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2-INNER PRODUCT ON FUZZY LINEAR SPACES OVER FUZZY FIELDS

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Abstract: In this paper we have studied the concept of 2-inner product on fuzzy linear space over fuzzy field. We have also discussed some fundamental properties of 2-inner product and have given relationship between 2-norm and 2-inner product function.

Key wards: fuzzy field; fuzzy linear space; 2-norm; 2-inner product.

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1. INTRODUCTION

The fundamental concept of fuzzy set was introduced by L. A. Zadeh [15] in 1965 and fuzzy topology was introduced by C.L. Chang [2] in 1968. There after many researchers introduced the notions of fuzzy norm and fuzzy inner product from different point of view. In 1984 Katsaras [7] defined a fuzzy norm on a linear space and there after Wu and Fang [12] introduced a fuzzy normed space. R. Biswas [1] in 1991 defined fuzzy norm and fuzzy inner product of elements on a linear space. In 1992, Felbin [6] introduced fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space. Another important approach of fuzzy norm on a linear space was introduced in 1994, by Cheng and Mordeson [3], on a parallel line as the corresponding fuzzy metric is of Kramosil and Michelek [9] type. There after Krishna and Sarma

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[8], Xiao and zhu [14] discussed fuzzy norms on linear spaces at different points of aspects. All these researchers have done their work in the area of crisp linear space. Gu Wenxiang and Lu Tu [13] were the first to introduce the concept of fuzzy fields and fuzzy linear spaces over fuzzy fields. In 2011, C.P. Santhosh and T.V. Ramakrishnan [11] introduced norm on fuzzy linear space over fuzzy field. In 2018, Noori F. AL-Mayahi and Suadad M. Abbas [10] defined fuzzy normed algebra over fuzzy field. A satisfactory theory of 2-inner product space and n-inner product space has been effectively constructed by C.R. Diminnie, S. Gähler and A. White [4,5]. In the present paper we introduce the idea of 2-norm and 2-inner product on fuzzy linear spaces over fuzzy fields and have also discussed some properties of it.

2. PRELIMINARIES

This section contains some definitions and preliminary results which are used in the paper.

Definition2.1.[11] Let X be a field and F a fuzzy set in X with the following conditions

(i) $F(x+y) \geq \min\{F(x), F(y)\}, x, y \in X$

(ii) $F(xy) \geq \min\{F(x), F(y)\}, x, y \in X$.

Then we call F a fuzzy field in X and denote it by (F, X). (F, X) is called a fuzzy field of X.

(It should be noted that $F(-x) = F(x)$ and $F(x^{-1}) = F(x)$)

Theorem 2.2. If (F, X) is a fuzzy field of X, then

(i) $F(0) \geq F(x), x \in X$

(ii) $F(1) \geq F(x),\ x(\neq 0) \in X$

(iii) $F(0) \geq F(1)$

Definition 2.3.[11] Let X be a field and (F, X) be a fuzzy field of X. Let Y be a linear space over X and V a fuzzy set of Y. Suppose the following condition hold:

(i) $V(x+y) \geq \min\{V(x), V(y)\}, x, y \in Y$

(ii) $V(\lambda x) \geq \min\{F(\lambda), V(x)\}, \lambda \in X, x \in Y$

(iii) $F(1) \geq V(0)$

Then (V, Y) is called a fuzzy linear space over (F, X).

(It should be noted that $V(-\mathrm{x}) = V(\mathrm{x}),\ \mathrm{x} \in Y$)

Theorem 2.4.[13] If (V, Y) is a fuzzy linear space over (F, X), then

(i) $F(0) \geq V(0)$

(ii) $V(0) \geq V(x), x \in Y$

(iii) $F(0) \geq V(x), x \in Y$

Theorem 2.5.[13] Let (F, X) be a fuzzy field of X and Y a linear space over X. Let V be a fuzzy set of Y. Then (V, Y) is a fuzzy linear space over (F, X) if and only if

(i) $V(\lambda x + \mu y) \geq \min\{F(\lambda), F(\mu), V(x), V(y)\}, \lambda, \mu \in X \; and \; x, y \in Y$

(ii) $F(1) \geq V(x), x \in Y$

Theorem 2.6. [13] Let Y and Z be linear space over the field X and f a linear transformation of Y into Z. Let (F, X) be a fuzzy field of X and (W, Z) be a fuzzy linear space over (F, X). Then $(f^{-1}(W), Y)$ is a fuzzy linear space over (F, X).

Theorem 2.7. Let Y and Z be linear spaces over the field X and f a linear transformation of Y into Z. Let (F, X) be a fuzzy field of X and (V, Y) be a fuzzy linear space over (F, X). Then (f(V), Z) is a fuzzy linear space over (F, X).

Definition 2.8. [11] Let (F, K) be a fuzzy field of K (K denotes either R or C), X be a linear space over K and (V, X) be a fuzzy linear space over (F, K). A norm on (V, X) is a function $\|.\|: X \to [0, \infty)$ such that

(i) $F(\|x\|) \geq V(x)$ for all $x \in X$

(ii) $\|x\| \geq 0 \;\; \forall \, x \in X$ and $\|x\| = 0$ if and only if $x = 0$

(iii) $\|x + y\| \leq \|x\| + \|y\| \;\; \forall \, x, y \in X$

(iv) $\|kx\| = |k|\|x\| \;\; \forall \, k \in K$ and $\forall \; x \in X$.

Thus $(V, X, \|.\|)$ is called a normed fuzzy linear space (NFLS).

Definition 2.9. [11] An inner product on a fuzzy linear space (V, X) over a fuzzy field (F, K) is a function $\langle \, , \rangle : X \times X \longrightarrow K$ such that for all $x, y, z \in X$ and $k \in K$

(i) $F(\langle x, y \rangle) \geq V \times V(x, y)$

(ii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$

(iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ and $\langle kx, y \rangle = k\langle x, y \rangle$

(iv) $\langle y, x\rangle = \overline{\langle x, y\rangle}$

Thus $(V, X, \langle\ ,\ \rangle)$ is called an inner product on fuzzy linear space.

3. 2-NORM ON FUZZY LINEAR SPACE OVER FUZZY FIELD

Here K denotes either R, the set of all real numbers or C, the set of all complex numbers.

Definition 3.1. Let (F, K) be a fuzzy field of K, X be a linear space over K and (V, X) be a fuzzy linear space over (F, K). A 2-norm on (V, X) is function $\|\ .\,,.\ \|: X \times X \to [0, \infty)$ such that

(i) $F(\|x, y\|) \geq \min\{V(x), V(y)\}$

(ii) $\|x, y\| = 0 \Leftrightarrow$ x and y are linearly dependent

(iii) $\|x, y\| = \|y, x\|$

(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$

(v) $\|kx, y\| \leq |k|\|x, y\|$ for all k ∈ K and for all $x, y \in X$.

Thus $(V, X, \|.\,,.\|)$ is called a 2-normed fuzzy linear space.

Theorem 3.2. Let $(V, X, \|.\,,.\|)$ be a 2- normed linear space over (F, K). Then

$$F(\|x, y\|)^2 \geq F(\|x, y\|) \text{ for all } x, y \in X.$$

Proof: - For all $k_1, k_2 \in K$

$$F(k_1, k_2) \geq \min\{F(k_1), F(k_2)\}$$

so, $F(k^2) \geq \min\{F(k), F(k)\}$

$$F(k^2) \geq F(k\ ), \forall\ k \in K$$

Hence $F(\|x, y\|^2) \geq F(\|x, y\|), \quad \forall\ x, y \in X$

Theorem 3.3. Let (V, X) be a fuzzy linear space over a fuzzy field (F, K), Y be a linear space over K and $T: X \to Y$ be an injective linear transformation. Then $T(V)\big(T(x)\big) = V(x), \forall\ x \in X$

Proof: - $T(V)\big(T(x)\big) = \underset{\substack{T(\alpha)=T(x) \\ \alpha \in X}}{Sup V(\alpha)}$

$$= \underset{\alpha \in X, \alpha = x}{sup\ V(\alpha)} \qquad \text{(As T is injective)}$$

$$= V(x).$$

Theorem 3.4. Let (V, X) be a fuzzy linear space over fuzzy field (F, K), Y be a linear space over K and T be an isomorphism of X onto Y. (V, X) is a 2- normed fuzzy linear space over (F, K) if and only if (T(V), Y) is a 2- normed fuzzy linear space over (F, K).

Proof: - Let $\|.,.\|_X$ be a 2-norm on (V, X). Let $x_i \in X$ so T $(x_i) \in Y$, $(i = 1,2)$. Now take $T(x_i) = y_i$ Now consider the 2-norm $\|.,.\|_Y$ on Y as $\|y_1, y_2\|_Y = \|x_1, x_2\|_X$. Then

$$F\,(\|y_1, y_2\|_Y) = F(\|x_1, x_2\|_X)$$
$$\geq \min\{V(x_1), V(x_2)\}$$
$$= \min\,\{T(V)T(x_1), T(V)T(x_2)\}$$
$$= \min\,\{T(V)y_1, T(V)y_2\}$$

Thus $\|.,.\|_Y$ is a 2-norm on (T(V), Y).

Conversely, assume that $\|.,.\|_Y$ is a 2-norm on (T(V), Y). Consider the 2- norm $\|.,.\|_X$ on X as $\|x_1, x_2\|_X = \|Tx_1, Tx_2\|_Y$

Then $F(\|x_1, x_2\|_X) = (\|Tx_{1,}Tx_2\|_Y$

$$\geq \min\,\{T(V)Tx_1, T(V)Tx_2\}$$
$$= \min\,\{V(x_1), V(x_2)$$

Thus $\|.,.\|_X$ is a 2-norm on (V, X).

Theorem 3.5. Let X be a linear space over K, (W, Y) a fuzzy linear space over a fuzzy field (F, K) and $T: X \to Y$ be an injective linear transformation. If (W, Y) is a 2-normed Fuzzy Linear space over (F, K). Then $(T^{-1}(W), X)$ is a 2-normed fuzzy linear space over (F, K).

Proof: - Let $\|.,.\|_Y$ be a 2- norm on (W, Y). Consider the 2-norm $\|.,.\|_X$ on X as

$$\|x_1, x_2\|_X = \|Tx_1, Tx_2\|_Y \quad \text{then}$$
$$F(\|x_1, x_2\|_X) = F(\|Tx_1, Tx_2\|_Y)$$
$$\geq \min\{WT(x_1), WT(x_2)\}$$
$$= \min\{T^{-1}W(x_1), T^{-1}W(x_2)\}$$

Hence $\|.,.\|_X$ is a 2- norm on $(T^{-1}(W), X)$.

Theorem 3.6. Let (V, X) be a 2-normed fuzzy linear space over a fuzzy field (F, K) and $T: X \to X$ be an injective linear transformation. Then $(T^{-1}(V), X)$ is a 2-normed fuzzy linear space over

(F, K).

Proof: - Obvious by theorem 3.5.

4. 2- INNER PRODUCT ON FUZZY LINEAR SPACE OVER FUZZY FIELD

Here K denotes either R, the set of all real numbers or C, the set of all complex numbers.

Definition 4.1. 2- inner product on a fuzzy linear space (V, X) over a fuzzy field (F, K). is a function $\langle .,.|.\rangle : X \times X \times X \to K$ such that for all $x, y, z \in X,\ k \in K$.

(i) $F\langle x, y|z\rangle \geq V \times V \times V(\,x, y, z)$.

(ii) $\langle x, x|z\rangle = \langle z, z|x\rangle$

(iii) $\langle x_1 + x_2, y|z\rangle = \langle x_1, y|z\rangle + \langle x_2, y|z\rangle$ and $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$

(iv) $\langle x, x|y\rangle \geq 0$ and $\langle x, x|y\rangle = 0 \Leftrightarrow x \& y$ are linearly dependent

(v) $\langle x, y|z\rangle = \overline{\langle y, x|z\rangle}$.

Thus $(V, X, \langle .,.|.\rangle)$ is called a 2-inner product on fuzzy linear space.

Example 4.2. Let F be a fuzzy field of R. the 2-inner product $\langle x, y|z\rangle = \sum_{i=0}^{n} x_i y_i z_i^2$ is a 2-inner product on a fuzzy linear space $(F \times F \times \dots \dots \times F, R^n)$.

Proof: -(i) Let $V = (F \times F \times \dots \dots . \times F, R^n)$.

$F(\langle x, y|z\rangle) = F(x_1 y_1 z_1^2 + x_2 y_2 z_2^2 + \cdots + x_n y_n z_n^2)$

$\geq \min\{F(x_1 y_1 z_1^2), F(x_2 y_2 z_2^2)\}, \dots \dots, F(x_n y_n z_n^2)\}$

$\geq \min[\min\{\,F(x_1), F(y_1), F(z_1^2)\}, \dots .., \min\{F(x_n) F(y_n) F(z_n^2)\}]$

$= \min[\min\{F(x_1), F(x_2) .., F(x_n)\}, \min\{F(y_1), F(y_2), .., F(y_n)\}, \min\{F(z_1^2), F(z_2^2), .., F(z_n^2)\}]$

$= \min[\min\{F(x_1), F(x_2) .., F(x_n)\}, \min\{F(y_1), F(y_2), .., F(y_n)\}, \min\{F(z_1), F(z_2), .., F(z_n)\}]$

$= \min\{V(x), V(y), V(z)\}$

$= V \times V \times V(x, y, z)$.

(ii) $\langle x, x|z\rangle = \sum_{i=0}^{n} x_i^2 z_i^2$

$= x_1^2 z_1^2 + x_2^2 z_2^2 + \cdots x_n^2 z_n^2$

$= z_1^2 x_1^2 + z_2^2 x_2^2 + \cdots z_n^2 x_n^2$

$= \langle z, z|x\rangle$

(iii) $\langle x + x', y|z\rangle = \sum (x_i + x_i')\, y_i z_i^2$

$= \sum x_i\, y_i z_i^2 + \sum x_i'\, y_i z_i^2$

$= \langle x, y|z\rangle + \langle x', y|z\rangle$

(iv) $\langle x, x|z\rangle \geq 0$ and $\langle x, x|z\rangle = 0 \iff x \,\&\, y$ are linearly dependent.

(v) $\langle x, y|z\rangle = \sum x_i\, y_i z_i^2$

$= \overline{\langle y, x|z\rangle}$.

So $\langle ., . |. \rangle$ is a 2- inner product on $(F \times F \times \dots \dots \times F,\ R^n)$.

Theorem 4.3. If $\langle ., . |. \rangle$ is a 2-inner product on the fuzzy linear space (V, X) Over the fuzzy field (F, K) then for all $x, y, z \in X$ and $k \in K$.

(i) $F(\langle x + x', y|z\rangle) \geq V \times V \times V \times V(x, x', y, z)$

(ii) $F(\overline{\langle x, y|z\rangle}) \geq V \times V \times V(y, x, z)$

(iii) $F(\lambda\langle x, y|z\rangle) \geq V \times V \times V(\lambda x, y, z)$

Proof: - (i) $F(\langle x + x', y|z\rangle) \geq V \times V \times V \times V(x, x', y, z)$

As $F(\langle x + x', y|z\rangle) = F(\langle x, y|z\rangle + \langle x', y|z\rangle)$

$\geq \min\, (F\langle x, y|z\rangle, F\langle x', y|z\rangle)$

$\geq \min\, \{ \mathrm{V} \times \mathrm{V} \times \mathrm{V}(\mathrm{x}, \mathrm{y}, \mathrm{z}),\ \mathrm{V} \times \mathrm{V} \times \mathrm{V}(x', y, z)\}$

$\geq \min\, [\min\{ V(x), V(y), V(z)\}, \min\{\mathrm{V}(x'), V(y), V(z)\}\}$

$= \min\, \{ V(x), V(x'), V(y), V(z)\}$

$= V \times V \times V \times V(x, x', y, z)$

(ii) $F(\overline{\langle x, y|z\rangle}) \geq V \times V \times V(y, x, z)$

As $F(\langle y, x|z\rangle) \geq V \times V \times V(y, x, z)$

(iii) $F(\lambda\langle x, y|z\rangle) \geq V \times V \times V(\lambda x, y, z)$

As $F(\langle \lambda x, y|z\rangle) \geq V \times V \times V(\lambda x, y, z)$.

Theorem 4.4. If $\langle ., . |. \rangle$ is a 2-inner product on the fuzzy linear space (V, X) Over the fuzzy field (F, K) then

(i) $F(\langle x + x', y|z\rangle) \geq \min\, \{ V(x), V(x'), V(y), V(z)\}$

(ii) $F(\langle kx, y|z\rangle) \geq \min\, \{ \mathrm{F}(\mathrm{k}), \mathrm{V}(\mathrm{x}), \mathrm{V}(\mathrm{y}), \mathrm{V}(\mathrm{z})\}$.

Theorem 4.5. Let (V, X) be a fuzzy linear space over a fuzzy field (F, K), Y a linear space K and T an isomorphism of X onto Y. Then there exists a 2-inner product on (V, X) if and only if there exists a 2-inner product on (T(V), Y).

Proof: - Let $\langle .,.|.\rangle_X$ be a 2-inner product on (V, X). Consider the 2- inner product $\langle .,.|.\rangle_Y$ on Y defined as $\langle y_1, y_2|y_3\rangle_Y = \langle x_1, x_2|x_3\rangle_X$ where $y_1 = Tx_1$, $y_2 = Tx_2$ and $y_3 = Tx_3$ then

$$F(\langle y_1, y_2|y_3\rangle_Y) = F(\langle x_1, x_2|x_3\rangle_X) \geq V \times V \times V(x_1, x_2, x_3)$$

$$= T(V) \times T(V) \times T(V)(Tx_1, Tx_2\ Tx_3)$$

$$= T(V) \times T(V) \times T(V)(y_1, y_2, y_3)$$

Thus $\langle .,.|.\rangle_Y$ is a 2-inner product on $(T(V), Y)$.

Conversely let $\langle .,.|.\rangle_Y$ be a 2-inner product on $(T(V), Y)$ consider the 2- inner product $\langle .,.|.\rangle_X$ on X defined as $\langle x_1, x_2|x_3\rangle_X =. \langle T(x_1),\ T(x_2)|T(x_3)\rangle_Y$

$$F(\langle x_1, x_2|x_3\rangle_X) = F(\langle T(x_1),\ T(x_2)|T(x_3)\rangle_Y)$$

$$\geq\ T(V) \times T(V) \times T(V)(Tx_1, Tx_2\ Tx_3)$$

$$= V \times V \times V(x_1, x_2, x_3)$$

So $\langle .,.|.\rangle_X$ is a 2-inner product on (V, X).

Theorem 4.6. Let X be a linear space over K, (W, Y) be a fuzzy linear space over a fuzzy field (F, K) and T: X → Y be an injective linear transformation. If there exists a 2-inner product on (W, Y), then there exists a 2-inner product on ($T^{-1}(W), X$).

Proof: - Let $\langle .,.|.\rangle_Y$ be a 2-inner product on (W, Y). Consider the 2-inner product $\langle .,.|.\rangle_X$ on X defined by $\langle x_1, x_2|x_3\rangle_X = \langle T(x_1), T(x_2)|T(x_3)\rangle_Y$.

Now consider $F(\langle x_1, x_2|x_3\rangle_X) = F(\langle T(x_1), T(x_2)|T(x_3)\rangle_Y)$

$$\geq W \times W \times W(T(x_1), T(x_2), T(x_3))$$

$$= \min\{W(Tx_1), W(Tx_2), W(Tx_3)\}$$

$$= \min\{T^{-1}W(x_1), T^{-1}W(x_2), T^{-1}W(x_3)\}$$

$$= T^{-1}W \times T^{-1}W \times T^{-1}W(x_1, x_2, x_3)$$

Therefore $\langle .,.|.\rangle_X$ is a 2-inner product on $(T^{-1}(W), X)$.

Theorem 4.7. Let (V, X) be a fuzzy linear space over (F, K) and T: X $\longrightarrow$ X be an injective linear transformation. If there exists a 2-inner product on (V, X), then there exists a 2-inner product on $T^{-1}(W), X)$.

5. RELATIONSHIP BETWEEN 2- NORM AND 2- INNER PRODUCT ON FUZZY LINEAR SPACES

Theorem 5.1. Let (V, X) be a fuzzy linear space over (F, K). A 2-norm on (V, X) satisfying the parallelogram law induces a 2-inner product on (V, X) if $\mathrm{F}(4), \mathrm{F}(\mathrm{i}) \geq \min\{\mathrm{V}(\mathrm{x}), \mathrm{V}(\mathrm{y}), \mathrm{V}(\mathrm{z})\}$ for all $x, y, z \in X$.

Proof: - If $\| .,. \|$ is a 2-norm on (V, X) satisfying the parallelogram law, then $F(\| x, y\|) \geq \min\{V(x), V(y)\}$ for all $x, y \in X$ and $\| .,. \|$ induces the 2-inner product $\langle .,. |. \rangle$ on X given by

$$\langle x, y|z\rangle = \frac{1}{4}(\|x+y,z\|^2 - \|x-y,z\|^2 + i\|x+iy,z\|^2 - i\|x-iy,z\|^2)$$

$$F(\langle x, y|z\rangle) = F(\frac{1}{4}(\|x+y,z\|^2 - \|x-y,z\|^2 + i\|x+iy,z\|^2 - i\|x-iy,z\|^2))$$

$$\geq \min\{F\left(\frac{1}{4}\right), F(\|x+y,z\|^2), \mathrm{F}(-\|x-y,z\|^2), \mathrm{F}(\mathrm{i}), \mathrm{F}(\|x+iy,z\|^2), \mathrm{F}(-\|x-iy,z\|^2)\}$$

$$= \min\{F(4), \mathrm{F}(\mathrm{i}), F(\|x+y,z\|^2), \mathrm{F}(\|x-y,z\|^2), \mathrm{F}(\|x+iy,z\|^2), \mathrm{F}(\|x-iy,z\|^2)\}$$

$$\geq \min\{F(4), \mathrm{F}(\mathrm{i}), F(\|x+y,z\|), \mathrm{F}(\|x-y,z\|), \mathrm{F}(\|x+iy,z\|), \mathrm{F}(\|x-iy,z\|)\}$$

$$\geq \min[F(4), F(i), \min\{V(x+y), V(z)\}, \min\{V(x-y), V(z)\},$$
$$\min\{V(x+iy), V(z)\}, \min\{V(x-iy), V(z)\}]$$

$$= \min\{F(4), F(i), V(x+y), V(x-y), V(x+iy), V(x-iy), V(z)\}$$

$$\geq \min\{F(4), F(i), V(x), V(y), V(z)\}$$

$$= \min\{V(x), V(y), V(z)\}$$

$$= V \times V \times V(x, y, z)$$

Hence $\| .,. \|$ induces a 2-inner product on (V, X) if F (4), F(i) $\geq \min\{V(x), V(y), V(z)\}$ for all $x, y, z \in X$.

Example 5.2. Prove that

$$4\langle x, y|z\rangle = (\|x+y,z\|^2 - \|x-y,z\|^2 + i\|x+iy,z\|^2 - i\|x-iy,z\|^2)$$

Proof: - We have $\|x+y,z\|^2-\|x-y,z\|^2+i\|x+iy,z\|^2-i\|x-iy,z\|^2$

$$= \langle x+y,x+y|z\rangle-\langle x-y,x-y|z\rangle+i\langle x+iy,x+iy|z\rangle-i\langle x-iy,x-iy|z\rangle$$

$$= \langle x,x+y|z\rangle+\langle y,x+y|z\rangle-\langle x,x-y|z\rangle+\langle y,x-y|z\rangle+i\langle x,x+iy|z\rangle+ i\langle iy,x+iy|z\rangle-i\langle x,x-iy|z\rangle-i\langle -iy,x-iy|z\rangle$$

$$= \langle x,x+y|z\rangle+\langle y,x+y|z\rangle-\langle x,x-y|z\rangle+\langle y,x-y|z\rangle+i\langle x,x+iy|z\rangle+ i^2\langle y,x+iy|z\rangle-i\langle x,x-iy|z\rangle+i^2\langle y,x-iy|z\rangle$$

$$= \langle x,x+y|z\rangle+\langle y,x+y|z\rangle-\langle x,x-y|z\rangle+\langle y,x-y|z\rangle+i\langle x,x+iy|z\rangle- \langle y,x+iy|z\rangle-i\langle x,x-iy|z\rangle-\langle y,x-iy|z\rangle$$

$$= \overline{\langle x+y,x|z\rangle}+\overline{\langle x+y,\ y|z\rangle}-\overline{\langle x-y,x|z\rangle}+\overline{\langle x-y,y|z\rangle}+i\,\overline{\langle x+\imath y,x|z\rangle}- \overline{\langle x+\ \imath y,y|z\rangle}-i\,\overline{\langle x-\imath y,x|z\rangle}-\overline{\langle x-\imath y,y|z\rangle}$$

$$= \overline{\langle x,x|z\rangle}+\overline{\langle y,x|z\rangle}+\overline{\langle x,y|z\rangle}+\overline{\langle y,y|z\rangle}-\overline{\langle x,x|z\rangle}+\overline{\langle y,x|z\rangle}+\overline{\langle x,y|z\rangle}-\overline{\langle y,y|z\rangle}+ i\overline{\langle x,x|z\rangle}+i\,\overline{\langle \imath y,x|z\rangle}-\overline{\langle x,y|z\rangle}-\overline{\langle \imath y,y|z\rangle}-i\,\overline{\langle x,x|z\rangle}-i\,\overline{\langle -\imath y,x|z\rangle}-\overline{\langle x,y|z\rangle}- \overline{\langle -\imath y,y|z\rangle}$$

$$= \langle x,x|z\rangle+\langle x,y|z\rangle+\langle y,x|z\rangle+\langle y,y|z\rangle-\langle x,x|z\rangle+\langle x,y|z\rangle+\langle y,x|z\rangle-\langle y,y|z\rangle+ i\langle x,x|z\rangle-i^2\langle x,y|z\rangle-\langle y,x|z\rangle+i\langle y,y|z\rangle-i\langle x,x|z\rangle+\langle x,y|z\rangle-\langle y,x|z\rangle-i\langle y,y|z\rangle$$

$$= 4\langle x,y|z\rangle.$$

Theorem 5.3. Let (V, X) be a 2-normed fuzzy linear space over (F, K), Y a linear space over K and T an isomorphism of X onto Y. Suppose that $\mathrm{F}(4),\mathrm{F}(\mathrm{i})\geq \min\{\mathrm{V}(x_1),\mathrm{V}(x_2),\mathrm{V}(x_3)\}$ for all $x_1,x_2,x_3\in X.$ The 2-norm on (V, X) induces a 2-inner product on (V, X) if and only if the 2-norm (T(V), Y).

Proof: - If the 2-norm $\|.,.\|_X$ on (V, X) induces a 2-inner product on (V, X) then $\|.,.\|_X$ satisfies the parallelogram law and by theorem 3.4, $\|.,.\|_Y$ defined by $\|y_1,y_2\|_Y=\|x_1,x_2\|_X$, where $y_1=Tx_1,\ y_2=Tx_2$ is a 2-norm on (T(V),Y).

If $y_1,y_2,y_3\in Y$ and $y_1=Tx_1,\ y_2=Tx_2,\ y_3=Tx_3$ then

$\|y_1+y_2,\ y_3\|_Y^2+\|y_1-y_2,\ y_3\|_Y^2=\|x_1+x_2,\ x_3\|_X^2+\|x_1-x_2,\ x_3\|_X^2=2(\ \|x_1,x_3\|_X^2\ + \|x_2,x_3\|_X^2)=\ 2(\ \|y_1,y_3\|_Y^2\ +\|y_2,y_3\|_Y^2\ ).$ That is $\|.,.\|_Y$ satisfices the parallelogram law.

Also $F(4),F(i)\geq \min\{V(x_1),V(x_2)\ V(x_3)\}$

$= \min\{T(V)(T(x_1), T(V)\big(T(x_2)\big), T(V)(T(x_3))\}$

$= \min\{T(V)(y_1), T(V)(y_2), T(V)(y_3)\}$

for all $y_1, y_2, y_3 \in Y$. Hence $\|.,.\|_Y$ induces a 2-inner product on (T(V), Y).

Similarly, the converse holds.

Note 5.4. A 2-inner product $\langle .,.|.\rangle$ on a fuzzy linear space (V, X) over (F, K) induces a 2-norm $\|.,.\|$ on X given by $\|x,z\| = \langle x,x|z\rangle^{1/2}$ and $\|.,.\|$ is a 2-norm on (V, X) only if $F(\|x,z\|) \geq \min\{V(x), V(z)\}$ for all $x, z \in X$.

Theorem 5.5. Let (V, X) be a fuzzy linear space over (F, K) with a 2-inner product on it. Y a linear space over K and T an isomorphism of X onto Y. the 2-inner product on (V, X) induces a 2-norm on (V, X) if and only if the 2-inner product on (T(V), Y) induces a 2-norm on (T(V), Y).

Proof: - If the 2-inner product $\langle .,.|.\rangle_X$ on (V, X) induces a 2- norm $\|.,.\|_X$ on (V, X) then $F(\|x_1, x_2\|_X) \geq \min\{V(x_1), V(x_2)\}$ for all $x_1, x_2 \in X$ and by theorem 4.5, $\langle .,.|.\rangle_Y$ defined by $\langle y_1, y_2|y_3\rangle_Y = \langle x_1, x_2|x_3\rangle_X$ $(y_1 = Tx_1,\ y_2 = Tx_2,\ y_3 = Tx_3)$ is 2-inner product on (T(V),Y). $\langle .,.|.\rangle_Y$ induces the 2-norm $\|.,.\|_Y$ on Y given by $\|y_1, y_2\|_Y = \|x_1,\ x_2\|_X$ $(y_1 = Tx_1, y_2 = Tx_2)$. $F(\|y_1, y_2\|_Y) = F(\|x_1, x_2\|_X) \geq \min\{V(x_1), V(x_2)\}$

$= \min\{T(V)\big(T(x_1)\big),\ T(V)(T(x_2))\}$

$= \min\{\ T(V)(y_1),\ T(V)(y_2)\}$

and hence $\|.,.\|_Y$ is a 2-norm on (T(V), Y).

Similarly, the converse holds.

Theorem 5.6. In theorem 3.5, if $F(4)F(i) \geq \min\{W(y_1), W(y_2), W(y_3)\}$ for all $y_1, y_2, y_3 \in Y$ and if $\|.,.\|_Y$ induces 2-inner product on (W, Y) then $\|.,.\|_X$ induces a 2-inner product on $(T^{-1}(W), X)$.

Proof: - Since $\|.,.\|_Y$ induces a 2-inner product on (W, Y), $\|.,.\|_Y$ satisfies the parallelogram law $\|y_1 + y_2,\ y_3\|_Y^2 + \|y_1 - y_2,\ y_3\|_Y^2 = 2(\ \|y_1, y_3\|_Y^2 + \|y_2, y_3\|_Y^2\ )$. Therefore if $x_1,\ x_2, x_3 \in X$ then $\|x_1 + x_2,\ x_3\|_X^2 + \|x_1 - x_2,\ x_3\|_X^2 = \|Tx_1 + Tx_2, Tx_3\|_Y^2 + \|Tx_1 - Tx_2, Tx_3\|_Y^2$

$= 2(\|Tx_1, Tx_3\|_Y^2 + \|Tx_2, Tx_3\|_Y^2)$

$= 2(\|x_1, x_3\|_X^2 + \|x_2, x_3\|_X^2)$

That is $\|.,.\|_X$ satisfies the parallelogram law.

Also, for all $x_1, x_2, x_3 \in X$, $F(4)F(i) \geq \min\{W(Tx_1), W(Tx_2), W(Tx_3)\}$

$$= \min\{T^{-1}(W)(x_1), T^{-1}(W)(x_1), T^{-1}(W)(x_3)\}.$$

Hence $\|.,.\|_X$ induces a 2-inner product on $(T^{-1}(W), X)$.

Theorem 5.7. In theorem 4.6 if $\langle .,.|.\rangle_Y$ induces a 2-norm on (W, Y), then $\langle .,.|.\rangle_X$ induces a 2-norm on $(T^{-1}(W), X)$.

Proof: -Assume that $\langle .,.|.\rangle_Y$ induces a 2-norm $\|.,.\|_Y$ given by $\|y_1, y_2\|_Y = \langle y_1, y_1|y_2\rangle_Y^{1/2}$ with $F(\|y_1, y_2\|_Y) \geq \min\{WT(y_1), WT(y_2)\}$, y_1, $y_2 \in Y$.

Consider the 2-norm $\|.,.\|_X$ on X induced by $\langle .,.|.\rangle_X$ given by $\|x_1, x_2,\|_X = \langle x_1, x_1|x_2\rangle_X^{1/2}$.

$\|x_1, x_2\|_X = \langle x_1, x_1|x_2\rangle_X^{1/2} = \langle Tx_1, Tx_1|Tx_2\rangle_Y^{1/2} = \|Tx_1, Tx_2\|_Y$. Therefore

F$(\|x_1, x_2\|_X) = F(\|Tx_1,, Tx_2\|_Y) \geq \min\{W(Tx_1), W(Tx_2)\}$

$$= \min\{T^{-1}(W)(x_1), T^{-1}(W)(x_2)\}$$

Therefore $\|.,.\|_X$ is a 2-norm on $(T^{-1}(W), X)$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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