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COMMON FIXED POINT RESULTS USING AN INTEGRAL TYPE CONTRACTIVE CONDITION ON S-METRIC SPACES

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Abstract. In this article, we adopt an integral type contraction to find fixed point results for four self mappings, which are weakly compatible in S-metric spaces. For this purpose, we use (E.A) / (CLR) - property alternatively. We provide befitting examples to justify our results.

Keywords: coincidence points; common fixed points; (E.A) property; weak compatibility; (CLR) property.

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1. INTRODUCTION

In 1986, the notion of compatibility was introduced by Gerald Jungck [3] as a generalization of commutative property. Later on, Jungck and Rhoades [4] came up with the idea of weak compatibility of mappings. They also proved that a pair of mappings which is compatible is always weakly compatible, but the other way not around. Aamri and Moutawakil [1], on the other hand, provided a new idea of (E.A) property in 2002. By applying this, a numerous results in fixed point theory have been established. As an alternative to (E.A) property, Sintunavarat and Kumam [9] recently introduced common limit in the range property, simply noted by (CLR)

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property, because of which the range of the mapping need not be closed for proving fixed point theorems.

In 2006, Z.Mustafa and B.sims [6] proposed G-metric space as an alternative and more appropriate generalization of metric spaces. Recently, S.Sedghi et al. [7] further expanded this concept and proposed a new class of metric spaces, that is, an S-metric space. Numerous fixed point results existing in the literature are inherently viable in S-metric spaces, as can be seen. The study of fixed points with integral type contractive condition has gotten a lot of attention in recent years. Certain existence outcomes in fixed point theory for a single mapping of a complete metric space with integral type inequality were shown by Branciari [2]. He proved the presence of a single fixed point of a self map on a complete metric space that meets a general integral type contractive condition, thereby generalising the Banach contraction principle. P.Vijayaraju et al.[10], on the other hand, found fixed point solutions for a pair of mappings with an integral type contraction. J.Kumar [5] extended these results to four self mappings with (E.A) and (CLR) properties. For this purpose, an integral type contraction was applied.

Inspired by the work of several authors, (see, e.g. [2], [5] and [10]), we prove certain new fixed point theorems for four self maps with pairwise (E.A) and (CLR) properties. In fact,we further expand and validate the findings of J.Kumar to S-metric spaces in this work. All of our assertions are supported by befitting examples.

2. PRELIMINARIES

Definition 2.1. [7] A function S: $X^3 \to [0,\infty)$ where X is a nonempty set is said to be an S-metric if for each $\upsilon,\nu,\omega,l \in X$,

(1) $S(\upsilon,\nu,\omega) = 0$ iff $\upsilon = \nu = \omega$,

(2) $S(\upsilon,\nu,\omega) \leq S(\upsilon,\upsilon,l)+S(\nu,\nu,l)+S(\omega,\omega,l)$.

The pair (X,S) called an S-metric space.

Example 2.2. [8] Let $X=\mathbb{R}$ and $S:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be a function defined by $S(\upsilon,\nu,\omega) = |\upsilon-\omega|+|\nu-\omega|$ for all $\upsilon,\nu,\omega\in\mathbb{R}$. Then S is an S-metric.

Lemma 2.3. [7] *Let X be an S-metric space. Then for all* $\upsilon,\nu \in X$, $S(\upsilon,\upsilon,\nu) = S(\nu,\nu,\upsilon)$.

Lemma 2.4. [7] *Let X be an S-metric space. If* $\{\upsilon_n\}$ *and* $\{\nu_n\}$ *are two sequences such that* $\upsilon_n \to \upsilon$ *and* $\nu_n \to \nu$*, then* $S(\upsilon_n, \upsilon_n, \nu_n) \to S(\upsilon, \upsilon, \nu)$.

Definition 2.5. [7] A sequence $\{\upsilon_n\}$ is an S-metric space X is said to converge to some υ in X if and only if $\lim_{n\to\infty} S(\upsilon_n, \upsilon_n, \upsilon) = 0$. In this case we write $\lim_{n\to\infty} \upsilon_n = \upsilon$.

Definition 2.6. Let X be an S-metric space. Then two mappings P and Q defined on X are said to

(1) be weakly compatible [4], if $\upsilon \in X$, $P\upsilon = Q\upsilon$ implies $PQ\upsilon = QP\upsilon$.

(2) satisfy property (E.A) [1], if there exists a sequence $\{\upsilon_n\}$ in X such that $\lim\limits_{n\to\infty} P\upsilon_n = \lim\limits_{n\to\infty} Q\upsilon_n = \upsilon, \upsilon \in X$.

(3) satisfy the common limit in the range of P (CLR_P) property [9], if there exists a sequence $\{\upsilon_n\}$ in X such that $\lim\limits_{n\to\infty} P\upsilon_n = \lim\limits_{n\to\infty} Q\upsilon_n = P\upsilon$, $\upsilon \in X$.

Example 2.7. Let $X = \mathbb{R}_+$ and let the mappings P and $Q : X \to X$ be defined by $P\upsilon = 2 + \upsilon^2$ and $Q\upsilon = 2^\upsilon + 1$ for all $\upsilon \in X$. Let the S-metric, S: $X^3 \to [0, \infty)$ be defined as in *Example 2.2.* Consider the sequence $\upsilon_n = \frac{1}{n\sqrt{n}}, n \in N$. Then

$$P\upsilon_n = 2 + \frac{1}{n^3} \text{ and } Q\upsilon_n = 2^{1/n\sqrt{n}} + 1,$$

$$S(P\upsilon_n, P\upsilon_n, 2) = S\left(2 + \frac{1}{n^3}, 2 + \frac{1}{n^3}, 2\right) = \frac{2}{n^3} \to 0 \text{ as } n \to \infty,$$

$$S(Q\upsilon_n, Q\upsilon_n, 2) = S\left(2^{1/n\sqrt{n}} + 1, 2^{1/n\sqrt{n}} + 1, 2\right)$$
$$= 2|2^{1/n\sqrt{n}} - 1| \to 0 \text{ as } n \to \infty.$$

Therefore, $\lim\limits_{n\to\infty} P\upsilon_n = \lim\limits_{n\to\infty} Q\upsilon_n = 2$, which implies (P, Q) satisfies (E.A) property.

Example 2.8. Let $X = \mathbb{R}_+$ and let the mappings P and $Q : X \to X$ be defined by $P\upsilon = e^\upsilon$ and $Q\upsilon = \upsilon^2 + 1$. Let the S-metric, S: $X^3 \to [0, \infty)$ be defined as in *Example 2.2.* Consider the sequence $\upsilon_n = \frac{1}{n}, n \in N$.Then

$$P\upsilon_n = e^{1/n} \text{ and } Q\upsilon_n = 1 + \frac{1}{n^2},$$

$$S(P\upsilon_n, P\upsilon_n, 1) = S\left(e^{1/n}, e^{1/n}, 1\right) = 2|e^{1/n} - 1| \to 0 \text{ as } n \to \infty,$$

$$S(Q\upsilon_n, Q\upsilon_n, 1) = S\left(1 + \frac{1}{n^2}, 1 + \frac{1}{n^2}, 1\right) = \frac{2}{n^2} \to 0 \text{ as } n \to \infty.$$

Therefore, $\lim_{n\to\infty} P\upsilon_n = \lim_{n\to\infty} Q\upsilon_n = 1 = P(0)$, which implies (P,Q) satisfies the (CLR_P)-property.

Throughout this paper, $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-negative, lebesgue integrable function which is summable and such that $\int_0^{\varepsilon} \lambda(\vartheta)d\vartheta > 0$ whenever $\varepsilon > 0$ and $\xi : [0,\infty) \to [0,\infty)$ is a right continuous function such that $\xi(0) = 0$ and $\xi(t) < t$ for $t > 0$.

3. Main Results

Theorem 3.1. *Let X be an S-metric space and F,G,P and Q be four self maps defined on X satisfying the following conditions,*

(1)

$$\int_0^{S(Fx,Fx,Gy)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(x,y)} \lambda(\vartheta)d\vartheta\right) \text{ for all } x,y \in X \text{ where}$$

(3.1.1)
$$\mu(x,y) = max\{S(Qx,Qx,Py), S(Qx,Qx,Fx), S(Py,Py,Gy), \frac{1}{2}[S(Qx,Qx,Gy) + S(Py,Py,Fx)]\},$$

(2) $F(X) \subseteq P(X)$, $G(X) \subseteq Q(X)$ and Q(X) or P(X) is closed,

(3)The pairs (F,Q) or (G,P) satisfy property (E.A),

(4) The pairs (F,Q) and (G,P) are weakly compatible.

Then F,G,P and Q have a unique common fixed point in X.

Proof. Firstly, we assume that the pair (F,Q) satisfies property (E.A) .
Therefore there must be a sequence $\{x_n\}$ in X such that

(3.1.2)
$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Qx_n = r,\ r \in X.$$

Given that $F(X) \subseteq P(X)$.Therefore $Fx_n = Py_n$ for each n, for some sequence $\{y_n\}$ in X. Hence

(3.1.3)
$$\lim_{n\to\infty} Py_n = r.$$

Now, we will prove that $\lim_{n\to\infty} Gy_n = r$.
To do this, we put $x = x_n$, $y = y_n$ in (3.1.1).Then

$$\int_0^{S(Fx_n,Fx_n,Gy_n)} \lambda(\vartheta)\,d\vartheta \le \xi\left(\int_0^{\mu(x_n,y_n)} \lambda(\vartheta)\,d\vartheta\right) \text{ for all } n \in N \text{ where,}$$

$$\mu(x_n,y_n) = max\{S(Qx_n,Qx_n,Py_n), S(Qx_n,Qx_n,Fx_n), S(Py_n,Py_n,Gy_n), \frac{1}{2}[S(Qx_n,Qx_n,Gy_n)+S(Py_n,Py_n,Fx_n)]\}.$$

On letting $n \to \infty$ and using (3.1.2) and (3.1.3), we will have

$$\begin{aligned}\lim_{n\to\infty} \mu(x_n,y_n) &= \lim_{n\to\infty} max\left\{0,0,S(r,r,Gy_n),\frac{1}{2}S(r,r,Gy_n)\right\} \\ &= \lim_{n\to\infty} S(r,r,Gy_n).\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\int_0^{S(Fx_n,Fx_n,Gy_n)} \lambda(\vartheta)\,d\vartheta &\le \lim_{n\to\infty}\xi\left(\int_0^{\mu(x_n,y_n)} \lambda(\vartheta)\,d\vartheta\right) \\ &= \lim_{n\to\infty}\xi\left(\int_0^{S(r,r,Gy_n)} \lambda(\vartheta)\,d\vartheta\right).\end{aligned}$$

This implies

$$\begin{aligned}\lim_{n\to\infty}\int_0^{S(r,r,Gy_n)} \lambda(\vartheta)\,d\vartheta &\le \lim_{n\to\infty}\xi\left(\int_0^{S(r,r,Gy_n)} \lambda(\vartheta)\,d\vartheta\right) \\ &< \lim_{n\to\infty}\int_0^{S(r,r,Gy_n)} \lambda(\vartheta)\,d\vartheta, \text{ if } \lim_{n\to\infty} S(r,r,Gy_n) \neq 0,\end{aligned}$$

a contradiction.

Therefore we must have $\lim_{n\to\infty} S(r,r,Gy_n) = 0$, which implies

$$\lim_{n\to\infty} Gy_n = r. \tag{3.1.4}$$

Suppose $Q(X)$ is closed.

Therefore by (3.1.2), we can find a point $u \in X$ such that

$$r = Qu. \tag{3.1.5}$$

We now claim that $Fu = r$.

To prove this, we put $x = u$, $y = y_n$ in (3.1.1). Then

$$\int_0^{S(Fu,Fu,Gy_n)} \lambda(\vartheta)\, d\vartheta \le \xi\left(\int_0^{\mu(u,y_n)} \lambda(\vartheta)\, d\vartheta\right) \text{ for all for all } n \in N \text{ where}$$

$$\mu(u,y_n) = \max\{S(Qu,Qu,Py_n), S(Qu,Qu,Fu), S(Py_n,Py_n,Gy_n), \frac{1}{2}[S(Qu,Qu,Gy_n)+S(Py_n,Py_n,Fu)]\}.$$

On letting $n \to \infty$ and using (3.1.3),(3.1.4) and (3.1.5)

$$\lim_{n\to\infty} \mu(u,y_n) = max\left\{0, S(r,r,Fu), 0, \frac{1}{2}S(r,r,Fu)\right\} = S(r,r,Fu).$$

Therefore,

$$\begin{aligned}\int_0^{S(Fu,Fu,r)} \lambda(\vartheta)d\vartheta &= \lim_{n\to\infty}\int_0^{S(Fu,Fu,Gy_n)} \lambda(\vartheta)d\vartheta \\ &\le \xi\left(\int_0^{S(r,r,Fu)} \lambda(\vartheta)d\vartheta\right) \\ &< \int_0^{S(r,r,Fu)} \lambda(\vartheta)d\vartheta, \text{ if } S(Fu,Fu,r) \ne 0,\end{aligned}$$

a contradiction and hence we must have $S(Fu,Fu,r) = 0$.

This implies

$$Fu = r. \tag{3.1.6}$$

From (3.1.5) and (3.1.6),

$$r = Qu = Fu. \tag{3.1.7}$$

(3.1.8) Since $r \in F(X) \subseteq P(X), r = Pv$ for some element v in X.

We now prove that $Gv = r$.

To prove this,we put $x = u$, $y = v$ in (3.1.1). Then

$$\int_0^{S(Fu,Fu,Gv)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(u,v)} \lambda(\vartheta)d\vartheta\right) \text{ where}$$

$$\begin{aligned}\mu(u,v) &= max\{S(Qu,Qu,Pv),S(Qu,Qu,Fu),S(Pv,Pv,Gv),\\ &\frac{1}{2}[S(Qu,Qu,Gv)+S(Pv,Pv,Fu)]\}\\ &= max\{0,0,S(r,r,Gv),\frac{1}{2}S(r,r,Gv)\}\\ &= S(r,r,Gv), \text{ by using (3.1.7) and (3.1.8)}\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^{S(r,r,Gv)} \lambda(\vartheta)d\vartheta &\leq \xi\left(\int_0^{S(r,r,Gv)} \lambda(\vartheta)d\vartheta\right)\\ &< \int_0^{S(r,r,Gv)} \lambda(\vartheta)d\vartheta, \text{ if } S(r,r,Gv)\neq 0\end{aligned}$$

which is a contradiction, Hence

(3.1.9) $$r = Gv.$$

From (3.1.8) and (3.1.9),

(3.1.10) $$r = Pv = Gv.$$

Similarly we can prove that (3.1.7) and (3.1.10) hold whenever $P(X)$ is closed. It is given that (F,Q) and (G,P) are weakly compatible.

Therefore from (3.1.7) and (3.1.10),we have $QFu = FQu$ and $PGv = GPv$.

This implies $Qr = Fr$ and $Pr = Gr$.

We now prove that $Gr = r$.

This can be done by taking $x = u$ and $y = r$ in (3.1.1) and using (3.1.7) and (3.1.10). Then we get

$$\int_0^{S(Fu,Fu,Gr)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(u,r)} \lambda(\vartheta)d\vartheta\right) \text{ where}$$

$$\begin{aligned}\mu(u,r) &= max\{S(Qu,Qu,Pr),S(Qu,Qu,Fu),S(Pr,Pr,Gr),\\ &\frac{1}{2}[S(Qu,Qu,Gr)+S(Pr,Pr,Fu)]\}\end{aligned}$$

$$= max\{S(r,r,Gr),0,0,\frac{1}{2}[S(r,r,Gr)+S(Gr,Gr,r)]\}$$
$$= max\{S(r,r,Gr),\frac{1}{2}[S(r,r,Gr)+S(Gr,Gr,r)]\}$$
$$= S(r,r,Gr).$$

Then,

$$\int_0^{S(r,r,Gr)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{S(r,r,Gr)} \lambda(\vartheta)d\vartheta\right) < \int_0^{S(r,r,Gr)} \lambda(\vartheta)d\vartheta, \text{ if } S(r,r,Gr) \neq 0,$$

a contradiction and hence $Gr = r$.

This implies $Qr = Gr = r$.

Similarly,it is easy to prove that $Pr = Fr = r$.

This implies $Pr = Fr = Qr = Gr = r$.

Therefore, r is a common fixed point of F,G,P and Q.

In order to establish the uniqueness of 'r', assume that $r^*(r \neq r^*)$ be other common fixed point of F,G,P and Q.

Then $Pr^* = Fr^* = Qr^* = Gr^* = r^*$.

By (3.1.1), we will get

$$\int_0^{S(r,r,r^*)} \lambda(\vartheta)d\vartheta = \int_0^{S(Fr,Fr,Gr^*)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(r,r^*)} \lambda(\vartheta)d\vartheta\right) \text{ where,}$$

$$\mu(r,r^*) = max\{S(Qr,Qr,Pr^*),S(Qr,Qr,Fr),S(Pr^*,Pr^*,Gr^*),$$
$$\frac{1}{2}[S(Qr,Qr,Gr^*)+S(Pr^*,Pr^*,Fr)]\}$$
$$= max\{S(r,r,r^*),\frac{1}{2}[S(r,r,r^*)+S(r^*,r^*,r)]\}$$
$$= S(r,r,r^*).$$

Then,

$$\int_0^{S(r,r,r^*)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{S(r,r,r^*)} \lambda(\vartheta)d\vartheta\right) < \int_0^{S(r,r,r^*)} \lambda(\vartheta)d\vartheta.$$

This contradicts our assumption that $r \neq r^*$ and therefore we must have $r = r^*$.

Similarly the proof follows from the (E.A) property of (G,P). □

Example 3.2. Suppose that $X = [0,1]$ and the maps F, G, P and Q of X are defined by

$$F(x) = 0, \qquad P(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0,1], \end{cases}$$

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{10} & \text{if } x \in (0,1], \end{cases} \qquad Q(x) = x.$$

Let the S-metric on X be given as in *Example 2.2.* We take $\lambda(\vartheta) = 1$ and $\xi(t) = \frac{t}{2}$. Then the inequality (3.1.1) will be

$$S(Fx, Fx, Gy) \le \xi(\mu(x,y)) = \frac{1}{2}\mu(x,y), \tag{3.2.1}$$

$$\text{where } \mu(x,y) = max\{S(Qx,Qx,Py), S(Qx,Qx,Fx), S(Py,Py,Gy), \frac{1}{2}[S(Qx,Qx,Gy) + S(Py,Py,Fx)]\}.$$

Case I: If $y = 0$, then $Fx = 0, Qx = x, Py = 0, Gy = 0$.

Therefore, $S(Fx,Fx,Gy) = S(0,0,0) = 0$.

Hence, inequality (3.2.1) holds.

Case II: If $y \in (0,1]$, then $Py = 1, Gy = \frac{1}{10}, Fx = 0, Qx = x$.

$$\text{Therefore, } S(Fx,Fx,Gy) = S\left(0,0,\frac{1}{10}\right) = 2\left|0 - \frac{1}{10}\right| = \frac{1}{5}.$$

$$S(Py,Py,Gy) = S\left(1,1,\frac{1}{10}\right) = 2\left|1 - \frac{1}{10}\right| = \frac{9}{5}.$$

Therefore $S(Fx,Fx,Gy) = \frac{1}{5} < \frac{9}{10} = \frac{1}{2}S(Py,Py,Gy) \le \frac{1}{2}\mu(x,y)$.

Hence the inequality (3.2.1) holds in both the cases.

$F(X) = \{0\} \subseteq \{0,1\} = P(X)$, $G(X) = \{0, \frac{1}{10}\} \subseteq [0,1] = Q(X)$ and $Q(X)$ is closed. Also for the sequence $x_n = \frac{1}{n^3}, n = 1,2,...$

$$Fx_n = 0,\ Qx_n = \frac{1}{n^3},$$

$$S(Fx_n, Fx_n, 0) = S(0,0,0) = 0 \text{ and}$$

$$S(Qx_n, Qx_n, 0) = S\left(\frac{1}{n^3}, \frac{1}{n^3}, 0\right) = \frac{2}{n^3} \to 0 \text{ as } n \to \infty.$$

Therefore, $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Qx_n = 0$.

Thus the maps (F,Q) satisfy (E.A) property.

We can easily see that (F,Q) and (G,P) are weakly compatible.

Also, 0 is the only common fixed point of F,G,P and Q.

Corollary 3.3. *Let X be an S-metric space and F,G and P be three self maps defined on X satisfying the following conditions,*

(1)

$$\int_0^{S(Fx,Fx,Gy)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(x,y)} \lambda(\vartheta)d\vartheta\right) \text{ for all } x,y \in X \text{ where,}$$

$$\mu(x,y) = max\{S(Px,Px,Py), S(Px,Px,Fx), S(Py,Py,Gy), \frac{1}{2}[S(Px,Px,Gy)+S(Py,Py,Fx)]\},$$

(2) $F(X) \subseteq P(X)$, $G(X) \subseteq P(X)$ and P(X) is closed,

(3)The pairs (F,P) or (G,P) satisfy property (E.A),

(4) The pairs (F,P) and (G,P) are weakly compatible.

Then F,G and P have a unique common fixed point in X.

Proof. The proof follows by taking $Q = P$ in Theorem 3.1. □

Corollary 3.4. *Let (X,S) be an S-metric space and G and P be two self maps defined on X satisfying the following conditions*

(1)

$$\int_0^{S(Gx,Gx,Gy)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(x,y)} \lambda(\vartheta)d\vartheta\right) \text{ for all } x,y \in X$$

where

$$\mu(x,y) = max\{S(Px,Px,Py), S(Px,Px,Gx), S(Py,Py,Gy), \frac{1}{2}[S(Px,Px,Gy)+S(Py,Py,Gx)]\},$$

(2) $G(X) \subseteq P(X)$ and P(X) is closed,

(3) The pair (G,P) satisfies property (E.A),

(4) The pairs (G,P) is weakly compatible. Then G and P have a unique common fixed point in X.

Proof. The proof follows by taking $Q=P$ and $F=G$ in Theorem 3.1. □

Theorem 3.5. *Let X be an S-metric space and F,G,P and Q be four self maps defined on X satisfying the following conditions*

(1)

$$\int_0^{S(Fx,Fx,Gy)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(x,y)} \lambda(\vartheta)d\vartheta\right) \text{ for all } x,y\in X \text{ where,}$$

$$\mu(x,y) = max\{S(Qx,Qx,Py), S(Qx,Qx,Fx), S(Py,Py,Gy), \frac{1}{2}[S(Qx,Qx,Gy)+S(Py,Py,Fx)]\}, \tag{3.5.1}$$

(2) $F(X)\subseteq P(X)$ *and* $G(X)\subseteq Q(X)$,

(3) The pairs (F,Q) *satisfy* (CLR_F) *property or* (G,P) *satisfy* (CLR_G) *property,*

(4) The pairs (F,Q) *and* (G,P) *are weakly compatible.*

Then the maps F,G,P and Q have a unique common fixed point in X.

Proof. Firstly, we suppose that the pair (F,Q) satisfies (CLR_F) property.

Therefore,there is a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Qx_n = Fz,\ z\in X. \tag{3.5.2}$$

It is given that $F(X)\subseteq P(X)$ and therefore $Fx_n = Py_n$ for all n, for some sequence $\{y_n\}$ in X.Then

$$\lim_{n\to\infty} Py_n = Fz. \tag{3.5.3}$$

Now, we prove that $\lim_{n\to\infty} Gy_n = Fz$.

This is done by taking $x=x_n$, $y=y_n$ in (3.5.1).

Then,

$$\int_0^{S(Fx_n,Fx_n,Gy_n)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(x_n,y_n)} \lambda(\vartheta)d\vartheta\right) \text{ for all } n\in N, \text{ where,}$$

$$\mu(x_n,y_n) = max\{S(Qx_n,Qx_n,Py_n), S(Qx_n,Qx_n,Fx_n), S(Py_n,Py_n,Gy_n), \frac{1}{2}[S(Qx_n,Qx_n,Gy_n)+S(Py_n,Py_n,Fx_n)]\}.$$

On letting $n \to \infty$ and using (3.5.2) and (3.5.3),

$$\lim_{n\to\infty} \mu(x_n, y_n) = \lim_{n\to\infty} max\{S(Fz, Fz, Gy_n), \frac{1}{2}S(Fz, Fz, Gy_n)\}$$
$$= \lim_{n\to\infty} S(Fz, Fz, Gy_n).$$

$$\begin{aligned}\lim_{n\to\infty} \int_0^{S(Fz,Fz,Gy_n)} \lambda(\vartheta)d\vartheta &= \lim_{n\to\infty} \int_0^{S(Fx_n,Fx_n,Gy_n)} \lambda(\vartheta)d\vartheta \\ &\leq \lim_{n\to\infty} \xi\left(\int_0^{S(Fz,Fz,Gy_n)} \lambda(\vartheta)d\vartheta\right) \\ &< \lim_{n\to\infty} \int_0^{S(Fz,Fz,Gy_n)} \lambda(\vartheta)d\vartheta, \\ &\text{if } \lim_{n\to\infty} S(Fz, Fz, Gy_n) \neq 0,\end{aligned}$$

a contradiction. Therefore $\lim_{n\to\infty} S(Fz, Fz, Gy_n) = 0$.

This implies

(3.5.4) $$\lim_{n\to\infty} Gy_n = Fz.$$

(3.5.5) Since $F(X) \subseteq P(X)$, we will have $Fz = Pv$ for some point v in X.

We claim that $Gv = Fz$.

To prove this,we put $x = x_n$, $y = v$ in (3.5.1). Then

$$\int_0^{S(Fx_n,Fx_n,Gv)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(x_n,v)} \lambda(\vartheta)d\vartheta\right) \text{ for all } n \in N, \text{ where,}$$

$$\mu(x_n, v) = max\{S(Qx_n, Qx_n, Pv), S(Qx_n, Qx_n, Fx_n), S(Pv, Pv, Gv),$$
$$\frac{1}{2}[S(Qx_n, Qx_n, Gv) + S(Pv, Pv, Fx_n)]\}.$$

On letting $n \to \infty$ and using (3.5.2) and (3.5.5), we get

$$\lim_{n\to\infty} \mu(x_n, v) = max\{S(Fz, Fz, Gv), \frac{1}{2}S(Fz, Fz, Gv)\}$$
$$= S(Fz, Fz, Gv).$$

Therefore,

$$\begin{aligned}\int_0^{S(Fz,Fz,Gv)}\lambda(\vartheta)d\vartheta &= \lim_{n\to\infty}\int_0^{S(Fx_n,Fx_n,Gv)}\lambda(\vartheta)d\vartheta \\ &\leq \xi\left(\int_0^{S(Fz,Fz,Gv)}\lambda(\vartheta)d\vartheta\right) \\ &< \int_0^{S(Fz,Fz,Gv)}\lambda(\vartheta)d\vartheta, \text{ if } \lim_{n\to\infty} S(Fz,Fz,Gv)\neq 0,\end{aligned}$$

a contradiction.Hence

$$Fz = Gv. \tag{3.5.6}$$

From (3.5.5) and (3.5.6), we have

$$Gv = Pv = r(\text{say}). \tag{3.5.7}$$

We have $GPv = PGv$, as it is given that (G,P) is weakly compatible .

This implies

$$Gr = Pr. \tag{3.5.8}$$

Since $G(X)\subseteq Q(X)$,by (3.5.7) there must be some $u\in X$ such that

$$r = Gv = Qu. \tag{3.5.9}$$

We claim that $Fu = r$. To prove this,we take $x = u$, $y = v$ in (3.5.1).

Therefore

$$\int_0^{S(Fu,Fu,Gv)}\lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(u,v)}\lambda(\vartheta)d\vartheta\right) \text{ where,}$$

$$\begin{aligned}\mu(u,v) = max\{&S(Qu,Qu,Pv), S(Qu,Qu,Fu), S(Pv,Pv,Gv),\\ &\frac{1}{2}[S(Qu,Qu,Gv)+S(Pv,Pv,Fu)]\}.\end{aligned}$$

On using (3.5.7) and (3.5.9)

$$\begin{aligned}\mu(u,v) &= max\{S(r,r,Fu), \frac{1}{2}S(r,r,Fu)\}\\ &= S(Fu,Fu,r).\end{aligned}$$

Therefore,

$$\int_0^{S(Fu,Fu,r)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{S(Fu,Fu,r)} \lambda(\vartheta)d\vartheta\right) < \int_0^{S(Fu,Fu,r)} \lambda(\vartheta)d\vartheta, \text{ if } S(Fu,Fu,r) \neq 0,$$

a contradiction. Hence

$$Fu = r. \tag{3.5.10}$$

From (3.5.9) and (3.5.10), $Fu = Qu = r$.

We have $FQu = QFu$, as it is given that (F,Q) is weakly compatible .

This further implies

$$Fr = Qr. \tag{3.5.11}$$

We now prove that $Fr = r$. To do this,we take $x = r$, $y = v$ in (3.5.1). Then

$$\int_0^{S(Fr,Fr,Gv)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{\mu(r,v)} \lambda(\vartheta)d\vartheta\right) \text{ where}$$

$$\mu(r,v) = max\{S(Qr,Qr,Pv), S(Qr,Qr,Fr), S(Pv,Pv,Gv), \frac{1}{2}[S(Qr,Qr,Gv) + S(Pv,Pv,Fr)]\}.$$

On (3.5.7) and (3.5.11),we get $\mu(r,v) = S(Fr,Fr,r)$.

Therefore,

$$\int_0^{S(Fr,Fr,r)} \lambda(\vartheta)d\vartheta \leq \xi\left(\int_0^{S(Fr,Fr,r)} \lambda(\vartheta)d\vartheta\right) < \int_0^{S(Fr,Fr,r)} \lambda(\vartheta)d\vartheta \text{ if } S(Fr,Fr,r) \neq 0,$$

a contradiction. Hence $Fr = r$.

similarly we can easily see that $Gr = r$.

From (3.5.8) and (3.5.11), $Fr = Qr = Gr = Pr = r$.

Hence r is the common fixed point of F, G, P and Q.

From the inequality (3.5.1), we can easily see that 'r' is unique. □

Example 3.6. Let $X = [0,1)$ and define the maps F, G, P and Q of X by

$$F(x) = 0, \quad G(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{10} & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

$$P(x) = \begin{cases} \text{x} & \text{if } x \in [0, \frac{1}{2}) \\ \frac{9}{10} & \text{if } x \in [\frac{1}{2}, 1), \end{cases} \quad Q(x) = x.$$

Let the S-metric on X be given as in *Example 2.2.* We take $\lambda(\vartheta) = 1$ and $\xi(t) = \frac{t}{2}$. Then the inequality (3.5.1) will be

$$S(Fx, Fx, Gy) \leq \xi(\mu(x,y)) = \frac{1}{2}\mu(x,y), \tag{3.6.1}$$

$$\text{where } \mu(x,y) = max\{S(Qx,Qx,Py), S(Qx,Qx,Fx), S(Py,Py,Gy), \frac{1}{2}[S(Qx,Qx,Gy) + S(Py,Py,Fx)]\}.$$

Case I: If $y \in [0, \frac{1}{2})$, then $Py = y, Gy = 0, Fx = 0, Qx = x$.

$$\text{Therefore, } S(Fx, Fx, Gy) = s(0,0,0) = 0$$

Hence, inequality (3.6.1) holds.

Case II: If $y \in [\frac{1}{2}, 1)$,then $Fx = 0, Qx = x, Py = \frac{9}{10}, Gy = \frac{1}{10}$.

$$\text{Therefore, } S(Fx, Fx, Gy) = S\left(0, 0, \frac{1}{10}\right) = 2\left|0 - \frac{1}{10}\right| = \frac{1}{5}$$

$$S(Py, Py, Gy) = S\left(\frac{9}{10}, \frac{9}{10}, \frac{1}{10}\right) = 2\left|\frac{9}{10} - \frac{1}{10}\right| = \frac{8}{5}$$

Hence, $S(Fx, Fx, Gy) = \frac{1}{5} < \frac{4}{5} = \frac{1}{2}S(Py, Py, Gy) \leq \frac{1}{2}\mu(x,y)$.

Hence the inequality (3.6.1) holds in both the cases.

Also, $F(X) = \{0\} \subseteq [0, \frac{1}{2}) \cup \{\frac{9}{10}\} = P(X)$, $GX) = \{0, \frac{1}{10}\} \subseteq [0,1) = Q(X)$.

Also,neither $P(X)$ nor $Q(X)$ are closed, as can be seen.

Also for the sequence $x_n = \frac{1}{n^{3/2}}, n = 1,2,...,$

$$Fx_n = 0 \ , \ Qx_n = \frac{1}{n^{3/2}}$$

$$S(Fx_n, Fx_n, 0) = S(0,0,0) = 0,$$

$$S(Qx_n, Qx_n, 0) = S\left(\frac{1}{n^{3/2}}, \frac{1}{n^{3/2}}, 0\right) = \frac{2}{n^{3/2}} \to 0 \text{ as } n \to \infty.$$

Therefore, $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Qx_n = 0 = F(0)$.

Thus the maps (F,Q) satisfies (CLR_F)-property.

We can easily see that the pairs (F,Q) and (G,P) are weakly compatible.

Also, '0' is the only common fixed point of F, G, P and Q.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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