{2\Delta^2(\Delta-1)^l}{(\Delta-2)^2}(\frac{l-2}{2}+1)-\frac{2\Delta^3(\Delta-1)^{(2l-2)}}{\Delta-2}(\frac{(\Delta-1)^{-2}}{((\Delta-1)^{-2}-1)^2}$$

$$(1+(((\Delta-1)^{-2}-1)\frac{l-2}{2}-1)((\Delta-1)^{-2})^{\frac{l-2}{2}})$$

$$+\frac{\Delta^3(2\Delta-3)\Delta-1)^{(l-2)}}{(\Delta-2)^2}((\frac{((\Delta-1)^{-2})^{(\frac{l-2}{2}+1)}-1}{(\Delta-1)^{-2}-1})$$

$$+(\frac{((\Delta-1)^{-2})^{(\frac{l-2}{2}+1)}-1}{(\Delta-1)^{-2}-1})\frac{\Delta^2(\Delta-1)^{(2l-2)}}{(\Delta-2)^2}((\Delta-2)(2\Delta l-3\Delta-1)-2\Delta))$$

$$= \frac{2\Delta^2 l(\Delta-1)^l}{(\Delta-2)^2} + \frac{2\Delta(\Delta-1)^l}{(\Delta-2)^3} + \frac{\Delta^2(\Delta-1)^l(2\Delta-3)}{(\Delta-2)^3} - \frac{2\Delta(\Delta-1)^{2l}}{(\Delta-2)^2}$$
$$- \frac{3\Delta^2(\Delta-1)^{2l}}{(\Delta-2)^2} - \frac{\Delta(\Delta-1)^{2l}}{(\Delta-2)^2} - \frac{2\Delta^2 l(\Delta-1)^l}{(\Delta-2)^2} + \frac{3\Delta^2(\Delta-1)^l}{(\Delta-2)^2} + \frac{\Delta(\Delta-1)^l}{(\Delta-2)^2}$$
$$= \frac{\Delta(\Delta-1)^l}{(\Delta-2)^3}(\Delta+3\Delta^2-6\Delta+2\Delta+2\Delta^2-3\Delta) + \frac{\Delta(\Delta-1)^{2l}}{(\Delta-2)^3}$$
$$(-2-2\Delta+2\Delta^2 l-4\Delta l-3\Delta^2+6\Delta-\Delta+2) - \frac{\Delta^2(2\Delta-3)}{(\Delta-2)^3}$$
$$= \frac{\Delta^2(\Delta-1)^l}{(\Delta-2)^3}(5\Delta-6) + \frac{\Delta^2(\Delta-1)^{2l}}{(\Delta-2)^3}(2l(\Delta-2)-3(\Delta-1)) - \frac{\Delta^2(2\Delta-3)}{(\Delta-2)^3}$$

□

Lemma 9. *Let's T_Δ^l be a Dendrimer's tree as previously described. Then :*

$$P_l \times dd(u_0, T_\Delta^l) = \frac{\Delta(\Delta-1)^{(l-1)}}{(\Delta-2)^2}(2\Delta^2-\Delta-2) + \frac{\Delta(\Delta-1)^{(2l-1)}}{(\Delta-2)^2}(\Delta+2-4\Delta l+2\Delta^2(l-1))$$

Proof. We using the lemmas 1 and 3 :

$$P_l \times dd(u_0, T_\Delta^l) = (\Delta(\Delta-1)^{(l-1)})(\frac{(\Delta-1)(3\Delta-2)}{(\Delta-2)^2} + \frac{(\Delta-1)^l}{(\Delta-2)^2}(\Delta+2-4\Delta l+2\Delta^2(l-1))-1)$$
$$= \frac{\Delta(\Delta-1)^{(l-1)}}{(\Delta-2)^2}(2\Delta^2-\Delta-2) + \frac{\Delta(\Delta-1)^{(2l-1)}}{(\Delta-2)^2}(\Delta+2-4\Delta l+2\Delta^2(l-1))$$

□

Lemma 10. *Let's T_Δ^l be a Dendrimer's tree as previously described. Then :*

$$P_0 \times dd(u_l, T_\Delta^l) = \frac{\Delta^2(\Delta-1)^l}{(\Delta-2)^2}((l-2)(\Delta-2)-1) + \frac{\Delta^2}{\Delta-2}(2\Delta-3)$$

Proof. We using the lemmas 1 and 4 :

$$P_0 \times dd(u_l, T_\Delta^l) = 1 \times \frac{\Delta^2}{(\Delta-2)^2}(2\Delta-3-(\Delta-1)^l+(\Delta-1)^l(l-2)(\Delta-2))$$
$$= \frac{\Delta^2(\Delta-1)^l}{(\Delta-2)^2}((l-2)(\Delta-2)-1) + \frac{\Delta^2}{\Delta-2}(2\Delta-3)$$

□

Lemma 11. *Let's T_Δ^l be a Dendrimer's tree as previously described. if l is odd, then:*

$$\sum_{j=1}^{\frac{l-1}{2}} P_{l-2j} \times dd(u_{2j}, T_\Delta^l) = \frac{\Delta^2(\Delta-1)^{(l-1)}}{(\Delta-2)^3}(2\Delta^2-\Delta-2) + \frac{\Delta^2(\Delta-1)^{(2l-1)}}{(\Delta-2)^3}(2\Delta l-4\Delta-4l+5) - \frac{\Delta^2(2\Delta-3)}{(\Delta-2)^3}$$

Proof. We using the lemmas 1, 2 and 5 :

$$\sum_{j=1}^{\frac{l-1}{2}} dd(u_{2j},T_\Delta^l)\times P_{l-2j} = \frac{2\Delta^2(\Delta-1)^l}{(\Delta-2)^2}\sum_{j=1}^{\frac{l-1}{2}} 1 - \frac{2\Delta^3(\Delta-1)^{(2l-1)}}{\Delta-2}\sum_{j=1}^{\frac{l-1}{2}}((\Delta-1)^{-2})^j j$$

$$+\frac{\Delta^2(\Delta-1)^{(l-1)}}{(\Delta-2)^2}((\Delta-1)^l(\Delta+2+2\Delta l(\Delta-2)-2\Delta^2)+2\Delta^2-3\Delta)\sum_{j=1}^{\frac{l-1}{2}}((\Delta-1)^{-2})^j$$

$$=\frac{2\Delta^2(\Delta-1)^l}{(\Delta-2)^2}\times\frac{l-1}{2}-\frac{2\Delta^3(\Delta-1)^{(2l-1)}}{\Delta-2}(\frac{(\Delta-1)^2}{((\Delta-1)^{-2}-1)^2}$$

$$(1+(((\Delta-1)^{-2}-1)\frac{l-1}{2}-1)((\Delta-1)^{-2})^{\frac{l-1}{2}}))+\frac{\Delta^2(\Delta-1)^{(l-1)}}{(\Delta-2)^2}$$

$$\times\frac{((\Delta-1)-2)^{\frac{l-1}{2}+1}-(\Delta-1)^{-2}}{(\Delta-1)^{-2}-1}((\Delta-1)^l(\Delta+2+2\Delta l(\Delta-2)-2\Delta^2)+2\Delta^2-3\Delta)$$

$$=\frac{\Delta(\Delta-1)^{l-1}}{(\Delta-2)^3}(\Delta(l-1)(\Delta-2)(\Delta-1)+\Delta l(\Delta-2)(\Delta-1)+(\Delta-1)^3+(\Delta-1)$$

$$-\Delta(\Delta-1)-2(\Delta-1)+2\Delta^2(\Delta-1)-2\Delta l(\Delta-2)(\Delta-1)+\Delta(2\Delta-3))$$

$$+\frac{\Delta(\Delta-1)^{(2l-1)}}{(\Delta-2)^3}(-2(\Delta-1)^2+\Delta+2-2\Delta^2+2\Delta l(\Delta-2))-\frac{\Delta^2(2\Delta-3)}{(\Delta-2)^3}$$

$$=\frac{\Delta^2(\Delta-1)^{(l-1)}}{(\Delta-2)^3}(2\Delta^2-\Delta-2)+\frac{\Delta^2(\Delta-1)^{(2l-1)}}{(\Delta-2)^3}(2\Delta l-4\Delta-4l+5)-\frac{\Delta^2(2\Delta-3)}{(\Delta-2)^3}$$

□

Lemma 12. *Let's T_Δ^l be a Dendrimer's tree as previously described. if l is odd, then:*

$$\sum_{j=0}^{\frac{l-3}{2}} P_{l-(2j+1)}\times dd(u_{2j+1},T_\Delta^l) = \frac{\Delta^2(\Delta-1)^l}{(\Delta-2)^3}(-\Delta^2 l+2\Delta^2+4\Delta l-2\Delta-4l)$$

$$+\frac{\Delta^2(\Delta-1)^{2l}}{(\Delta-2)^3}(2\Delta l-3\Delta-4l+3)-\frac{\Delta^2(\Delta-1)(2\Delta-3)}{(\Delta-2)^3}$$

Proof. We using the lemmas 1, 2 and 6:

$$\sum_{j=0}^{\frac{l-3}{2}} P_{l-(2j+1)}dd(u_{2j+1},T_\Delta^l) = \frac{2\Delta^2(\Delta-1)^l}{(\Delta-2)^2}\sum_{j=0}^{\frac{l-3}{2}} 1-\frac{2\Delta^3(\Delta-1)^{(2l-2)}}{\Delta-2}\sum_{j=0}^{\frac{l-3}{2}}((\Delta-1)^{-2})^j j$$

$$+\sum_{j=0}^{\frac{l-3}{2}}((\Delta-1)^{-2})^j(\frac{\Delta^3(2\Delta-3)(\Delta-1)^{(l-2)}}{(\Delta-2)^2})$$

$$+\frac{\Delta^2(\Delta-1)^{(2l-2)}}{(\Delta-2)^2}((\Delta-2)(2\Delta l-3\Delta-1)-2\Delta)\sum_{j=0}^{\frac{l-3}{2}}((\Delta-1)^{-2})^j$$

$$= \frac{2\Delta^2(\Delta-1)^l}{(\Delta-2)^2}(\frac{l-3}{2}+1) - \frac{2\Delta^3(\Delta-1)^{(2l-2)}}{\Delta-2}$$
$$(\frac{(\Delta-1)^{-2}}{((\Delta-1)^{-2}-1)^2})(1+(((\Delta-1)^{-2}-1)\frac{l-3}{2}-1)((\Delta-1)^{-2})^{\frac{l-3}{2}})$$
$$+\frac{\Delta^3(2\Delta-3)(\Delta-1)^{(l-2)}}{(\Delta-2)^2}(\frac{((\Delta-1)^{-2})^{\frac{l-3}{2}+1}-1}{(\Delta-1)^{-2}-1})$$
$$+(\frac{((\Delta-1)^{-2})^{\frac{l-3}{2}+1}-1}{(\Delta-1)^{-2}-1})\frac{\Delta^2(\Delta-1)^{(2l-2)}}{(\Delta-2)^2}((\Delta-2)(2\Delta l-3\Delta-1)-2\Delta)$$
$$= \frac{\Delta^2(\Delta-1)^l}{(\Delta-2)^3}(-\Delta^2 l+2\Delta^2+4\Delta l-2\Delta-4l)+\frac{\Delta^2(\Delta-1)^{2l}}{(\Delta-2)^3}(2\Delta l-3\Delta-4l+3)$$
$$-\frac{\Delta^2(\Delta-1)(2\Delta-3)}{(\Delta-2)^3}$$

□

5. CONCLUSION

We have rewote in this paper the degree distance's formula of the Dendrimer's tree in function of the Δ, its levels' number l, its size m and its order n. The proof which is long got using the new formula of the degree distance index that is proofed in [2].

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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