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EXISTENCE OF SOLUTION TO FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS USING TOPOLOGICAL DEGREE THEORY

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Abstract. This paper attempts to examine the existence and uniqueness of solution for the fractional Hybrid differential equations. It is based on applying topological degree techniques for certain reasonable conditions in Banach space. An example of this can be confirmed in the results.

Keywords: Fractional derivatives and integral; Fixed point theorems; Hybrid initial value problem; Topological methods.

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1. Introduction

Fractional differential equations are important tools in the modeling of nonlinear real-world phenomena corresponding to a great variety of events, in relation with several fields of the physical sciences and technology, see [1, 7, 4, 5, 6, 19, 20]. Hybrid differential equations have recently become a lot more interesting and there have also been significant contributions to the theory of hybrid differential equations, see [1, 7, 15, 16]. Hybrid fixed point theory can also be used to construct the existence principle for the hybrid equation. Topological processes have proven to be an excellent tool for studying a wide range of nonlinear analytic problems.

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Results on the existence and uniqueness of solutions are also included, see [2, 3, 11, 12, 17, 20]. Dhage, Lakshmikantham, and Krasnoselskii examined the Hybrid fractional differential equations system extensively, see [9, 8, 13]. The existence of an ordinary hybrid differential equation with first-type linear perturbations was established by the authors in a study

$$\begin{cases} \frac{d}{dt}[\frac{x(t)}{\xi(t,x(t))}] = \eta(t,x(t)), \quad t \in \mathscr{J} = [0,1] \\ \qquad x(t_0) = x_0 \end{cases}$$

where $\xi \in \mathscr{C}(\mathscr{J} \times \mathscr{R}, \mathscr{R} \backslash \{0\})$ and $\eta \in \mathscr{C}(\mathscr{J} \times \mathscr{R}, \mathscr{R})$.

In [22], the above result of hybrid differential equations was extended to fractional order differential equations involving Riemann-Liouville differential operators by the researchers as

$$\begin{cases} D_0^{\alpha}[\frac{x(t)}{\xi(t,x(t))}] = \eta(t,x(t)), \quad t \in \mathscr{J} = [0,T] \\ \qquad x(0) = 0 \end{cases}$$

where $0 < \alpha < 1$. For the above mentioned class of hybrid fractional differential equations, the researchers extended some sufficient conditions for existence and uniqueness of solution. Additionally, in [9] researchers applied the above results to hybrid fractional differential equations using Caputo's derivative boundary conditions. In [22], the concept of fractional hybrid differential equations, which are related to Riemann–Liouville differential operators of order $q \in (0,1)$, was developed by the authors. Under mixed Lipschitz and Carathéodory conditions, they derived an existence theorem for fractional hybrid differential equations. Melliani et. al. [14], investigated a hybrid differential equation of boundary value problem with linear and nonlinear disturbances. They broadened the scope of an existing second-type problem. The Leray–Schauder alternative was used to generate the existence result, and Banach's fixed-point theorem ensured uniqueness. Hussain and Khan [10], established the existence of a solution to the hybrid differential equation with $1 < \alpha \leq 2$ that satisfied certain growth conditions. The results in question were produced using the topological degree method.

In this paper, we consider the generalisation of fractional Hybrid differential equations (FHDEs) in Banach space $\mathscr{X}$, motivated by the problems indicated above:

(1)
$$\begin{cases} {}^c\mathscr{D}^q[x(t)\xi(t,x(t))-\eta(t,x(t))]=\zeta(t,x(t)), & t\in\mathscr{J}:=[0,T] \\ x(0)\xi(0,x(0))+\alpha x(T)\xi(T,x(T))=\eta(0,x(0))+\alpha\eta(T,x(T))+\beta \end{cases}$$

where ${}^c\mathscr{D}^q$ is the Caputo fractional derivative of order $q\in(0,1)$ and α,β are elements of $\mathscr{X}$. $\xi,\eta:\mathscr{J}\times\mathscr{X}\to\mathscr{X}$ are continuous maps, $\zeta:\mathscr{C}(\mathscr{J},\mathscr{X})\to\mathscr{X}$ is a given continuous map and $\mathscr{C}(\mathscr{J},\mathscr{X})$ is a Banach space of all continuous functions from $\mathscr{J}$ into $\mathscr{X}$ with the norm $\|x\|_c:=sup\{\|x(t)\|:x\in\mathscr{C}(\mathscr{J},\mathscr{X})\}$ for $t\in\mathscr{J}$ and $\mathscr{J}=[0,T],T>0$.

2. Preliminaries

We introduce several specific definitions, Lemma and propositions in this section that will be used throughout this study.

Definition 2.1. [15, 18] *The qth fractional order integral of a continuous function ξ on a closed interval $[a,b]$ is defined as follows:*

$$\mathscr{I}^q_{a+}\xi(t)=\frac{1}{\Gamma(q)}\int_a^t(t-s)^{q-1}\xi(s)ds, \tag{2}$$

where Γ is the gamma function.

Definition 2.2. [15, 18] *The qth Riemann- Liouville fractional order derivative of a continuous function ξ on a closed interval $[a,b]$ is defined as:*

$$(\mathscr{D}^q_{a+}\xi)(t)=\frac{1}{\Gamma(n-q)}(\frac{d}{dt})^n\int_a^t(t-s)^{n-q-1}\xi(s)ds, \tag{3}$$

where $n=[q]+1$ and $[q]$ denotes the integer part of q.

Definition 2.3. [15, 18] *The Caputo fractional order derivative of a continuous function ξ on a closed interval $[a,b]$ is defined as follows:*

$$({}^c\mathscr{D}^q_{a+}\xi)(t)=\frac{1}{\Gamma(n-q)}\int_a^t(t-s)^{n-q-1}\xi^{(n)}(s)ds, \tag{4}$$

where $n = [q] + 1$.

Definition 2.4. [18, 21] *Let* $\Omega \subset \mathscr{X}$ *and* $\mathscr{F} : \Omega \to \mathscr{X}$ *be a continuous bounded map. One can say that* $\mathscr{F}$ *is* α*-Lipschitz if there exists* $k \geq 0$ *such that*

$$\alpha(\mathscr{F}(B)) \leq k\alpha(B) \ \ (\forall)\ B \subset \Omega \text{ bounded}.$$

In case, $k < 1$*, then we call* $\mathscr{F}$ *is a strict* α*-contraction . One can say that* $\mathscr{F}$ *is* α*-condensing if*

$$\alpha(\mathscr{F}(B)) < \alpha(B) \ \ (\forall)\ B \subset \Omega \text{ bounded with } \alpha(B) > 0.$$

We recall that $\mathscr{F} : \Omega \to \mathscr{X}$ *is Lipschitz if there exists* $k > 0$ *such that*

$$\|\mathscr{F}_x - \mathscr{F}_y\| \leq k\|x - y\| \quad (\forall) x, y \subset \Omega$$

and if $k < 1$ *then* $\mathscr{F}$ *is a strict contraction.*

Lemma 2.1. [20] *Let* $n - 1 < q \leq n$ *, then*

$$\mathscr{I}^q({}^c\mathscr{D}^q \xi)(t) = \xi(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathscr{X}, i = 0, 1, 2, ..., n-1, n = [q] + 1$.

Proposition 2.1. [18, 21] *If* $\psi, \varphi : \Omega \to \mathscr{X}$ *are* α*-Lipschitz maps with constants* k *and* k'*, then* $\psi + \varphi : \Omega \to \mathscr{X}$ *is* α*-Lipschitz with constant* $k + k'$.

Proposition 2.2. [18, 21] ψ *is* α*-Lipschitz with zero constant if* $\psi : \Omega \to \mathscr{X}$ *is compact.*

Proposition 2.3. [18, 21] *If* $\psi : \Omega \to \mathscr{X}$ *is Lipschitz with constant* k*, then* ψ *is* α*-Lipschitz, with the same constant* k*.*

3. Main Results

To give the existence results to FHDE (1), we should first make the following assumptions:

H1: $\xi, \eta : \mathscr{J} \times \mathscr{X} \to \mathscr{X}$ are continuous.

H2: For arbitrary $(t,x) \in \mathscr{C}(\mathscr{J},\mathscr{X})$ there exist positive constants γ_ξ, γ_η and $\gamma_\zeta > 0$, such that

$$\begin{aligned} \|\xi(t,x)\| &\geq \gamma_\xi, \\ \|\eta(t,x)\| &\leq \gamma_\eta, \\ \|\zeta(t,x)\| &\leq \gamma_\zeta. \end{aligned} \tag{5}$$

H3: For arbitrary $x, y \in \mathscr{X}$, there exist positive constants $\delta_\xi, \delta_\eta, \delta_\zeta \in (0,1)$, such that

$$\begin{aligned} \|\xi(t,x) - \xi(t,y)\| &\leq \delta_\xi \|x-y\|, \\ \|\eta(t,x) - \eta(t,y)\| &\leq \delta_\eta \|x-y\|, \\ \|\zeta(t,x) - \zeta(t,y)\| &\leq \delta_\zeta \|x-y\|. \end{aligned} \tag{6}$$

Definition 3.1. *A function $x \in \mathscr{C}(\mathscr{J},\mathscr{X})$ is called a solution of FHDE (1) if x satisfies the equation ${}^c\mathscr{D}^q[x(t)\xi(t,x(t)) - \eta(t,x(t))] = \zeta(t,x(t))$ almost everywhere on $\mathscr{J}$ and the condition $x(0)\xi(0,x(0)) + \alpha x(T)\xi(T,x(T)) = \eta(0,x(0)) + \alpha\eta(T,x(T)) + \beta$.*

Lemma 3.1. *Let $\zeta : \mathscr{C}(\mathscr{J},\mathscr{X}) \to \mathscr{X}$, then a function $x \in \mathscr{C}(\mathscr{J},\mathscr{X})$ is said to be a solution of FHDE (1) if and only if it satisfies the fractional integral equation:*

$$\begin{aligned} x(t) = \frac{\beta}{(1+\alpha)\xi(t,x(t))} - \frac{\alpha}{(1+\alpha)\xi(t,x(t))} \cdot \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1}\zeta(s,x(s))ds \\ + \frac{\eta(t,x(t))}{\xi(t,x(t))} + \frac{1}{\Gamma(q)\xi(t,x(t))} \int_0^t (t-s)^{q-1}\zeta(s,x(s))ds. \end{aligned} \tag{7}$$

Proof. Assume that x is a solution for FHDE (1), then we have to show that x is also a solution for FIE(7). We have,

$$x(t)\xi(t,x(t)) = \eta(t,x(t)) + x(0)\xi(0,x(0)) - \eta(0,x(0)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\zeta(s,x(s))ds. \tag{8}$$

Then,

$$x(T)\xi(T,x(T)) = \eta(T,x(T)) + x(0)\xi(0,x(0)) - \eta(0,x(0)) + \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1}\zeta(s,x(s))ds. \tag{9}$$

By using the condition in FHDE(1) in equation (9), we get $x(T)\xi(T,x(T)) - \eta(T,x(T)) = \frac{1}{\alpha}[\beta - x(0)\xi(0,x(0)) + \eta(0,x(0))]$. Then,

$$\beta - x(0)\xi(0,x(0)) + \eta(0,x(0)) = \alpha x(0)\xi(0,x(0)) - \alpha\eta(0,x(0)) + \frac{\alpha}{\Gamma(q)} \int_0^T (T-s)^{q-1}\zeta(s,x(s))ds$$

which implies

$$x(0)\xi(0,x(0))-\eta(0,x(0))=\frac{\beta}{(1+\alpha)}-\frac{\alpha}{(1+\alpha)\Gamma(q)}\int_0^T(T-s)^{q-1}\zeta(s,x(s))ds.$$

Replacing in equation (8), we get

$$\begin{aligned}x(t)\xi(t,x(t))=\eta(t,x(t))+\frac{\beta}{(1+\alpha)}-\frac{\alpha}{(1+\alpha)\Gamma(q)}\int_0^T(T-s)^{q-1}\zeta(s,x(s))ds\\+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\zeta(s,x(s))ds.\end{aligned}$$

Hence,

$$\begin{aligned}x(t)=\frac{\eta(t,x(t))}{\xi(t,x(t))}+\frac{\beta}{(1+\alpha)\xi(t,x(t))}-\frac{\alpha}{(1+\alpha)\Gamma(q)\xi(t,x(t))}\int_0^T(T-s)^{q-1}\zeta(s,x(s))ds\\+\frac{1}{\Gamma(q)\xi(t,x(t))}\int_0^t(t-s)^{q-1}\zeta(s,x(s))ds.\end{aligned}$$

Conversely, assume that $x\in\mathscr{C}(\mathscr{J},\mathscr{X})$ satisfies the FIE(7). If $t=0$, it is easy to obtain $x(0)\xi(0,x(0))+\alpha x(T)\xi(T,x(T))=\eta(0,x(0))+\alpha\eta(T,x(T))+\beta$. For $t\in\mathscr{J}$ by using the both facts that Caputo fractional derivative ${}^c\mathscr{D}_t^q$ is the left inverse of the fractional integral $\mathscr{I}_t^q$ and the Caputo derivative of a constant is equal to zero, we can deduce that ${}^c\mathscr{D}^q[x(t)\xi(t,x(t))-\eta(t,x(t))]=\zeta(t,x(t))$ which completes the proof.

□

Lemma 3.2. *The opertor $\mathscr{F}:\mathscr{C}(\mathscr{J},\mathscr{X})\to\mathscr{C}(\mathscr{J},\mathscr{X})$ defined as*

$$(\mathscr{F}x)(t)=\frac{\beta}{(1+\alpha)\xi(t,x(t))}+\frac{\eta(t,x(t))}{\xi(t,x(t))},\qquad t\in\mathscr{J}$$

is Lipschitz with constant $[\frac{|\beta|}{|1+\alpha|}+\gamma_\eta]\delta_\xi+\gamma_\xi\delta_\eta$. Consequently, $\mathscr{F}$ is $\alpha-$Lipschitz with the same constant.

Proof. By using (H2) and (H3), we have

$$
\begin{aligned}
\|(\mathscr{F}x)(t)-(\mathscr{F}y)(t)\| &\leq \|\frac{\beta}{(1+\alpha)\xi(t,x(t))}+\frac{\eta(t,x(t))}{\xi(t,x(t))}-\frac{\beta}{(1+\alpha)\xi(t,y(t))}-\frac{\eta(t,y(t))}{\xi(t,y(t))}\| \\
&\leq \frac{|\beta|}{|1+\alpha|}\|\frac{1}{\xi(t,x(t))}-\frac{1}{\xi(t,y(t))}\|+\|\frac{\eta(t,x(t))}{\xi(t,x(t))}-\frac{\eta(t,y(t))}{\xi(t,y(t))}\| \\
&\leq \frac{|\beta|}{|1+\alpha|}\|\frac{\xi(t,y(t))-\xi(t,x(t))}{\xi(t,x(t))\xi(t,y(t))}\|+\|\frac{\eta(t,x(t))\xi(t,y(t))-\eta(t,y(t))\xi(t,x(t))}{\xi(t,x(t))\xi(t,y(t))}\| \\
&\leq \frac{|\beta|}{|1+\alpha|}\|\xi(t,y(t))-\xi(t,x(t))\|+\|\eta(t,x(t))\xi(t,y(t))-\eta(t,y(t))\xi(t,x(t))\| \\
&\leq \frac{|\beta|}{|1+\alpha|}\|\xi(t,y(t))-\xi(t,x(t))\|+\|\eta(t,x(t))\xi(t,y(t))-\eta(t,x(t))\xi(t,x(t))\| \\
&\quad +\|\eta(t,x(t))\xi(t,x(t))-\eta(t,y(t))\xi(t,x(t))\| \\
&\leq \frac{|\beta|}{|1+\alpha|}\|\xi(t,y(t))-\xi(t,x(t))\|+\|\eta(t,x(t))\|\|\xi(t,y(t))-\xi(t,x(t))\| \\
&\quad +\|\xi(t,x(t))\|\|\eta(t,x(t))-\eta(t,y(t))\|
\end{aligned}
$$

$$
\therefore \|(\mathscr{F}x)(t)-(\mathscr{F}y)(t)\| \leq [\frac{|\beta|}{|1+\alpha|}\delta_\xi+\gamma_\eta\delta_\xi+\gamma_\xi\delta_\eta]\|x-y\|.
$$

Hence, the operator $\mathscr{F}$ is Lipschitz with constant $[\frac{|\beta|}{|1+\alpha|}+\gamma_\eta]\delta_\xi+\gamma_\xi\delta_\eta$. By proposition (2.3), $\mathscr{F}$ is also $\alpha-$Lipschitz with the same constant. □

Lemma 3.3. *$\mathscr{G}:\mathscr{C}(\mathscr{J},\mathscr{X})\to\mathscr{C}(\mathscr{J},\mathscr{X})$ defined as*

$$
\begin{aligned}
(\mathscr{G}x)(t) &= -\frac{\alpha}{(1+\alpha)\xi(t,x(t))}\cdot\frac{1}{\Gamma(q)}\int_0^T(T-s)^{q-1}\zeta(s,x(s))ds \\
&\quad +\frac{1}{\Gamma(q)\xi(t,x(t))}\int_0^t(t-s)^{q-1}\zeta(s,x(s))ds, \qquad t\in\mathscr{J}
\end{aligned}
$$

is continuous. Moreover, $\mathscr{G}$ satisfies the following growth condition:

$$
\|(\mathscr{G}x)\| \leq \frac{(2\alpha+1)T^q}{(1+\alpha)\Gamma(q+1)}\frac{\gamma_\zeta}{\gamma_\xi} \tag{10}
$$

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of a bounded subset $\mathscr{B} \subseteq \mathscr{C}(\mathscr{J},\mathscr{X})$ such that $x_n \to x$ in $\mathscr{B}$. Then,

$$\begin{aligned}
\|\mathscr{G}x_n - \mathscr{G}x\| = \| &- \frac{\alpha}{(1+\alpha)\xi(t,x_n(t))} \cdot \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1}\zeta(s,x_n(s))ds \\
&+ \frac{1}{\Gamma(q)\xi(t,x_n(t))} \int_0^t (t-s)^{q-1}\zeta(s,x_n(s))ds + \frac{\alpha}{(1+\alpha)\xi(t,x(t))} \cdot \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1}\zeta(s,x(s))ds \\
&- \frac{1}{\Gamma(q)\xi(t,x(t))} \int_0^t (t-s)^{q-1}\zeta(s,x(s))ds\| \\
\leq \frac{\alpha}{(1+\alpha)\Gamma(q)} &\| \frac{1}{\xi(t,x_n(t))} \int_0^T (T-s)^{q-1}\zeta(s,x_n(s))ds - \frac{1}{\xi(t,x(t))} \int_0^T (T-s)^{q-1}\zeta(s,x(s))ds\| \\
&+ \frac{1}{\Gamma(q)} \| \frac{1}{\xi(t,x_n(t))} \int_0^t (t-s)^{q-1}\zeta(s,x_n(s))ds - \frac{1}{\xi(t,x(t))} \int_0^t (t-s)^{q-1}\zeta(s,x(s))ds\|
\end{aligned}$$

By means of the Lebesgue Dominated Convergence Theorem, for each $t \in \mathscr{J}$ and $s \in [0,t]$, we get $\|\frac{1}{\xi(t,x_n(t))}\int_0^t (t-s)^{q-1}\zeta(s,x_n(s))ds - \frac{1}{\xi(t,x(t))}\int_0^t (t-s)^{q-1}\zeta(s,x(s))ds\| \to 0$. Therefore, for all $t \in \mathscr{J}$, $\|\mathscr{G}x_n - \mathscr{G}x\| \to 0$ as $n \to \infty$ which means that $\mathscr{G}$ is continuous. It is easy to get relation (10) as a simple consequence of (H2) as follow:

$$\begin{aligned}
\|(\mathscr{G}x)(t)\| &\leq \| \frac{\alpha}{(1+\alpha)\xi(t,x(t))} \cdot \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1}\zeta(s,x(s))ds\| \\
&\quad + \| \frac{1}{\Gamma(q)\xi(t,x(t))} \int_0^t (t-s)^{q-1}\zeta(s,x(s))ds\| \\
&\leq \frac{\alpha}{(1+\alpha)\Gamma(q)} \| \frac{1}{\xi(t,x(t))} \int_0^T (T-s)^{q-1}\zeta(s,x(s))ds\| \\
&\quad + \frac{1}{\Gamma(q)} \| \frac{1}{\xi(t,x(t))} \int_0^t (t-s)^{q-1}\zeta(s,x(s))ds\| \\
&\leq \frac{\alpha}{(1+\alpha)\Gamma(q)} \frac{\gamma_\zeta}{\gamma_\xi} \int_0^T (T-s)^{q-1}ds + \frac{1}{\Gamma(q)} \frac{\gamma_\zeta}{\gamma_\xi} \int_0^t (t-s)^{q-1}ds \\
&\leq \frac{\alpha T^q}{(1+\alpha)\Gamma(q+1)} \frac{\gamma_\zeta}{\gamma_\xi} + \frac{t^q}{\Gamma(q+1)} \frac{\gamma_\zeta}{\gamma_\xi}
\end{aligned}$$

For $0 \leq t \leq T$, we get

$$\|(\mathscr{G}x)\| \leq \frac{(2\alpha+1)T^q}{(1+\alpha)\Gamma(q+1)} \frac{\gamma_\zeta}{\gamma_\xi}$$

□

Lemma 3.4. *The operator $\mathscr{G}: \mathscr{C}(\mathscr{J},\mathscr{X}) \to \mathscr{C}(\mathscr{J},\mathscr{X})$ is compact. Consequently, $\mathscr{G}$ is $\alpha-$Lipschitz with zero constant.*

Proof. Consider a bounded subset $\mathscr{M} \subseteq \mathscr{C}(\mathscr{J}, \mathscr{X})$, then we have to show that $\mathscr{G}(\mathscr{M})$ is relatively compact in $\mathscr{C}(\mathscr{J}, \mathscr{X})$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of a bounded subset $\mathscr{M} \subseteq \mathscr{C}(\mathscr{J}, \mathscr{X})$.

$$\begin{aligned}
\|(\mathscr{G}x_n)(t)\| &\leq \| \frac{-\alpha}{(1+\alpha)\xi(t,x_n(t))} \cdot \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \zeta(s,x_n(s))ds \\
&\quad + \frac{1}{\Gamma(q)\xi(t,x_n(t))} \int_0^t (t-s)^{q-1} \zeta(s,x_n(s))ds \| \\
&\leq \| \frac{\alpha}{(1+\alpha)\Gamma(q)} \cdot \frac{1}{\xi(t,x_n(t))} \int_0^T (T-s)^{q-1} \zeta(s,x_n(s))ds \| \\
&\quad + \| \frac{1}{\Gamma(q)} \cdot \frac{1}{\xi(t,x_n(t))} \int_0^t (t-s)^{q-1} \zeta(s,x_n(s))ds \| \\
&\leq \frac{\alpha}{(1+\alpha)\Gamma(q)} \cdot \frac{\gamma_\zeta}{\gamma_\xi} \int_0^T (T-s)^{q-1} ds + \frac{1}{\Gamma(q)} \cdot \frac{\gamma_\zeta}{\gamma_\xi} \int_0^t (t-s)^{q-1} ds \\
&\leq \frac{\alpha}{(1+\alpha)\Gamma(q)} \cdot \frac{\gamma_\zeta}{\gamma_\xi} (\frac{T^q}{q}) + \frac{1}{\Gamma(q)} \cdot \frac{\gamma_\zeta}{\gamma_\xi} (\frac{t^q}{q}) \\
\therefore \|(\mathscr{G}x_n)(t)\| &\leq [\frac{\alpha T^q + (1+\alpha)t^q}{(1+\alpha)\Gamma(q+1)}] \frac{\gamma_\zeta}{\gamma_\xi} := K
\end{aligned}$$

For all $t \in \mathscr{J}$, we get $\mathscr{G}$ map bounded sets into bounded sets in $\mathscr{C}(\mathscr{J}, \mathscr{X})$.

Now, we show that $\{\mathscr{G}x_n\}$ is equicontinuous. For $t_1, t_2 \in \mathscr{J}$ and $0 \leq t_1 \leq t_2 \leq 1$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence of a bounded subset $\mathscr{M} \subseteq \mathscr{C}(\mathscr{J}, \mathscr{X})$. Then,

$$\|(\mathscr{G}x_n)(t_2) - (\mathscr{G}x_n)(t_1)\| = \| - \frac{\alpha}{(1+\alpha)\xi(t_2,x_n(t_2))} \cdot \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \zeta(s,x_n(s))ds$$

$$+ \frac{1}{\Gamma(q)\xi(t_2,x_n(t_2))} \int_0^{t_2} (t_2-s)^{q-1} \zeta(s,x_n(s))ds + \frac{\alpha}{(1+\alpha)\xi(t_1,x_n(t_1))}$$

$$\cdot \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \zeta(s,x_n(s))ds - \frac{1}{\Gamma(q)\xi(t_1,x_n(t_1))} \int_0^{t_1} (t_1-s)^{q-1} \zeta(s,x_n(s))ds \|$$

$$\leq \frac{\alpha}{(1+\alpha)\Gamma(q)} \| \frac{1}{\xi(t_2,x_n(t_2))} \int_0^T (T-s)^{q-1} \zeta(s,x_n(s))ds - \frac{1}{\xi(t_1,x_n(t_1))} \int_0^T (T-s)^{q-1} \zeta(s,x_n(s))ds \|$$

$$+ \| \frac{1}{\Gamma(q)} \frac{1}{\xi(t_2,x_n(t_2))} \int_0^{t_1} (t_2-s)^{q-1} \zeta(s,x_n(s))ds - \frac{1}{\Gamma(q)} \frac{1}{\xi(t_1,x_n(t_1))}$$

$$\cdot \int_0^{t_1} (t_1-s)^{q-1} \zeta(s,x_n(s))ds \| + \| \frac{1}{\Gamma(q)} \frac{1}{\xi(t_2,x_n(t_2))} \int_{t_1}^{t_2} (t_2-s)^{q-1} \zeta(s,x_n(s))ds \|$$

$$\leq \frac{\alpha}{(1+\alpha)\Gamma(q)}\|\frac{\xi(t_1,x_n(t_1))-\xi(t_2,x_n(t_2))}{\xi(t_1,x_n(t_1))\xi(t_2,x_n(t_2))}\int_0^T(T-s)^{q-1}\zeta(s,x_n(s))ds\|$$
$$+\frac{1}{\Gamma(q)}\|\frac{1}{\xi(t_1,x_n(t_1))\xi(t_2,x_n(t_2))}\int_0^{t_1}[\xi(t_1,x_n(t_1))(t_2-s)^{q-1}-\xi(t_2,x_n(t_2))(t_1-s)^{q-1}]\zeta(s,x_n(s))ds\|$$
$$+\frac{1}{\Gamma(q)}\|\frac{1}{\xi(t_2,x_n(t_2))}\int_{t_1}^{t_2}(t_2-s)^{q-1}\zeta(s,x_n(s))ds\|$$

As $t_2 \to t_1$, we can get $\|(\mathscr{G}x_n)(t_2)-(\mathscr{G}x_n)(t_1)\| \to 0$ that means $\{\mathscr{G}x_n\}$ is equicontinuous.

As consequence of the above results together with the Arzela Ascoli theorem, we can get $\mathscr{G}$ is a relatively compact. By Proposition (2.2), $\mathscr{G}$ is $\alpha-$Lipschitz with zero constant.

□

Theorem 3.1. *Assume that hypotheses (H1)-(H3) hold, and*

$$[\frac{|\beta|}{|1+\alpha|}+\gamma_\eta]\delta_\xi+\gamma_\xi\delta_\eta<1 \tag{11}$$

then, the FHDE (1) has at least one solution $x \in \mathscr{C}(\mathscr{J},\mathscr{X})$, and the set of the solution is bounded in $\mathscr{C}(\mathscr{J},\mathscr{X})$.

Proof. Define operator $\mathscr{T}:\mathscr{C}(\mathscr{J},\mathscr{X})\to\mathscr{C}(\mathscr{J},\mathscr{X})$ as $\mathscr{T}=\mathscr{F}+\mathscr{G}$. It is clear that $\mathscr{T}$ is well defined. Then, the fractional integral equation (7) can be written as the following operator equation:

$$\mathscr{T}x=\mathscr{F}x+\mathscr{G}x \tag{12}$$

Thus, the existence of a solution for FHDE (1) is also equivalent to the existence of a fixed point satisfies the operator $\mathscr{T}$. As we proved in Lemma (3.2), $\mathscr{F}$ is $\alpha-$ Lipschitz with constant $[\frac{|\beta|}{|1+\alpha|}+\gamma_\eta]\delta_\xi+\gamma_\xi\delta_\eta$, and by Lemma (3.4), $\mathscr{G}$ is $\alpha-$ Lipschitz with zero constant. Consequently, $\mathscr{T}$ is $\alpha-$ Lipschitz with constant $[\frac{|\beta|}{|1+\alpha|}+\gamma_\eta]\delta_\xi+\gamma_\xi\delta_\eta$. Hence, $\mathscr{T}$ is a strict $\alpha-$ Contraction with constant $[\frac{|\beta|}{|1+\alpha|}+\gamma_\eta]\delta_\xi+\gamma_\xi\delta_\eta$. By assumption, we have $[\frac{|\beta|}{|1+\alpha|}+\gamma_\eta]\delta_\xi+\gamma_\xi\delta_\eta<1$, which implies $\mathscr{T}$ is $\alpha-$ Condensing. Now, Consider the following set

$$S=\{x\in\mathscr{C}(\mathscr{J},\mathscr{X}): \textit{there exist } \lambda\in[0,1],\ \textit{such that } x=\lambda\mathscr{T}x\}$$

We need to prove S is bounded. For $x \in S$, we have

$$\|x\|=\|\lambda\mathscr{T}x\|=\|\lambda[\mathscr{F}x+\mathscr{G}x]\|$$

$$= \lambda \| \frac{\beta}{(1+\alpha)\xi(t,x(t))} + \frac{\eta(t,x(t))}{\xi(t,x(t))} - \frac{\alpha}{(1+\alpha)\xi(t,x(t))} \cdot \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \zeta(s,x(s))ds$$

$$+ \frac{1}{\Gamma(q)\xi(t,x(t))} \int_0^t (t-s)^{q-1} \zeta(s,x(s))ds \|$$

$$\leq \lambda [\frac{\beta}{(1+\alpha)\gamma_\xi} + \frac{\gamma_\eta}{\gamma_\xi} + \frac{(2\alpha+1)T^q}{(1+\alpha)\Gamma(q+1)} \frac{\gamma_\zeta}{\gamma_\xi}]$$

The above inequality with $q \in (0,1)$ show that S is bounded. Thus $\mathscr{T}$ has at least one fixed point, which corresponds to a solution of FHDE (1). Then the set of solutions is bounded. $\square$

Example 3.1. *Consider the following problem:*

$$\begin{cases} {}^c\mathscr{D}^{\frac{1}{2}}[x(t)(3+\frac{cost}{4}|x(t)|) - \frac{1}{12e^t}\frac{1}{1+|x(t)|}] = \frac{2}{\sqrt{\pi}}\sqrt{x(t)}, \qquad t \in [0,\pi] \\ x(0) + 2x(\pi) - \frac{1}{4}x(\pi)|x(\pi)| = \frac{1}{12e^\pi}\frac{1}{1+|x(\pi)|} - \frac{1}{12}\frac{1}{1+|x(0)|} + \frac{11}{4} \end{cases} \tag{13}$$

Set $\xi(t,x) = 3 + \frac{cost}{4}|x(t)|$, $\eta(t,x) = \frac{1}{12e^t}\frac{1}{1+|x(t)|}$, $\zeta(t,x) = \frac{2}{\sqrt{\pi}}\sqrt{x(t)}$, $\alpha = 2$ and $\beta = \frac{11}{4}$. We have, for $t \in [0,\pi]$

$$|\xi(t,x)| = |3 + \frac{cost}{4}|x(t)|| \geq 3 \Rightarrow \gamma_\xi = 3,$$

$$|\eta(t,x)| = |\frac{1}{12e^t}\frac{1}{1+|x(t)|}| \leq \frac{1}{12} \Rightarrow \gamma_\eta = \frac{1}{12},$$

$$|\xi(t,x) - \xi(t,y)| = \frac{cost}{4}|x(t)-y(t)| \Rightarrow \delta_\xi = \frac{1}{4},$$

$$|\eta(t,x) - \eta(t,y)| = \frac{1}{12e^t}|\frac{x(t)-y(t)}{(1+|x(t)|)(1+|y(t)|)}| \leq \frac{1}{12}|x(t)-y(t)| \Rightarrow \delta_\eta = \frac{1}{12}.$$

Then,

$$[\frac{|\beta|}{|1+\alpha|} + \gamma_\eta]\delta_\xi + \gamma_\xi \delta_\eta = [\frac{(\frac{11}{4})}{1+2} + \frac{1}{12}](\frac{1}{4}) + 3(\frac{1}{12}) = \frac{1}{2} < 1$$

Obviously, all assumptions in Theorem (3.1) are satisfied so the given problem (13) has at least one solution.

CONCLUSION

The present paper studies some necessary conditions for the existence and uniqueness of solution to a fractional Hybrid differential equation problem. It deals with the Lebesgue Dominated Convergence Theorem and the Arzela Ascoli Theorem, as well as the topological techniques of approximation solutions, which are to obtain result. At last, an example has been given to understand our results.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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