Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:167
https://doi.org/10.28919/jmcs/7394
ISSN: 1927-5307

EXISTENCE RESULTS FOR QUASILINEAR PARABOLIC SYSTEMS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. This paper propose new approaches to the investigation of the work studing the Quasilinear Parabolic Equations With Nonlocal Boundary Conditions , this study is a generalization of there results, where we prove the existence of a generalized solution for a class of quasilinear equations with nonlocal boundary conditions By using Feado-Galerkin approximation.

Keywords: parabolic system; generalized solution; nonlocal boundary conditions.

2010 AMS Subject Classification: 58J35, 35D30.

1. Introduction

Parabolic partial differential equations $(PDEs)$ have been the focus of many scientist and researchers attention [5], [29], [31] and became very active in recent years. The attention has been given to prove the existence results for this type of problems. Recently, this interest is focused on the nonlinear [14], [33], [31], [28] and quasilinear problem [12], [13], [23]. For the nonlinear parabolic problem, there is the study of Alexander Karakov in 2021 [20], where they studied the existence of the solutions for the nonlinear evolutionary partial equation of diffusion wave type and proved a new existence and uniqueness theorem for this type of problem. In the

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Received March 26, 2022

study of the quasilinear problem, there is the work of Yongqiang Fu and Mingqi Xiang [15] in 2015, which they gave a new class of quasilinear parabolic equation involving $p(x,t)-$ Laplacian operator and nonlocal term in a bounded domain and obtained the local existence of weak solutions by applying Galerkin's approximation method. Furthermore, the study in 2020 by Dolan et all [22], which proved the globel existence of weak solutions to a class of quasilinear parabolic equations with nonlinearities depending on first order terms and integrable data in a moving domain invertigated, this class includes the $p-$Laplace equation as a special case. In the same year, Suping and Zhong [32], proved some new nonexistence theorems for a class of quasilinear parabolic differential inequalities with a singular potential term and nonlocal source term in the case of homogeneous and non-homogeneous by the test function method. Azroul et all in 2020 [12] studied the existence of solution for some quasilinear parabolic systems with weight and weak monotonicity. However, the different methods has been used for the existence for this type of problem, such as Topological degree method [2], compactness method [23, 24, 25], Feado-Galarkin method [3, 21, 27, 29, 30, 15]. Especially, the Feado-Galarkin method found to be a convenient tool for this sort of problems, which, in 2008, Bouziani et all [4] showed the existence of a unique weak solutions for linear parabolic equation with nonlocal boundary conditions, such as the nonlocal boundary conditions used to replace the local conditions that gives better effect than the local because the measurement given by a nonlocal conditions is usually more precise than the only one measurement given by local conditions. See in instance, in 2004 [17] Hong-Ming Yin studied a class of parabolic equations subject to a nonlocal boundary condition by using energy method, there problem is a generalized model for a theory of reaction-diffusion in channels. Gladkov and Kavitova [18] in 2020 proved global existence and blow-up of solutions of initial-boundary value problem for nonlinear nonlocal parabolic equation with nonlinear nonlocal boundary condition. In 2011, Chen [9], used the Faedo-Galerkin approximation to prove the existence of a generalized solution for quasilinear parabolic equation with nonlocal boundary conditions, given by:

$$\frac{\partial u}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big(|u|^{p-2}\frac{\partial u}{\partial x_i}\big)+|u|^{p-2}u=f(x,t), \quad x\in\Omega, t\in[0,T] \tag{1}$$

$$u(x,t)=\int_{\Omega}k(x,y)u(y,t)dy, \quad x\in\Gamma \tag{2}$$

$$u(x,0) = u_0(x). \tag{3}$$

The main difficulty of there problem is related to the presence of both quasilinear term and nonlocal boundary condition, where is the extension of the problem in Lion's book [[26], p.140] in which the boundary conditions are homegeneous, but this kind of problem are very limited, where we only found a study in [10], for a quasilinear parabolic equation with nonlocal boundary conditions different from (2).

The aim of our study, is to give a generalization to the study of Chen in 2011 [9] in system, where we prove the existence of a generalized solution to the proposed model by constructing approximate solution using Faedo-Galerkin method and applying a weak convergence and compactness arguments. The proposed problem is the follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(|u|^{\rho}\frac{\partial u}{\partial x_i}\big) + |v|^{\rho} v = f_1(x,t), \quad x \in \Omega \times t \in [0,T] \\ \\ \dfrac{\partial v}{\partial t} - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(|v|^{\rho}\frac{\partial v}{\partial x_i}\big) + |u|^{\rho} u = f_2(x,t), \quad x \in \Omega \times t \in [0,T] \\ \\ u(x,t) = \displaystyle\int_{\Omega} k(x,y)u(y,t)dy, \quad x \in \Gamma \\ \\ v(x,t) = \displaystyle\int_{\Omega} k(x,y)v(y,t)dy, \quad x \in \Gamma \\ \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{4}$$

where Ω is a regular and bounded domain of $\mathbb{R}^n$ with boundary Γ, T is a fixed real number $(T > 0)$ and $\rho = p-2$. Where $(u,v)(x,t)$ are the solution of this system.

In this study, we need the following assumptions:

(H_1) $n \geq 2,\ p > n,\ r > \frac{n}{2}+2.$

(H_2) $\frac{1}{p}+\frac{1}{q} = 1.$

(H_3) $f = (f_1, f_2) \in L^q(0,T,L^q(\Omega))^2$ and u_0, v_0 are in $L^\infty(\Omega)$.

(H_4) For any $x \in \Gamma, K(x) < \infty$ and $K_i(x) < \infty$.

(H_5) $\sum_{i=1}^{n} \int_\Gamma K(x)^{p-1} K_i(x) d\Gamma < 1 - \frac{1}{p}$.

The paper, is organized as follows:

We began with an introduction in section 1, then we state the art method for *PDEs* with the definition of the generalized solution in section 2, then we demonstrate the construction of an approximate solution and we derive a priori estimates for the approximate solution. Finally, we show the convergence of the approximate solution.

Before stating our result, we need to introduce some notations.

Notations. We are ready to present some notations that will be used throughout this paper. First, let

$(.,.)$: usual inner product in $L^2(\Omega)$;

$W^{k,p}(\Omega)$: Sobolev space on Ω, $H^r(\Omega)$: Sobolev space $W^{r,2}(\Omega)$;

$H^{-r}(\Omega)$: dual space of $H^r(\Omega)$, $|.|_{H^{-r}(\Omega)}$: norm in $H^{-r}(\Omega)$;

$K(x)$: norm of $k(x,y)$ in $L^q(\Omega)$ with respect to y, i.e., $K(x) = \left(\int_\Omega |k(x,y)|^q dy \right)^{1/q}$;

$K_i(x)$: norm of $D_i k(x,y)$ in $L^q(\Omega)$ with respect to y, i.e., $K_i(x) = \left(\int_\Omega |\frac{\partial k(x,y)}{\partial x_i}|^q dy \right)^{1/q}$.

Now, we define the space:

$$\mathrm{W} = L^2(\Omega) \times L^2(\Omega).$$

which is a Banach space endowed with the norm

$$|(u,v)|^2_{\mathrm{W}} = |u|^2_{L^2(\Omega)} + |v|^2_{L^2(\Omega)},$$

and let $\mathrm{V} = L^p(\Omega) \times L^p(\Omega)$, $\mathrm{Y} = L^q(\Omega) \times L^q(\Omega)$. In the sequel, $|\cdot|_{L^q(\Omega)}$, $|\cdot|_{L^2(\Omega)}$, $|\cdot|_{L^p(\Omega)}$ and $|\cdot|_{L^p(\Gamma)}$ will denote the usual norms of $L^q(\Omega)$, $L^2(\Omega)$, $L^p(\Omega)$ and $L^p(\Gamma)$, respectively.

2. Main Results

In this section, we discuss the notation of a generalized solutions, and we present the approximate solutions and a priori estimates, then we prove that the approximate solution converge to the solution of the problem. First, let we clearly state our definition to a generalized solution of the problem (4), we define the following space:

$\mathrm{U} = \{\forall(\psi,\varphi) \in (H^r(\Omega))^2,\ \psi(x) = \int_\Omega k(x,y)\psi(y)dy$ and $\varphi(x) = \int_\Omega k(x,y)\varphi(y)dy$, for $x \in \Gamma\}$.

Definition 1. *Let* (u,v) *the generalized solution of the problem* (4) *if:*

(i) $(u,v) \in (L^\infty(0,T,L^2(\Omega)))^2 \cap (C(0,T,H^{-r}(\Omega)))^2$,

(ii) $(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}) \in (L^q(0,T,H^{-r}(\Omega)))^2$,

(iii) $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$,

(iv) for all $(\psi,\varphi) \in U$ *and a.e.* $t \in [0,T]$,

$$\int_\Omega \frac{\partial u}{\partial t}\psi dx + \int_\Omega \frac{\partial v}{\partial t}\varphi dx - \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|u|^{p-2}\frac{\partial u}{\partial x_i}\big)\psi dx$$

$$- \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|v|^{p-2}\frac{\partial v}{\partial x_i}\big)\varphi dx + \int_\Omega |v|^{p-2}v\psi dx \tag{5}$$

$$+ \int_\Omega |u|^{p-2}u\varphi dx = \int_\Omega f_1(x,t)\psi dx + \int_\Omega f_2(x,t)\varphi dx,$$

2.1. The approximate solution and a priori estimates. It is easy to see that U is a subspace of $(H^r(\Omega))^2$, which is separable, then we can choose a countable set of distinct basis elements $(w_j,\widetilde{w}_j)$, where $j = 1,2,\ldots$, which generate U and are orthonormal in W. Let U_m be the subspace of U generated by the first m elements, we constuct the approximate solution of the problem in the following form:

$$\begin{cases} u_m(x,t) = \sum_{j=1}^m g_{jm}(t)w_j(t), \quad (x,t) \in \Omega\times[0,T], \\ \\ v_m(x,t) = \sum_{j=1}^m h_{jm}(t)\widetilde{w}_j(t), \quad (x,t) \in \Omega\times[0,T], \end{cases} \tag{6}$$

where $\big((g_{jm}(t))_{j=1}^m,(h_{jm}(t))_{j=1}^m\big)$ remains to be determined.

Let the orthogonal projection of (u_0,v_0) on U_m given by $(u_m^0,v_m^0) = (P\mathrm{U}_m u_0, P\mathrm{U}_m v_0)$, then $(u_m^0,v_m^0) \longrightarrow (u_0,v_0)$, as $m \longrightarrow \infty$ in U. Let $\big((g_{jm}^0)_{j=1}^m,(h_{jm}^0)_{j=1}^m\big)$ be the coordinate of (u_m^0,v_m^0) in the basis $\big((w_j)_{j=1}^m,(\widetilde{w}_j(t))_{j=1}^m\big)$ of U_m, such as

$$\begin{cases} u_m^0 = \sum_{j=1}^m g_{jm}^0 w_j(t), \\ \\ v_m^0 = \sum_{j=1}^m h_{jm}^0 \widetilde{w}_j(t), \end{cases}$$

where

$$\begin{cases} g_{jm}^0 = g_{jm}(0), \\ \\ h_{jm}^0 = h_{jm}(0), \end{cases}$$

we have to determine $\big((g_jm(t))_{j=1}^m, (h_jm(t))_{j=1}^m\big)$ to satisfy

(7)
$$\begin{aligned} &\int_\Omega \frac{\partial u_m}{\partial t} w_j dx - \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big) w_j dx \\ + \quad &\textstyle\int_\Omega |v_m|^{p-2} v_m w_j dx = \int_\Omega f_1(x,t) w_j dx, \quad 1 \le j \le m, \\ &\int_\Omega \frac{\partial v_m}{\partial t} \widetilde{w}_j dx - \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big) \widetilde{w}_j dx \\ + \quad &\textstyle\int_\Omega |u_m|^{p-2} u_m \widetilde{w}_j dx = \int_\Omega f_2(x,t) \widetilde{w}_j dx, \quad 1 \le j \le m. \end{aligned}$$

The system (7) is a system of ordinary differential equations in $g_{jm}(t)$ respectively $h_{jm}(t)$, by Caratheodory theorem [8]. there exists solution $(g_{jm}(t), h_{jm}(t))_{j=1}^m$, $t \in [0, t_m)$.

Multiply both sides of (7) by $g_{jm}(t)$ and $h_{jm}(t)$ respectively, then sum over j from 0 to m, getting:

(8)
$$\begin{aligned} &\int_\Omega \frac{\partial u_m}{\partial t} u_m dx + \int_\Omega \frac{\partial v_m}{\partial t} v_m dx - \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big) u_m dx \\ &\textstyle - \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big) v_m dx + \int_\Omega |v_m|^{p-2} v_m u_m dx \\ &\textstyle + \int_\Omega |u_m|^{p-2} u_m v_m dx = \int_\Omega f_1(x,t) u_m dx + \int_\Omega f_2(x,t) v_m dx, \quad 1 \le j \le m. \end{aligned}$$

Integrating by parts (8), we have:

(9)
$$\begin{aligned} &\int_\Omega \frac{\partial u_m}{\partial t} u_m dx + \int_\Omega \frac{\partial v_m}{\partial t} v_m dx + \sum_{i=1}^n \int_\Omega \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\frac{\partial u_m}{\partial x_i} dx \\ &\textstyle + \sum_{i=1}^n \int_\Omega \big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\frac{\partial v_m}{\partial x_i} dx + \int_\Omega |v_m|^{p-2} v_m u_m dx \\ &\textstyle + \int_\Omega |u_m|^{p-2} u_m v_m dx = \int_\Omega f_1(x,t) u_m dx + \int_\Omega f_2(x,t) v_m dx \\ &\textstyle + \sum_{i=1}^n \int_\Gamma \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big) u_m d\Gamma + \sum_{i=1}^n \int_\Gamma \big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big) v_m d\Gamma. \end{aligned}$$

Integrating with respect to t from 0 to T on the both sides of (9), we have:

$$
\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u_m}{\partial t}u_m dxdt+\int_0^T\int_\Omega \frac{\partial v_m}{\partial t}v_m dxdt+\int_0^T\sum_{i=1}^n\int_\Omega \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\frac{\partial u_m}{\partial x_i}dxdt\\
&+\int_0^T\sum_{i=1}^n\int_\Omega \big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\frac{\partial v_m}{\partial x_i}dxdt+\int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt\\
&+\int_0^T\int_\Omega |u_m|^{p-2}u_m v_m dxdt=\int_0^T\int_\Omega f_1(x,t)u_m dxdt+\int_0^T\int_\Omega f_2(x,t)v_m dxdt\\
&+\int_0^T\sum_{i=1}^n\int_\Gamma \big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)u_m d\Gamma dt+\int_0^T\sum_{i=1}^n\int_\Gamma \big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)v_m d\Gamma dt,
\end{aligned}
\tag{10}
$$

then

$$
\begin{aligned}
&\frac{1}{2}\Big[|u_m(T)|^2_{L^2(\Omega)}+|v_m(T)|^2_{L^2(\Omega)}\Big]+\frac{4}{p^2}\Big[\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|u_m|^{\frac{p-2}{2}}u_m)\big)^2dxdt\\
+\;&\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|v_m|^{\frac{p-2}{2}}v_m)\big)^2dxdt\Big]+\int_0^T\int_\Omega |u_m|^{p-2}u_m v_m dxdt\\
+\;&\int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt=\int_0^T\int_\Omega f_1(x,t)u_m dxdt+\int_0^T\int_\Omega f_2(x,t)v_m dxdt\\
+\;&\int_0^T\int_\Gamma\sum_{i=1}^n\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)u_m d\Gamma dt+\int_0^T\int_\Gamma\sum_{i=1}^n\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)v_m d\Gamma dt\\
+\;&\frac{1}{2}\Big[|u_m(0)|^2_{L^2(\Omega)}+|v_m(0)|^2_{L^2(\Omega)}\Big],
\end{aligned}
$$

this gives

$$
\begin{aligned}
&|(u_m(T),v_m(T)|^2_{\mathrm{W}}+\int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt+\int_0^T\int_\Omega |u_m|^{p-2}u_m v_m dxdt\\
+\;&\frac{4}{p^2}\Big[\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|u_m|^{\frac{p-2}{2}}u_m)\big)^2dxdt+\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|v_m|^{\frac{p-2}{2}}v_m)\big)^2dxdt\Big]\\
\le\;&\int_0^T\int_\Omega f(x,t)(u_m,v_m)dxdt+\int_0^T\int_\Gamma\sum_{i=1}^n\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)u_m d\Gamma dt\\
+\;&\int_0^T\int_\Gamma\sum_{i=1}^n\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)v_m d\Gamma dt+|(u_m(0),v_m(0))|^2_{\mathrm{W}}.
\end{aligned}
\tag{11}
$$

We derive some a priori estimates for the approximate solution as follows:

The first term in the right-hand side of (11) can be estimated as follows:

$$
\begin{aligned}
\Big|\int_0^T\int_\Omega (f_1(x,t),f_2(x,t))(u_m,v_m)dxdt\Big| &\le \int_0^T\int_\Omega |f||(u_m,v_m)|dxdt\\
&\le \int_0^T\Big[\big(\int_\Omega |f|^q dx\big)^{1/q}\big(\int_\Omega |(u_m,v_m)|^p dx\big)^{1/p}\Big]dt\\
&\le \int_0^T |f|_{\mathrm{Y}}|(u_m,v_m)|_{\mathrm{V}}dt\\
&\le \frac{1}{p}\int_0^T |(u_m,v_m)|^p_{\mathrm{V}}dt+\frac{p-1}{p}\int_0^T |f|^{\frac{p}{p-1}}_{\mathrm{Y}}dt,
\end{aligned}
$$

hence

$$|\int_0^T\int_\Omega (f_1(x,t),f_2(x,t))(u_m,v_m)dxdt| \leq \frac{1}{p}\int_0^T |(u_m,v_m)|_{\mathrm{V}}^p dt + \frac{p-1}{p}\int_0^T |f|_{\mathrm{Y}}^{\frac{p}{p-1}}dt. \tag{12}$$

For the second and the third terms in the right-hand side of (11), we have for $x\in\Gamma$:

$$\begin{cases} |u_m(x,t)| \leq K(x)|u_m|_p, \\ \\ |v_m(x,t)| \leq K(x)|v_m|_p, \end{cases}$$

and

$$\begin{cases} |\frac{\partial}{\partial x_i}u_m(x,t)| \leq K_i(x)|u_m|_p, \\ \\ |\frac{\partial}{\partial x_i}v_m(x,t)| \leq K_i(x)|v_m|_p. \end{cases}$$

Using Hölder's inequality and the assumptions $(H_4)$and (H_5), we have:

$$\begin{aligned} |\int_0^T\sum_{i=1}^n\int_\Gamma \left(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\right)u_m d\Gamma dt| &\leq \int_0^T\sum_{i=1}^n\int_\Gamma |u_m|^{p-1}|\frac{\partial u_m}{\partial x_i}|d\Gamma dt \\ &\leq \int_0^T\left[\sum_{i=1}^n\int_\Gamma K(x)^{p-1}K_i(x)d\Gamma\right]|u_m|_{L^p(\Omega)}^p dt, \end{aligned}$$

hence

(13)
$$|\int_0^T\sum_{i=1}^n\int_\Gamma \left(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\right)u_m d\Gamma dt + \int_0^T\sum_{i=1}^n\int_\Gamma \left(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\right)v_m d\Gamma dt| \leq c\int_0^T |(u_m,v_m)|_V^p dt,$$

where $c = 2\sum_{i=1}^n\int_\Gamma K(x)^{p-1}K_i(x)d\Gamma < 2(1-\frac{1}{p})$

For the second and the third terms in the left-hand side of (11), we have:

$$\begin{aligned} |\int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt| &\leq \int_0^T\int_\Omega |v_m|^{p-1}|u_m|dxdt \\ &\leq \int_0^T\left[\left(\int_\Omega |v_m^{p-1}|^q dx\right)^{\frac{1}{q}}\int_\Omega |u_m|^p dx\right)^{\frac{1}{p}}\right]dt \\ &\leq \int_0^T |v_m|_{L^p(\Omega)}^{\frac{p}{q}}|u_m|_{L^p(\Omega)}dt, \\ &\leq \frac{p-1}{p}\int_0^T |v_m|_{L^p(\Omega)}^p dt + \frac{1}{p}\int_0^T |u_m|_{L^p(\Omega)}^p dt, \end{aligned}$$

hence

$$|\int_0^T\int_\Omega |v_m|^{p-2}v_m u_m dxdt+\int_0^T\int_\Omega |u_m|^{p-2}u_m v_m dxdt|\le 2\int_0^T |(u_m,v_m)|_{\mathrm{V}}^p dt. \tag{14}$$

From the form (12), (13) and (14), we have:

$$\begin{aligned} &|(u_m(T),v_m(T)|_{\mathrm{W}}^2+\frac{4}{p^2}\Big[\int_0^T\int_\Omega\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|u_m|^{\frac{p-2}{2}}u_m)\big)^2dxdt \\ + &\int_0^T\int_\Omega\textstyle\sum_{i=1}^n\big(\frac{\partial}{\partial x_i}(|v_m|^{\frac{p-2}{2}}v_m)\big)^2dxdt\Big]+C\int_0^T|(u_m,v_m)|_{\mathrm{V}}^p dt \\ \le &\tfrac{p-1}{p}\int_0^T|f|_{\mathrm{Y}}^{\frac{p}{p-1}}dt+|(u_m(0),v_m(0))|_{\mathrm{W}}^2, \end{aligned} \tag{15}$$

where $C=2+c+\frac{1}{p}$.

From the above estimates (12), (13) and (14) and for any finite $T>0$, we have the following a priori estimates:

(i) (u_m,v_m) are bounded in $(L^\infty(0,T,L^2(\Omega)))^2$.

(ii) $|u_m|^{\frac{p-2}{2}}u_m$ and $|v_m|^{\frac{p-2}{2}}v_m$ are bounded in $L^2(0,T,H^1(\Omega))$.

(iii) (u_m,v_m) are bounded in $(L^p(0,T,L^p(\Omega)))^2$.

Since T is an artbitrary positive number, we have

$$|(u_m,v_m)|_{\mathrm{V}}^p<\infty,\quad a.e.t \tag{16}$$

2.2. The existence of a generalized solution. The main focus of this section is to prove existence of a generalized solution. First, we have to prove the following lemma.

Lemma 1. *Let* (u_m,v_m) *be the approximate solution of the problem* (4) *in the sense of the definition*(1)*, then* $(\frac{\partial u_m}{\partial t},\frac{\partial v_m}{\partial t})$ *are bounded in* $(L^q(0,T,H^{-r}(\Omega)))^2$.

Proof. Let U is a subset of $H^r(\Omega)$ and for $(\psi,\varphi)\in$ U satisfy:

$$\begin{aligned} &\int_\Omega\frac{\partial u_m}{\partial t}\psi dx+\int_\Omega\frac{\partial v_m}{\partial t}\varphi dx+\sum_{i=1}^n\int_\Omega\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\frac{\partial\psi}{\partial x_i}dx \\ &+\textstyle\sum_{i=1}^n\int_\Omega\big(|v_m|^{p-2}\frac{\partial\varphi}{\partial x_i}\big)\frac{\partial v_m}{\partial x_i}dx+\int_\Omega|v_m|^{p-2}v_m\psi dx \\ &+\int_\Omega|u_m|^{p-2}u_m\varphi dx=\int_\Omega f_1(x,t)\psi dx+\int_\Omega f_2(x,t)\varphi dx \\ &+\textstyle\sum_{i=1}^n\int_\Gamma\big(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\big)\psi d\Gamma+\sum_{i=1}^n\int_\Gamma\big(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\big)\varphi d\Gamma. \end{aligned} \tag{17}$$

Using Hölder inequality to estimate the two terms in the left-hand side in the last of (17), this give:

$$\begin{aligned}
|\int_{\Omega} |v_m|^{p-2} v_m \varphi dx + \int_{\Omega} |u_m|^{p-2} u_m \varphi dx| &\leq |v_m|^{\frac{p}{q}}_{L^p(\Omega)} |\psi|_{L^p(\Omega)} + |u_m|^{\frac{p}{q}}_{L^p(\Omega)} |\varphi|_{L^p(\Omega)} \\
&\leq 2|(u_m, v_m)|^{\frac{p}{q}}_{\mathrm{V}} |(\psi, \varphi)|_{\mathrm{V}}.
\end{aligned}$$

Since $H^r(\Omega) \hookrightarrow L^p(\Omega)$, then

$$|\int_{\Omega} |v_m|^{p-2} v_m \varphi dx + \int_{\Omega} |u_m|^{p-2} u_m \varphi dx| \leq c_1 |(u_m, v_m)|^{\frac{p}{q}}_{\mathrm{V}} |(\psi, \varphi)|_{(H^r(\Omega))^2}, \quad c_1 > 0.$$

Then

$$||v_m|^{p-2} v_m|_{H^{-r}(\Omega)} + ||u_m|^{p-2} u_m|_{H^{-r}(\Omega)} \leq c_1 |(u_m, v_m)|^{\frac{p}{q}}_{\mathrm{V}}, \quad c_1 > 0.$$

The norm of $|v_m|^{p-2} v_m$ and $|u_m|^{p-2} u_m$ in $L^q(0, T, H^{-r}(\Omega))$ are bounded by:

$$\left(\int_0^T (c_1 (|(u_m, v_m)|^{p}_{\mathrm{V}})^{1/q})^q dt \right)^{\frac{1}{q}} = \left(\int_0^T c_1^q |(u_m, v_m)|^{p}_{\mathrm{V}} dt \right)^{\frac{1}{q}} < \infty, \tag{18}$$

Therefore, $|u_m|^{p-2} u_m$ and $|v_m|^{p-2} v_m$ are bounded in $L^q(0, T, H^{-r}(\Omega))$.

Next, we estimate the last two terms in the right-hand side of (17) by:

$$\big(a(u_m, v_m), (\psi, \varphi)\big) = \int_{\Gamma} \sum_{i=1}^{n} \big(|u_m|^{p-2} \frac{\partial u_m}{\partial x_i}\big) \psi d\Gamma + \int_{\Gamma} \sum_{i=1}^{n} \big(|v_m|^{p-2} \frac{\partial v_m}{\partial x_i}\big) \varphi d\Gamma,$$

$$\begin{aligned}
|\big(a(u_m, v_m), (\psi, \varphi)\big)| &= |\int_{\Gamma} \sum_{i=1}^{n} \big(|u_m|^{p-2} \frac{\partial u_m}{\partial x_i}\big) \psi d\Gamma + \int_{\Gamma} \sum_{i=1}^{n} \big(|v_m|^{p-2} \frac{\partial v_m}{\partial x_i}\big) \varphi d\Gamma| \\
&\leq \sum_{i=1}^{n} |(|u_m|^{p-2} \frac{\partial u_m}{\partial x_i})|_{L^q(\Gamma)} |\psi|_{L^p(\Gamma)} + \sum_{i=1}^{n} |(|v_m|^{p-2} \frac{\partial v_m}{\partial x_i})|_{L^q(\Gamma)} |\varphi|_{L^p(\Gamma)} \\
&\leq \sum_{i=1}^{n} |K(x)^{p-2} K_i(x)|_{L^q(\Gamma)} |K(x)|_{L^p(\Gamma)} |u_m|^{p-1}_{L^p(\Omega)} |\psi|_{L^p(\Omega)} \\
&+ \sum_{i=1}^{n} |K(x)^{p-2} K_i(x)|_{L^q(\Gamma)} |K(x)|_{L^p(\Gamma)} |v_m|^{p-1}_{L^p(\Omega)} |\varphi|_{L^p(\Omega)} \\
&\leq 2 \sum_{i=1}^{n} |K(x)^{p-2} K_i(x)|_{L^q(\Gamma)} |K(x)|_{L^p(\Gamma)} |(u_m, v_m)|^{p-1}_{\mathrm{V}} |(\psi, \varphi)|_{\mathrm{V}} \\
&\leq 2 \sum_{i=1}^{n} |K(x)^{p-2} K_i(x)|_{L^q(\Gamma)} |K(x)|_{L^p(\Gamma)} |(u_m, v_m)|^{p-1}_{\mathrm{V}} c_1 |(\psi, \varphi)|_{(H^r(\Omega))^2}.
\end{aligned}$$

Hence

$$|a(u_m,v_m)|_{(H^{-r}(\Omega))^2} \leq 2\sum_{i=1}^{n} |K(x)^{p-2}K_i(x)|_{L^q(\Gamma)}|K(x)|_{L^p(\Gamma)}|(u_m,v_m)|_{\mathrm{V}}^{p-1}c_1 < \infty.$$

The norm of $a(u_m,v_m)$ in $(L^q(0,T,H^{-r}(\Omega)))^2$ is bounded by

$$\left(2\int_0^T \sum_{i=1}^{n} \left(|K(x)^{p-2}K_i(x)|_{L^q(\Gamma)}|K(x)|_{L^p(\Gamma)}c_1\right)^q |(u_m,v_m)|_{\mathrm{V}}^{p} dt\right)^{\frac{1}{q}} < \infty, \tag{19}$$

Therefore, $a((u_m,v_m))$is bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$.

Next, we consider the third and the fourth terms in the left-hand side of (17) and by integrating by parts, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\Omega}\left(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}\right)\frac{\partial \psi}{\partial x_i}dx + \sum_{i=1}^{n}\int_{\Omega}\left(|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i}dx \\
&= \tfrac{1}{p-1}\Bigg(\textstyle\sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p-2}u_m\frac{\partial \psi}{\partial x_i}d\Gamma - \int_{\Omega}|u_m|^{p-2}u_m\Delta\psi dx \\
&+ \textstyle\sum_{i=1}^{n}\int_{\Gamma}|v_m|^{p-2}v_m\frac{\partial \varphi}{\partial x_i}d\Gamma - \int_{\Omega}|v_m|^{p-2}v_m\Delta\varphi dx\Bigg).
\end{aligned} \tag{20}$$

Let

$$\big(b(u_m,v_m),(\psi,\varphi)\big) = \sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p-2}u_m\frac{\partial \psi}{\partial x_i}d\Gamma + \sum_{i=1}^{n}\int_{\Gamma}|v_m|^{p-2}v_m\frac{\partial \varphi}{\partial x_i}d\Gamma,$$

$$\begin{aligned}
|\big(b(u_m,v_m),(\psi,\varphi)\big)| &= |\sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p-2}u_m\frac{\partial \psi}{\partial x_i}d\Gamma + \sum_{i=1}^{n}\int_{\Gamma}|v_m|^{p-2}v_m\frac{\partial \varphi}{\partial x_i}d\Gamma| \\
&\leq 2\sum_{i=1}^{n}|K(x)^{p-1}K_i(x)|_{L^q(\Gamma)}|(u_m,v_m)|_{\mathrm{V}}^{p-1}|(\psi,\varphi)|_{\mathrm{V}} \\
&\leq \sum_{i=1}^{n}|K(x)^{p-1}K_i(x)|_{L^q(\Gamma)}|(u_m,v_m)|_{\mathrm{V}}^{p-1}c_1|(\psi,\varphi)|_{(H^r(\Omega))^2}.
\end{aligned}$$

Then

$$|b(u_m,v_m)|_{(H^{-r}(\Omega))^2} \leq 2\sum_{i=1}^{n}|K(x)^{p-1}|_{L^q(\Gamma)}|K_i(x)|_{L^p(\Gamma)}|(u_m,v_m)|_{\mathrm{V}}^{p-1}c_1 < \infty.$$

The norm of $b(u_m,v_m)$ in $(L^q(0,T,H^{-r}(\Omega)))^2$ is bounded by

$$\left(2\int_0^T \sum_{i=1}^{n}\left(|K(x)^{p-1}|_{L^q(\Gamma)}|K_i(x)|_{L^p(\Gamma)}c_1\right)^q|(u_m,v_m)|_{\mathrm{V}}^{p}dt\right)^{\frac{1}{q}} < \infty. \tag{21}$$

Hence $b(u_m,v_m)$is bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$.

Next, we consider

$$\big(\tilde{b}(u_m,v_m),(\psi,\varphi)\big)=\int_\Omega |u_m|^{p-2}u_m\Delta\psi dx+\int_\Omega |v_m|^{p-2}v_m\Delta\varphi dx,$$

$$\begin{aligned}|\big(\tilde{b}(u_m,v_m),(\psi,\varphi)\big) &= |\int_\Omega |u_m|^{p-2}u_m\Delta\psi dx+\int_\Omega |v_m|^{p-2}v_m\Delta\varphi dx|\\ &\leq \int_\Omega |u_m|^{p-1}|\Delta\psi|dx+\int_\Omega |v_m|^{p-1}|\Delta\varphi|dx\\ &\leq |u_m|^{p-1}_{L^p(\Omega)}|\Delta\psi|_{L^p(\Omega)}+|v_m|^{p-1}_{L^p(\Omega)}|\Delta\varphi|_{L^p(\Omega)}\\ &\leq |(u_m,v_m)|^{\frac{p}{q}}_{\mathrm{V}}|(\Delta\psi,\Delta\varphi)|_{\mathrm{V}};\end{aligned}$$

From the proof of [[26], Theorem 12.2 p 140] we have

$$|\big(\tilde{b}(u_m,v_m),(\psi,\varphi)\big)|\leq c_1|(u_m,v_m)|^{\frac{p}{q}}_{\mathrm{V}}|(\psi,\varphi)|_{(H^r(\Omega))^2}<\infty. \tag{22}$$

Therefore

$$\left(\int_0^T |\tilde{b}(u_m,v_m)|^q_{(H^{-r}(\Omega))^2}dt\right)^{\frac{1}{q}}\leq\left(\int_0^T c_1^q|(u_m,v_m)|^p_{\mathrm{V}}dt\right)^{\frac{1}{q}}<\infty. \tag{23}$$

So, we have $\tilde{b}(u_m,v_m)$ is bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$.

Finally, we consider

$$\begin{aligned}|\int_\Omega (f_1(x,t),f_2(x,t))(\psi,\varphi)dx| &= \int_\Omega |f(x,t)||(\psi,\varphi)|dx\\ &\leq |f|_{\mathrm{Y}}|(\psi,\varphi)|_{\mathrm{V}}\\ &\leq |f|_{\mathrm{Y}}c_1|(\psi,\varphi)|_{H^r(\Omega)}.\end{aligned}$$

Then

$$|f|_{(H^{-r}(\Omega))^2}\leq c_1|f|_{\mathrm{Y}}<\infty. \tag{24}$$

The norm of f in $(L^q(0,T,H^{-r}(\Omega)))^2$ is bounded by

$$\left(\int_0^T |f|^q_{(H^{-r}(\Omega))^2}dt\right)^{\frac{1}{q}}\leq\left(\int_0^T c_1^q|f|^q_{\mathrm{Y}}dt\right)^{\frac{1}{q}}<\infty. \tag{25}$$

Then, we have that f is bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$.

From the result (18), (19), (21), (23) and (25), we have $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$ are bounded in $L^q(0,T,H^{-r}(\Omega))^2$ □

2.3. **The convergence.** From the lemmas 4.3 to lemma 4.7 of [9], we quote the following lemmas.

Lemma 2. *Let* (u_m, v_m) *be the approximate solution of* (4)*, constructed as in* (6)*, then*

$$(u_m, v_m) \to (u, v),$$

in $(L^p(0,T,L^p(\Omega)))^2$ *strongly and almost everywhere.*

Proof. We quote the Theorem here from the Lemma(1) and the use of [[26], Theorem 12.1] to prove the previous Lemma.

Theorem 1. *Let B,* B_1 *be the Banach spaces, and S be a set, we define:*

$$M(\varphi_1, \varphi_2) = \left(\sum_{i=1}^{n} \int_\Omega |\varphi_1|^{p-2} (\frac{\partial \varphi_1}{\partial x_i})^2 dx\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \int_\Omega |\varphi_2|^{p-2} (\frac{\partial \varphi_2}{\partial x_i})^2 dx\right)^{\frac{1}{p}},$$

On S with:

$(a)-$ $S \subset B \subset B_1$*, and* $M(\varphi_k) \geq 0$ *on S,* $M(\lambda_k \varphi_k) = |\lambda_k| M(\varphi_k)$*, where* $k = 1, 2$*.*

$(b)-$ *The set* $\{\varphi_k | \varphi_k \in S, M(\varphi_k) \leq 1\}$ *is relatively compact in B.*

Define the set $F = \{\varphi_k : \varphi_k$ *are locally summable on* $[0,T]$ *with value in* B_1*,* $\int_0^T \big(M(\varphi_k(t))\big)^{p_0} dt \leq C, \varphi_k'$ *bounded in* $(L^{p_1}(0,T,B_1))^2\}$*, where* $1 < p_j < \infty$*,* $j = 0, 1$ *and* $k = 1, 2$*. Then* $F \subset L^{p_0}(0,T,B)$ *and F is relatively compact in* $L^{p_0}(0,T,B)$*.*

We define the set S as follows:

$S = \{\varphi_k : |\varphi_k|^{\frac{p-2}{2}} \varphi_k \in (H^1(\Omega))^2$, where $k = 1, 2\}$.

$H^1(\Omega)$ is also compactly embedded in $L^2(\Omega)$, with the proof of [Proposition 12.1 p 143, [26]], we have (b) in the Theorem(1).

From the Lemma(1), $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$ are bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$, then we have to prove now that

$$\int_0^T \big(M(u_m(t), v_m(t)\big)^{p_0} dt \leq c, \quad c > 0.$$

Let $B=L^p(\Omega)$, $B_1=H^{-r}(\Omega)$, $p_0=p$ and $p_1=q$ then

$$\int_0^T \big(M(u_m(t),v_m(t))\big)^{p_0}dt=\frac{4}{p^2}\Bigg[\int_0^T\sum_{i=1}^n\int_\Omega\left(\frac{\partial}{\partial x_i}(|u_m|^{\frac{p-2}{2}}u_m)\right)^2dxdt$$
$$+\quad \textstyle\sum_{i=1}^n\int_\Omega\left(\frac{\partial}{\partial x_i}(|v_m|^{\frac{p-2}{2}}v_m)\right)^2dxdt\Bigg].$$

Let $|u_m|^{\frac{p-2}{2}}u_m$ and $|v_m|^{\frac{p-2}{2}}v_m$ are bounded in $L^2(0,T,H^1(\Omega))$, so

$$\int_0^T \big(M(u_m(t),v_m(t))\big)^{p_0}dt<\infty.$$

Therefore, conclusion follows easily from application of Theorem(1), $F\subset L^p(0,T,L^p(\Omega))$ and F is relatively compact in $L^p(0,T,L^p(\Omega))$. □

Next, we prove that we can pass to the limit on (17).

Lemma 3. *Let* (u_m,v_m) *be the approximate solution of* (4)*, constructed as in* (6)*, then*

$$(|u_m|^{p-2}u_m+|v_m|^{p-2}v_m,\varphi)\to(|u|^{p-2}u+|v|^{p-2}v,\varphi),\quad m\to\infty.$$

Proof. To prove this Lemma, we need to show that $|u_m|^{p-2}u_m\rightharpoonup|u|^{p-2}u$ and $|v_m|^{p-2}v_m\rightharpoonup|v|^{p-2}v$ weakly in $L^q(\Omega)$, this is a consequence of [[26], Lemma 1.3].

First, let prove that.

$$\begin{aligned}||u_m|^{p-2}u_m|_{L^q(\Omega)} &\le \Big(\int_\Omega(|u_m|^{p-1})^qdx\Big)^{\frac{1}{q}}\\ &\le \Big(\int_\Omega|u_m|^pdx\Big)^{\frac{1}{q}}\\ &\le \big(|u_m|^p_{L^p(\Omega)}\big)^{\frac{1}{q}}.\end{aligned}$$

Then, from the Lemma(2) we have $|u_m|^{p-2}u_m\to|u_m|^{p-2}u_m$ almost everywhere, for $x\in\Omega$. Similarly for $|v_m|^{p-2}v_m$.

□

Lemma 4. *Let* (u_m,v_m) *be the approximate solution of* (4)*, constructed as in* (6)*, then*

$$\int_\Gamma\left(|u_m|^{p-2}\frac{\partial u_m}{\partial x_i}+|v_m|^{p-2}\frac{\partial v_m}{\partial x_i}\right)\varphi d\Gamma\to\int_\Gamma\left(|u|^{p-2}\frac{\partial u}{\partial x_i}+|v|^{p-2}\frac{\partial v}{\partial x_i}\right)\varphi d\Gamma,\quad m\to\infty.$$

Proof. From the result, (u_m, v_m) are bounded $(L^p(\Omega))^2$, for almost every where t, then there exists a subsequence of (u_m, v_m) still denote (u_m, v_m) converge in $(L^p(\Omega))^2$, for almost every where t.

From the assumptions, $K(x) = \big(\int_\Omega |k(x,t)|^q dx\big)^{\frac{1}{q}} < \infty$, for fixed $x \in \Gamma$, where

$$\int_\Omega k(x,y)u_m(y,t)dy \to \int_\Omega k(x,y)u(y,t)dt, \quad m \to \infty,$$

$$\int_\Omega \frac{\partial k(x,y)}{\partial x_i} u_m(y,t)dy \to \int_\Omega \frac{\partial k(x,y)}{\partial x_i} u(y,t)dt, \quad m \to \infty.$$

similarly for $v_m(x,t)$ for fixed $x \in \Gamma$.

First we prove that $\int_\Gamma |u_m|^{p-2} \frac{\partial u_m}{\partial x_i} d\Gamma \rightharpoonup \int_\Gamma |u|^{p-2} \frac{\partial u}{\partial x_i}$ weakly in $L^q(\Gamma)$.

Since, $K(x) \in L^p(\Gamma)$,we have $|u_m|_{L^p(\Gamma)} < \infty$. Similarly, we have $|(\frac{\partial u_m}{\partial x_i}|_{L^q(\Gamma)} < \infty$ then

$$\begin{aligned} ||u_m|^{p-2} \frac{\partial u_m}{\partial x_i}|_{L^q(\Gamma)} &= ||u_m|^{p-2} \frac{\partial u_m}{\partial x_i}|_{L^{\frac{p}{p-2}+p}(\Gamma)} \\ &\leq ||u_m|^{p-2}|_{L^{\frac{p}{p-2}}(\Gamma)} |\frac{\partial u_m}{\partial x_i}|_{L^p(\Gamma)} \\ &\leq |u_m|^{p-2}_{L^p(\Gamma)} |\frac{\partial u_m}{\partial x_i}|_{L^p(\Gamma)} < \infty. \end{aligned}$$

According to the lemma 1.3 [26], $\int_\Gamma |u_m|^{p-2} \frac{\partial u_m}{\partial x_i} d\Gamma \rightharpoonup \int_\Gamma |u|^{p-2} \frac{\partial u}{\partial x_i}$ weakly in $L^q(\Gamma)$, for $a.e, t \in [0,T]$, since $|\varphi|_{L^p(\Gamma)}$. Similarly, we have $\int_\Gamma |v_m|^{p-2} \frac{\partial v_m}{\partial x_i} d\Gamma \rightharpoonup \int_\Gamma |v|^{p-2} \frac{\partial v}{\partial x_i}$ weakly in $L^q(\Gamma)$, for $a.e, t \in [0,T]$, since $|\varphi|_{L^p(\Gamma)}$.

□

Lemma 5. *Let* (u_m, v_m) *be the approximate solution of* (4)*, constructed as in* (6)*, then*

$$\int_\Omega \left(|u_m|^{p-2} \frac{\partial u_m}{\partial x_i} + |v_m|^{p-2} \frac{\partial v_m}{\partial x_i}\right)(\frac{\partial \varphi}{\partial x_i})dx \to \int_\Omega \left(|u|^{p-2} \frac{\partial u}{\partial x_i} + |v|^{p-2} \frac{\partial v}{\partial x_i}\right)(\frac{\partial \varphi}{\partial x_i})dx, \quad m \to \infty.$$

Proof. From (20), we have to prove

(i) $$\int_\Gamma \left(|u_m|^{p-2}u_m + |v_m|^{p-2}v_m\right)\frac{\partial \varphi}{\partial x_i} d\Gamma \to \int_\Gamma \left(|u|^{p-2}u + |v|^{p-2}v\right)\frac{\partial \varphi}{\partial x_i} d\Gamma, \quad m \to \infty.$$

(ii) $$\int_\Omega \left(|u_m|^{p-2}u_m + |v_m|^{p-2}v_m\right)\Delta\varphi dx \to \int_\Omega \left(|u|^{p-2}u + |v|^{p-2}v\right)\Delta\varphi dx, \quad m \to \infty.$$

(i) From the proof of the lemma (2), lemma (4) and for fixed $x \in \Gamma$,

$$|u_m(x,t)|^{p-2}u_m(x,t) \to |u(x,t)|^{p-2}u(x,t),$$

almost everywhere, and

$$||u_m(x,t)|^{p-2}u_m(x,t)|_{L^q(\Gamma)} \quad \leq \quad |u_m|^{p-1}_{L^p(\Gamma)} < \infty.$$

According to the consequence of lemma1.3 [26], we have $|u_m(x,t)|^{p-2}u_m(x,t) \rightharpoonup |u(x,t)|^{p-2}u(x,t)$, weakly in $L^q(\Gamma)$, since $\frac{\partial \varphi}{\partial x_i} \in L^q(\Gamma)$, and the same for $|v_m(x,t)|^{p-2}v_m(x,t) \rightharpoonup |v(x,t)|^{p-2}v(x,t)$, weakly in $L^q(\Gamma)$, since $\frac{\partial \varphi}{\partial x_i} \in L^q(\Gamma)$.

(ii) From the proof of lemma (2), we have the convergence almost everywhere, for $x \in \Omega$, then

$$||u_m(x,t)|^{p-2}u_m(x,t)|_{L^q(\Omega)} \quad \leq \quad |u_m|^{p-1}_{L^p(\Omega)} < \infty.$$

From the lemma1.3 [26], $|u_m(x,t)|^{p-2}u_m(x,t) \rightharpoonup |u(x,t)|^{p-2}u(x,t)$, weakly in $L^q(\Omega)$, since $\Delta\varphi \in L^q(\Omega)$.

Similarly, $|v_m(x,t)|^{p-2}v_m(x,t) \rightharpoonup |v(x,t)|^{p-2}v(x,t)$, weakly in $L^q(\Omega)$, since $\Delta\varphi \in L^q(\Omega)$.

□

Lemma 6. *Let* (u_m, v_m) *be the approximate solution of* (4)*, constructed as in* (6)*, then*

$$\Big((\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t}), (\psi, \varphi) \Big) \to \Big((\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}), (\psi, \varphi) \Big), \quad m \to \infty.$$

And $(u(t), v(t))$ *are continuous on* $[0,T]$.

Proof. From the previous result, we have that $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$ are bounded in $(L^q(0,T,H^{-r}(\Omega)))^2$, by Alaoglu's Theorem, there exists a subsequences, still denoted by $(\frac{\partial u_m}{\partial t}, \frac{\partial v_m}{\partial t})$, converging to $(\chi, \tilde{\chi})$ weak star in $L^q(0,T,H^{-r})$. Then by modifying the proof of [[4], Theorem 1] (with the space $L^q(0,T,H^{-r})$, instead of $L^2(0,T,B^{\frac{1}{2}}(0,1)))$. We have $(\chi, \tilde{\chi}) = (\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t})$ and $(u(t), v(t))$ are continuous on $[0,T]$. □

Acknowledgements

We are grateful to the anonymous referees for their comments.

Funding Information

There are no funders to report for this submission.

Conflict of Interests

The authors declare that there is no conflict of interests.

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