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COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS USING AN INTEGRAL TYPE CONTRACTIVE CONDITION IN A S-METRIC SPACE

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Abstract. In this paper, by employing a contractive condition of integral type, we obtain a unique common fixed point for four weakly compatible self maps of a S-metric space which satisfy common limit range property.

Keywords: fixed point; weakly compatibility; common limit range property; S-metric space.

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1. Introduction

Gerald Jungck [6] introduced the concept of compatibility to generalize the notion of commutative property. Further Jungck and Rhoades [7] proposed weakly compatibility of mappings. Also they proved that for a pair of mappings compatibility always implies weakly compatibility but not conversely.

To prove common fixed point theorems, Sintunavarat et al [14] initiated common limit range (CLR) property.

Several authors Dhage, Gahler, Sedghi, Mustafa [2,3,4,8,13] generalized the notion of metric space by introducing 2-metric space, D^*-metric spaces and G-metric spaces.

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Shaban Sedghi et al [12] proposed S-metric space as further generalization of metric spaces. This concept of S-metric spaces generated lot of interest among many researches.

In this paper, we prove a common fixed point theorem for four weakly compatible self maps of S-metric space satisfying common limit range property along with an integral type contractive condition [1]. Our result generalizes the results already proved in literature [15]. A suitable example is provided to validate our theorem.

2. Preliminaries

Definition 2.1. [12] Let M be non empty set. A function $S : M^3 \longrightarrow [0,\infty)$ is said to be an S-metric on M,if for each $\nu,\omega,\vartheta,\rho \in M$

1. $S(\nu,\omega,\vartheta) \geq 0$
2. $S(\nu,\omega,\vartheta) = 0 \Leftrightarrow \nu = \omega = \vartheta$
3. $S(\nu,\omega,\vartheta) \leq S(\nu,\nu,\rho)+S(\omega,\omega,\rho)+S(\vartheta,\vartheta,\rho)$

Then (M,S) is called an S-metric space.

Lemma 2.1. *[10] In a S-metric space we have $S(\nu,\nu,\omega) = S(\omega,\omega,\nu)$ for all $\nu,\omega \in M$*

Definition 2.2. [11] Let (M,S) be a S- metric space.

(a) A sequence (ν_n) in M converge to ν if $S(\nu_n,\nu_n,\nu) \to 0$ as $n \to \infty$ then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,$S(\nu_n,\nu_n,\nu) < \varepsilon$ and we denote this by writing $\lim\limits_{n\to\infty} \nu_n = \nu$.

(b) A sequence (ν_n) be a Cauchy sequence if for each $\varepsilon > 0$,there exists $n_0 \in \mathbb{N}$ such that $S(\nu_n,\nu_n,\nu_m) < \varepsilon$ for each $n,m \geq n_0$.

(c) By a complete S-metric space we mean a S-metric space in which every Cauchy sequence is convergent.

Lemma 2.2. *[11] In a S-metric space (M,S), if there exist two sequences (ν_n) and (ω_n) such that $\lim\limits_{n\to\infty} \nu_n = \nu$ and $\lim\limits_{n\to\infty} \omega_n = \omega$, then $\lim\limits_{n\to\infty} S(\nu_n,\nu_n,\omega_n) = S(\nu,\nu,\omega)$*

Definition 2.3. [7] The self mappings E,F of a S-metric space (M,S) are called weakly compatible if $EF\nu = FE\nu$ whenever $E\nu = F\nu$ for any ν in M.

Definition 2.4. [9] In a S-metric space (M,S), the two pairs of self mappings (E,G) and (F,H) on M are said to satisfy common (E.A)-property if there exist two sequences (ν_n) and (ω_n) in M such that

$$\lim_{n\to\infty} E\nu_n = \lim_{n\to\infty} G\nu_n = \lim_{n\to\infty} F\omega_n = \lim_{n\to\infty} H\omega_n = \tau, \text{ where } \tau \in M.$$

Definition 2.5. [14] In a S-metric space (M,S), the two pairs of self mappings (E,G) and (F,H) on M are said to satisfy common limit range property with respect to G and H,denoted by (CLR_{GH}) if there exists two sequences (ν_n) and (ω_n) in M such that

$$\lim_{n\to\infty} E\nu_n = \lim_{n\to\infty} G\nu_n = \lim_{n\to\infty} F\omega_n = \lim_{n\to\infty} H\omega_n = \tau, \text{ where } \tau \in G(M)\cap H(M).$$

Remark 2.1. *Throughout this paper* $\varphi : [0,\infty) \longrightarrow [0,\infty)$ *is a Lebesgue integrable function which is summable on compact subset of* $[0,\infty)$ *with* $\int_0^{\varepsilon} \varphi(\alpha)d\alpha > 0$*, for any* $\varepsilon > 0$.

3. Main Results

Now we state our main theorem.

Theorem 3.1. *In a S-metric space (M,S), suppose E,F,G,H are self mappings of M satisfying the following conditions*

(i) The pairs (E,G) and (F,H) are weakly compatible

(ii) The pairs (E,G) and (F,H) share (CLR_{GH})*-property*

(iii) $\int_0^{S(E\nu,E\nu,F\omega)} \varphi(\alpha)d\alpha \leq \lambda \int_0^{\frac{S(F\omega,F\omega,H\omega)[1+S(E\nu,E\nu,G\nu)]}{[1+S(G\nu,G\nu,H\omega)]}} \varphi(\alpha)d\alpha + \mu \int_0^{S(G\nu,G\nu,H\omega)} \varphi(\alpha)d\alpha$

where $\lambda,\mu > 0$ *with* $\lambda+\mu < 1$

then E,F,G and H have a unique common fixed point in M.

Proof. From the (CLR_{GH})-property of the pairs (E,G) and (F,H), we have two sequences (ν_n) and (ω_n) in M such that

$$\lim_{n\to\infty} E\nu_n = \lim_{n\to\infty} G\nu_n = \lim_{n\to\infty} F\omega_n = \lim_{n\to\infty} H\omega_n = \tau, \textit{ where } \tau \in G(M)\cap H(M) \tag{1}$$

Also there exists a point $\eta \in M$ such that $G\eta = \tau$, from (1),we have

$$\lim_{n\to\infty} E\nu_n = \lim_{n\to\infty} G\nu_n = \lim_{n\to\infty} F\omega_n = \lim_{n\to\infty} H\omega_n = \tau = G\eta \tag{2}$$

We now claim that $E\eta=\tau$,if $E\eta\neq\tau$ then $S(E\eta,E\eta,\tau)>0$

keeping $\nu=\eta$ and $\omega=\omega_n$ in condition (iii) of the Theorem 3.1 we get

$$\int_0^{S(E\eta,E\eta,F\omega_n)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{\frac{S(F\omega_n,F\omega_n,H\omega_n)[1+S(E\eta,E\eta,G\eta)]}{[1+S(G\eta,G\eta,H\omega_n)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(G\eta,G\eta,H\omega_n)}\varphi(\alpha)d\alpha \tag{3}$$

on passing to the limits

$$\int_0^{S(E\eta,E\eta,\tau)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{\frac{S(\tau,\tau,\tau)[1+S(E\eta,E\eta,\tau)]}{[1+S(\tau,\tau,\tau)]}}\varphi(\alpha)d\alpha+\mu\int_0^{S(\tau,\tau,\tau)}\varphi(\alpha)d\alpha$$

$$\int_0^{S(E\eta,E\eta,\tau)}\varphi(\alpha)d\alpha=0 \tag{4}$$

giving that $S(E\eta,E\eta,\tau)=0$

leading to a contradiction to the fact that $S(E\eta,E\eta,\tau)>0$

proving $E\eta=\tau$

$$\text{Giving } \ G\eta=E\eta=\tau \tag{5}$$

Also $H\zeta=\tau$. Again from (1) we obtain

$$\lim_{n\to\infty}E\nu_n=\lim_{n\to\infty}G\nu_n=\lim_{n\to\infty}F\omega_n=\lim_{n\to\infty}H\omega_n=\tau=H\zeta \tag{6}$$

We now claim that $F\zeta=\tau$. For if $F\zeta\neq\tau$ then $S(F\zeta,F\zeta,\tau)>0$

on taking $\nu=\nu_n$ and $\omega=\zeta$ in condition (iii) of the Theorem 3.1, we obtain

$$\int_0^{S(E\nu_n,E\nu_n,F\zeta)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{\frac{S(F\zeta,F\zeta,H\zeta)[1+S(E\nu_n,E\nu_n,G\nu_n)]}{[1+S(G\nu_n,G\nu_n,H\zeta)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(G\nu_n,G\nu_n,H\zeta)}\varphi(\alpha)d\alpha \tag{7}$$

on passing to the limits

$$\int_0^{S(\tau,\tau,F\zeta)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{\frac{S(F\zeta,F\zeta,\tau)[1+S(\tau,\tau,\tau)]}{[1+S(\tau,\tau,\tau)]}}\varphi(\alpha)d\alpha+\mu\int_0^{S(\tau,\tau,\tau)}\varphi(\alpha)d\alpha$$

$$\int_0^{S(\tau,\tau,F\zeta)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{S(F\zeta,F\zeta,\tau)}\varphi(\alpha)d\alpha+0$$

$$(1-\lambda)\int_0^{S(F\zeta,F\zeta,\tau)}\varphi(\alpha)d\alpha \leq 0 \tag{8}$$

which gives $S(F\zeta,F\zeta,\tau)=0$

Again leading to a contradiction to the fact that $S(F\zeta,F\zeta,\tau)>0$

proving $F\zeta=\tau$

$$\text{Therefore } \; F\zeta=H\zeta=\tau \tag{9}$$

Further,we obtain

$$E\eta=G\eta=F\zeta=H\zeta=\tau \tag{10}$$

Now we established τ is a common fixed point of E,F,G and H.

$$\text{clearly } \; GE\eta=EG\eta$$

from which we get

$$G\tau=E\tau \tag{11}$$

and

$$HF\zeta=FH\zeta$$

$$H\tau=F\tau \tag{12}$$

we have $E\tau=\tau$,For if $E\tau\neq\tau$ then $S(E\tau,E\tau,\tau)>0$

substituting $\nu=\tau$ and $\omega=\zeta$ in condition (iii) of the Theorem 3.1 , we get

$$\int_0^{S(E\tau,E\tau,F\zeta)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{\frac{S(F\zeta,F\zeta,H\zeta)[1+S(E\tau,E\tau,G\tau)]}{[1+S(G\tau,G\tau,H\zeta)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(G\tau,G\tau,H\zeta)}\varphi(\alpha)d\alpha$$

$$\int_0^{S(E\tau,E\tau,\tau)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{\frac{S(\tau,\tau,\tau)[1+S(E\tau,E\tau,E\tau)]}{[1+S(E\tau,E\tau,\tau)]}}\varphi(\alpha)d\alpha+\mu\int_0^{S(E\tau,E\tau,\tau)}\varphi(\alpha)d\alpha$$

$$\int_0^{S(E\tau,E\tau,\tau)}\varphi(\alpha)d\alpha \leq \mu\int_0^{S(E\tau,E\tau,\tau)}\varphi(\alpha)d\alpha$$

$$(1-\mu)\int_0^{S(E\tau,E\tau,\tau)}\varphi(\alpha)d\alpha \leq 0 \tag{13}$$

which gives $S(E\tau,E\tau,\tau)=0$

contradicting the fact that $S(E\tau,E\tau,\tau)>0$

proving $E\tau=\tau$

$$\text{therefore } \; G\tau=E\tau=\tau \tag{14}$$

similarly, we can prove that

$$F\tau=H\tau=\tau \tag{15}$$

from(14) and (15), it follows that

$$E\tau=F\tau=G\tau=H\tau=\tau \tag{16}$$

proving τ is a common fixed point of E,F,G and H.

For if $\varsigma(\varsigma\neq\tau)$ is in M such that

$$E\varsigma=F\varsigma=G\varsigma=H\varsigma=\varsigma$$

Then on taking $\nu=\tau$ and $\omega=\varsigma$ in condition (iii) of the Theorem 3.1, we get

$$\int_0^{S(E\tau,E\tau,F\varsigma)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{\frac{S(F\varsigma,F\varsigma,H\varsigma)[1+S(E\tau,E\tau,G\tau)]}{[1+S(G\tau,G\tau,H\varsigma)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(G\tau,G\tau,H\varsigma)}\varphi(\alpha)d\alpha$$

$$\int_0^{S(\tau,\tau,\varsigma)}\varphi(\alpha)d\alpha \leq \lambda\int_0^{\frac{S(\varsigma,\varsigma,\varsigma)[1+S(\tau,\tau,\tau)]}{[1+S(\tau,\tau,\varsigma)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(\tau,\tau,\varsigma)}\varphi(\alpha)d\alpha$$

$$(1-\mu)\int_0^{S(\tau,\tau,\varsigma)}\varphi(\alpha)d\alpha \leq 0 \tag{17}$$

giving $S(\tau,\tau,\varsigma)=0$

from which it follows that $\tau=\varsigma$

proving that E,F,G and H have a unique common fixed point in M. □

As an illustration we have the following example.

Example 1. Let $M=[0,2]$ be a S-metric space with $S(\nu,\omega,\vartheta)=|\nu-\vartheta|+|\omega-\vartheta|$,where $\nu,\omega,\vartheta\in M$ and E,F,G and H be self maps on M,defined by

$$E(\nu)=\begin{cases}1, & \nu\in[0,1],\\ \frac{1}{3}, & \nu\in(1,2].\end{cases}\qquad F(\nu)=\begin{cases}1, & \nu\in[0,1],\\ \frac{1}{2}, & \nu\in(1,2].\end{cases}$$

$$G(\nu)=\begin{cases}1, & \nu\in[0,1],\\ \frac{3}{4}, & \nu\in(1,2].\end{cases}\qquad H(\nu)=\begin{cases}1, & \nu\in[0,1],\\ \frac{3}{2}, & \nu\in(1,2].\end{cases}$$

Also take $\varphi(\alpha)=3\alpha^2$ for $\alpha\in[0,\infty)$

Let (ν_n) and (ω_n) be sequences in M with $\nu_n=\frac{n}{n+1}$ and $\omega_n=\frac{n}{n+2}$, where $n\geq 1$,then

$$\lim_{n\to\infty}E\nu_n=\lim_{n\to\infty}E(\frac{n}{n+1})=1=E(1)$$

$$\lim_{n\to\infty}G\nu_n=\lim_{n\to\infty}G(\frac{n}{n+1})=1=G(1)$$

$$\lim_{n\to\infty}F\omega_n=\lim_{n\to\infty}F(\frac{n}{n+2})=1=F(1)$$

$$\lim_{n\to\infty}H\omega_n=\lim_{n\to\infty}H(\frac{n}{n+2})=1=H(1)$$

$$thus\lim_{n\to\infty}E\nu_n=\lim_{n\to\infty}G\nu_n=\lim_{n\to\infty}F\omega_n=\lim_{n\to\infty}H\omega_n=1\ and\ 1\in G(M)\cap H(M)$$

proving (E,G) and (F,H) satisfy (CLR_{GH})-property.

Now we verify condition (iii) of Theorem 3.1 in different cases.

Case(i). Let $\nu,\omega\in[0,1]$

then $E\nu=G\nu=F\omega=H\omega=1$ and from (iii)

$$\int_0^{S(E\nu,E\nu,F\omega)}\varphi(\alpha)d\alpha=\int_0^{S(1,1,1)}3\alpha^2d\alpha=0$$

$$\lambda\int_0^{\frac{S(F\omega,F\omega,H\omega)[1+S(E\nu,E\nu,G\nu)]}{[1+S(G\nu,G\nu,H\omega)]}}\varphi(\alpha)d\alpha+\mu\int_0^{S(G\nu,G\nu,H\omega)}\varphi(\alpha)d\alpha$$

$$\lambda\int_0^{\frac{S(1,1,1)[1+S(1,1,1)]}{[1+S(1,1,1)]}}3\alpha^2d\alpha+\mu\int_0^{S(1,1,1)}3\alpha^2d\alpha=0$$

therefore

$$\int_0^{S(E\nu,E\nu,F\omega)}\varphi(\alpha)d\alpha = \lambda\int_0^{\frac{S(F\omega,F\omega,H\omega)[1+S(E\nu,E\nu,G\nu)]}{[1+S(G\nu,G\nu,H\omega)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(G\nu,G\nu,H\omega)}\varphi(\alpha)d\alpha$$

case(ii). Let $\nu,\omega\in(1,2]$
then $E\nu=\frac{1}{3}$,$G\nu=\frac{3}{4}$,$F\omega=\frac{1}{2}$, $H\omega=\frac{3}{2}$ and from (iii)

$$\int_0^{S(E\nu,E\nu,F\omega)}\varphi(\alpha)d\alpha = \int_0^{S(\frac{1}{3},\frac{1}{3},\frac{1}{2})}3\alpha^2 d\alpha = \frac{1}{27}$$

$$\lambda\int_0^{\frac{S(F\omega,F\omega,H\omega)[1+S(E\nu,E\nu,G\nu)]}{[1+S(G\nu,G\nu,H\omega)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(G\nu,G\nu,H\omega)}\varphi(\alpha)d\alpha$$

$$\lambda\int_0^{\frac{S(\frac{1}{2},\frac{1}{2},\frac{3}{2})[1+S(\frac{1}{3},\frac{1}{3},\frac{3}{4})]}{[1+S(\frac{3}{4},\frac{3}{4},\frac{3}{2})]}}3\alpha^2 d\alpha + \mu\int_0^{S(\frac{3}{4},\frac{3}{4},\frac{3}{2})}3\alpha^2 d\alpha = \lambda\int_0^{\frac{22}{15}}3\alpha^2 d\alpha + \mu\int_0^{\frac{3}{2}}3\alpha^2 d\alpha$$
$$= \lambda\frac{10648}{3375} + \mu\frac{27}{8}$$

since $\lambda,\mu>0$ with $\lambda+\mu<1$

therefore

$$\int_0^{S(E\nu,E\nu,F\omega)}\varphi(\alpha)d\alpha < \lambda\int_0^{\frac{S(F\omega,F\omega,H\omega)[1+S(E\nu,E\nu,G\nu)]}{[1+S(G\nu,G\nu,H\omega)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(G\nu,G\nu,H\omega)}\varphi(\alpha)d\alpha$$

Case(iii). Let $\nu\in[0,1]$ and $\omega\in(1,2]$
then $E\nu=1, G\nu=1, F\omega=\frac{1}{2}, H\omega=\frac{3}{2}$ and from (iii)

$$\int_0^{S(E\nu,E\nu,F\omega)}\varphi(\alpha)d\alpha = \int_0^{S(1,1,\frac{1}{2})}3\alpha^2 d\alpha = 1$$

$$\lambda\int_0^{\frac{S(F\omega,F\omega,H\omega)[1+S(E\nu,E\nu,G\nu)]}{[1+S(G\nu,G\nu,H\omega)]}}\varphi(\alpha)d\alpha + \mu\int_0^{S(G\nu,G\nu,H\omega)}\varphi(\alpha)d\alpha$$

$$= \lambda \int_0^{\frac{S(\frac{1}{2},\frac{1}{2},\frac{3}{2})[1+S(1,1,1)]}{[1+S(1,1,\frac{3}{2})]}} 3\alpha^2 d\alpha + \mu \int_0^{S(1,1,\frac{3}{2})} 3\alpha^2 d\alpha = \lambda \int_0^2 3\alpha^2 d\alpha + \mu \int_0^1 3\alpha^2 d\alpha$$

$$= \lambda 8 + \mu$$

since $\lambda, \mu > 0$ with $\lambda + \mu < 1$

thus we have

$$\int_0^{S(Ev,Ev,F\omega)} \varphi(\alpha)d\alpha \leq \lambda \int_0^{\frac{S(F\omega,F\omega,H\omega)[1+S(Ev,Ev,Gv)]}{[1+S(Gv,Gv,H\omega)]}} \varphi(\alpha)d\alpha + \mu \int_0^{S(Gv,Gv,H\omega)} \varphi(\alpha)d\alpha$$

Any λ, μ satisfying conditions obtained in case (ii) and case(iii) with $\lambda, \mu > 0$ and $\lambda + \mu < 1$ will work here.

Similarly we can check condition (iii) of Theorem 3.1 in case if $\omega \in [0,1]$ and $v \in (1,2]$. Hence condition (iii) is satisfied in various cases.

Observe that 1 is the unique common fixed point of E,F,G,H.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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