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COMMON FIXED POINT THEOREMS FOR THREE SELF-MAPS UNDER DIFFERENT CONTRACTION PRINCIPLES IN $\mathbf{M}_{\nu}^{\mathbf{b}}-$ COMPLETE METRIC SPACE

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Abstract: In this paper, it is shown that in M_{ν}^{b} complete metric space, coincidence points exist and a unique common fixed point is established for three self-maps under Banach, Kannan Reich, and modified Hardy-Rogers type contraction principles, which is further applied to solve two problems.

Keywords: complete M_{ν}^{b} metric space; self maps; m_{ν}^{b}- convergent; coincidence point; fixed point.
2010 AMS Subject Classification: 47H10.

1. INTRODUCTION

Recently, start-ups in the food delivery industry have experienced a boom. One of the reasons for this growth is a series of lockdowns due to covid-19. Almost two-thirds of the world's logistics costs go into transportation planning. The vehicle routing problem is used to solve delivery decisions such as the sequence of orders, choice of vehicles, delivery routes, and exact time of delivery, and so on. The main objective here is to minimize the total duration of routes, the total distance travelled, the number of vehicles used, and to address some of the accompanying challenges including rush hour, no rush hour, vehicle capacity, and refrigeration

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[1]. The metric spaces and distance functions defined on these concepts are continuously evolving, where distance is usually defined as the difference between two points, that is, mod (x - y) such that the distance between the same points is zero. However, if the distance function is max (x, y), then the distance between the same points is not zero.

In addressing the aforementioned issues, this work is organized as follows. In Main section, a metric space is considered with non-zero distance between the same points. The Hardy-Rogers contraction principle was modified to consider three self-maps under different contractions, such as Banach, Kannan, Reich, and modified Hardy-Rogers. The existence of coincidence points and the uniqueness of the point of coincidence are further proved such that if the self-maps are weakly compatible, then there is a unique common fixed point. This theorem is applied to examples for verification.

2. PRELIMINARIES

Definition 2.1 [2] Let (M, d) be a metric space and A, B : M → M be two single valued maps. The maps A and B have coincidence point x if Ax = Bx = w for $w \in$ M, and w is called a point of coincidence of A and B. If $w = x$, then x is called a common fixed point of A and B.

Definition 2.2 [3] In 2021, Joshi et al. introduced the M_{v}^{b} - metric space.

For a non-empty set M with a real number s ≥ 1 and a map m_{v}^{b}: M x M → $\mathbb{R}^{+}$ satisfying

1) $m_{v}^{b}(x, y) = m_{v}^{b}(x, x) = m_{v}^{b}(y, y) \iff x = y$,

2) $m_{v\,x,y}^{b} \leq m_{v}^{b}(x, y)$ where $m_{v\,x,y}^{b} = \min\{m_{v}^{b}(x, x), m_{v}^{b}(y, y)\}$,

and $M_{v\,x,y}^{b} = \max\{m_{v}^{b}(x, x), m_{v}^{b}(y, y)\}$.

3) $m_{v}^{b}(x, y) = m_{v}^{b}(y, x)$

4) $[m_{v}^{b}(x, y) - m_{v\,x,y}^{b}] \leq s\,\{ [m_{v}^{b}(x,z_1) - m_{v\,x,z_1}^{b}] + [m_{v}^{b}(z_1, z_2) - m_{v\,z_1,z_2}^{b}] + \ldots\ldots. +$

$$[m_{v}^{b}(z_{v}, y) - m_{v\,z_{v},y}^{b}] \} - \sum_{j=1}^{v} m_{v}^{b}(z_j, z_j).$$

where x, z_1, z_2, ... z_v, y are distinct and belong to M, the pair (M, m_{v}^{b}) is a M_{v}^{b} - metric space.

Definition 2.3 [3]

i) A sequence $\{y_n\}$ in (M, m_{v}^{b}) is m_{v}^{b}-convergent to y $\in$ M iff

$$\lim_{n\to\infty}(m_{v}^{b}(y_n, y) - m_{v\,y_n,y}^{b}) = 0$$

ii) A sequence $\{y_n\}$ in (M, m_{v}^{b}) is m_{v}^{b}- Cauchy Sequence iff $\lim_{n,m\to\infty}(m_\nu^b(y_n, y_m) - m_{\nu\, y_n,y_m}^b)$ and $\lim_{n,m\to\infty}(M_{\nu\, y_n,y_m}^b - m_{\nu\, y_n,y_m}^b)$

exists and are finite.

iii) If each m_ν^b - Cauchy sequence $\{y_n\}$ in (M, m_ν^b) converges to $y \in M$ such that

$\lim_{n,m\to\infty}(m_\nu^b(y_n, y_m) - m_{\nu\, y_n,y_m}^b) = 0$ and $\lim_{n,m\to\infty}(M_{\nu\, y_n,y_m}^b - m_{\nu\, y_n,y_m}^b) = 0.$

Lemma 2.4 [3] Let (M, m_ν^b) be an M_ν^b - metric space with coefficient $s \geq 1$ and self map $\mathcal{B}$: M→ M satisfies $m_\nu^b(\mathcal{B}x, \mathcal{B}y) \leq \mu\,(m_\nu^b(x, y)$ with $0 < \mu < \frac{1}{2s}$ and $x, y \epsilon M$

Consider the sequence $\{y_n\}$ defined by $y_{n+1} = \mathcal{B}y_n$. If $y_n \to y$ as $n \to \infty$, then $\mathcal{B}y_n \to \mathcal{B}y$ as $n \to \infty$.

Theorem 2.5 [3] Let (M, m_ν^b) be an M_ν^b - complete metric space with coefficient $s \geq 1$ and self-map $\mathcal{B}$: $M \to M$ satisfies $m_\nu^b(\mathcal{B}x, \mathcal{B}y) \leq \mu\,[m_\nu^b(x, \mathcal{B}x) + m_\nu^b(y, \mathcal{B}y)]$ with $\mu < \frac{1}{2s}$ and $x, y \,\epsilon\, M$. Then, $\mathcal{B}$ has a unique fixed point y^* such that $m_\nu^b(y^*, y^*) = 0$ and the sequence of iterates $\{\mathcal{B}^n y_0\} \subseteq M$ converges to $y^* \in M$.

Definition 2.6 [2] Let (M, d) be a metric space and A, B: M → M be two single valued maps. The maps A and B are weakly compatible if they commute at their coincidence points, that is Ax = Bx implies that ABx = BAx.

Definition 2.7 [4] Let f: X → X be a self-map and (X, d) be a metric space.

1) The mapping f is called Banach-type contraction

if for all x, y in X, and $\alpha \,\epsilon\, [0, 1)$ such that $\alpha < 1$ and

$d(fx, fy) \leq \alpha\, d(x, y).$

2) The mapping f is called Kannan-type contraction

if for all x, y in X, and $\beta, \gamma \,\epsilon\, [0, 1)$ such that $\beta + \gamma < 1$ and

$d(fx, fy) \leq \beta\, d(x, fx) + \gamma\, d(y, fy)$

3) The mapping f is called Reich-type contraction

if for all x, y in X, and $\alpha, \beta, \gamma \,\epsilon\, [0, 1)$ such that $\alpha + \beta + \gamma < 1$ and

$d(fx, fy) \leq \alpha\, d(x, y) + \beta\, d(x, fx) + \gamma\, d(y, fy).$

4) The mapping f is called Hardy-Rogers-type contraction

if for all x, y in X and $\alpha, \beta, \gamma, \delta, \eta \,\epsilon\, [0, 1)$ such that $\alpha + \beta + \gamma + \delta + \eta < 1$ and

$d(fx, fy) \leq \alpha\, d(x, y) + \beta\, d(x, fx) + \gamma\, d(y, fy) + \delta\, d(x, fy) + \eta\, d(y, fx).$

Examples 2.8:

i) If W = [0, 1], and a map is defined as $m_{\nu}^{b}(x, y) = |x - y|$.

By simple calculation, it can be shown that the map is an M_{ν}^{b} −metric space for any ν and s, it is also a metric space.

ii) If W = [0, 1] and the map is defined as $m_{\nu}^{b}(x, y) = |x - y| + \max\{|x|, |y|\}$.

It is clear that the map is a M_{ν}^{b} −metric space for any ν and s.

This is not a metric space because $m_{\nu}^{b}(x, x) = |x|$ and $m_{\nu}^{b}(x, x) \neq 0$.

iii) Every metric space is M_{ν}^{b} −metric space,

but not every M_{ν}^{b} −metric space is a metric space.

iv) If W = [0, 1] and a map is defined as $m_{\nu}^{b}(x, y) = |x - y| + |x|$.

It is not an M_{ν}^{b} −metric space. It does not satisfy the symmetry condition

$m_{\nu}^{b}(x, y) = |x - y| + |x| \neq m_{\nu}^{b}(y, x) = |y - x| + |y|$.

v) Detailed information can be found in the literature [1-8] for the reference.

3. MAIN RESULTS

3.1 Definition. We are introducing a modified Hardy -Rogers contraction principle for three self-maps in M_{ν}^{b} −complete metric space as

$m_{\nu}^{b}(Px, Qy) \leq \alpha m_{\nu}^{b}(Rx, Ry) + \beta m_{\nu}^{b}(Px, Rx) + \gamma m_{\nu}^{b}(Qy, Ry) + \delta m_{\nu}^{b}(Rx, Qx) + \eta m_{\nu}^{b}(Py, Ry)$.

whereas the Hardy-Rogers contraction principle is given in [2] as

$m_{\nu}^{b}(Px, Qy) \leq \alpha m_{\nu}^{b}(Rx, Ry) + \beta m_{\nu}^{b}(Px, Rx) + \gamma m_{\nu}^{b}(Qy, Ry) + \delta m_{\nu}^{b}(Rx, Qy) + \eta m_{\nu}^{b}(Px, Ry)$.

3.2 Theorem:

Let (U, m_{ν}^{b}) be an $M_{\nu}^{b}-$ complete metric space with a real number $s \geq 1$ and $\nu \in \mathbb{N}$. Suppose P, Q, R : U $\rightarrow$ U are the self-mappings satisfying a modified Hardy -Rogers contraction principle.

$m_{\nu}^{b}(Px, Qy) \leq \alpha m_{\nu}^{b}(Rx, Ry) + \beta m_{\nu}^{b}(Px, Rx) + \gamma m_{\nu}^{b}(Qy, Ry) + \delta m_{\nu}^{b}(Rx, Qx) + \eta m_{\nu}^{b}(Py, Ry)$.

for all x, y ϵ U where $\alpha, \beta, \gamma, \delta, \eta \in [0, 1)$ such that $(\alpha + \beta + \gamma + \delta + \eta) < 1/s$.

If P (U) $\subseteq$ R (U), Q (U) $\subseteq$ R (U), and either P (U) union Q (U) or R (U) is a complete subspace of U, then P, Q, R have a unique point of coincidence in U. Moreover, if (P, R) and (Q, R) are weakly compatible, then P, Q, and R have a unique common fixed point.

Proof

Let x_0 be an arbitrary point in U. $x_1 \in$ U exists such that $y_0 = Px_0 = Rx_1$ because P (U) $\subseteq$ R (U) and $x_2 \in$ U exists such that $y_1 = Qx_1 = Rx_2$ because Q (U) $\subseteq$ R (U).

In this way we collect the sequence $\{y_n\}$ in U where

$y_{2n} = Px_{2n} = Rx_{2n+1}$ and $y_{2n+1} = Qx_{2n+1} = Rx_{2n+2}$ for all $n \geq 0$.

For y_{2n}, y_{2n+1}.

$$m_{\nu}^{b}(y_{2n}, y_{2n+1}) = m_{\nu}^{b}(Px_{2n}, Qx_{2n+1})$$
$$\leq \alpha m_{\nu}^{b}(Rx_{2n}, Rx_{2n+1}) + \beta m_{\nu}^{b}(Px_{2n}, Rx_{2n}) + \gamma m_{\nu}^{b}(Qx_{2n+1}, Rx_{2n+1})$$
$$+ \delta m_{\nu}^{b}(Rx_{2n}, Qx_{2n}) + \eta m_{\nu}^{b}(Px_{2n+1}, Rx_{2n+1})$$

$$m_{\nu}^{b}(y_{2n}, y_{2n+1}) \leq \alpha m_{\nu}^{b}(y_{2n-1}, y_{2n}) + \beta m_{\nu}^{b}(y_{2n}, y_{2n-1}) + \gamma m_{\nu}^{b}(y_{2n+1}, y_{2n}) + \delta m_{\nu}^{b}(y_{2n-1}, y_{2n}) +$$
$$\eta m_{\nu}^{b}(y_{2n+1}, y_{2n})$$

$$(1-\gamma-\eta)\ m_{\nu}^{b}(y_{2n}, y_{2n+1}) \leq (\alpha+\beta+\delta)\ m_{\nu}^{b}(y_{2n-1}, y_{2n})$$

$$m_{\nu}^{b}(y_{2n}, y_{2n+1}) \leq \zeta m_{\nu}^{b}(y_{2n-1}, y_{2n}) \qquad \text{where } \zeta = \frac{(\alpha+\beta+\delta)}{(1-\gamma-\eta)} < 1$$

Using the same process we can show for different values of y_n, y_m

$$m_{\nu}^{b}(y_n, y_m) \leq \zeta\ m_{\nu}^{b}(y_{n-2}, y_m) \leq \zeta\zeta\ m_{\nu}^{b}(y_{n-4}, y_m)$$

Depending on whether n is even or odd this reduces either to

$$\zeta^{(n/2)}\ m_{\nu}^{b}(y_0, y_m) \text{ or } \zeta^{(n-1)/2}\ m_{\nu}^{b}(y_1, y_m)$$

Again for m (even or odd) it further reduces to

$$\zeta^{(n+m)/2}\ m_{\nu}^{b}(y_0, y_0) \text{ or } \zeta^{(n-1+m)/2}\ m_{\nu}^{b}(y_1, y_0)$$
$$\text{or } \zeta^{(n+m-1)/2}\ m_{\nu}^{b}(y_0, y_1) \text{ or } \zeta^{(n-1+m-1)/2}\ m_{\nu}^{b}(y_1, y_1) \quad \text{where } (\zeta < 1)$$

If Limit $n, m \to \infty$ is applied then we get $\lim_{n,m\to\infty} m_{\nu}^{b}(y_n, y_m) \to 0, \lim_{n\to\infty} m_{\nu}^{b}(y_n, y_n) \to 0$

$$\lim_{n,m\to\infty} m_{\nu}^{b}(y_n, y_m) - m_{\nu y_n y_m}^{b} \to 0,\quad \lim_{n,m\to\infty} M_{\nu y_n,y_m}^{b} \to 0,\quad \lim_{n,m\to\infty} M_{\nu y_n,y_m}^{b} - m_{\nu y_n y_m}^{b} \to 0$$

This implies that $\{y_n\}$ in U is a $m_{\nu}^{b}-$Cauchy Sequence.

Given that (U, m_{ν}^{b}) is an $M_{\nu}^{b}-$complete metric space and $\{y_n\}$ is a Cauchy sequence then there exists a limit in U. Let the limit be l.

Given that R (U) is a complete subspace of U and P (U) $\subseteq$ R (U).

There exists $l \in R(U)$ such that limit y_{2n+1} = limit $Rx_{2n+2} = l$

we can always find x such that $Rx = l$. To prove $Px = l$. It is clear from definition that

$[m_{\nu}^{b}(Px, l) - m_{\nu\ Px,l}^{b}] \leq 0$, and $[m_{\nu}^{b}(Px, l) - m_{\nu\ Px,l}^{b}] \geq 0$. Thus $[m_{\nu}^{b}(Px, l) - m_{\nu\ Px,l}^{b}] = 0$

Hence we obtain $Px = l$ and $l = Rx = Px$. x is the coincidence point of (P, R).

For Q (U) $\subseteq$ R (U), we can show that $l = Qx = Rx$. x is the coincidence point of (Q, R).

Now, we obtain $l = Rx = Px = Qx$. x is the coincidence point of P, Q, and R.

Similarly, the proof follows if P (U) union Q (U) is a complete subspace of U.

Coincidence points exist in P, Q, R, and the point of coincidence is l.

Let there be another point of coincidence say l', $l \neq l'$, and l' = Rx' = Px' = Qx'.

$$\mathrm{m}_{\nu}^{\mathrm{b}}(l, l') = \mathrm{m}_{\nu}^{\mathrm{b}}(\mathrm{Px}, \mathrm{Qx'})$$
$$\leq \alpha\mathrm{m}_{\nu}^{\mathrm{b}}(\mathrm{Rx}, \mathrm{Rx'}) + \beta\mathrm{m}_{\nu}^{\mathrm{b}}(\mathrm{Px}, \mathrm{Rx}) + \gamma\mathrm{m}_{\nu}^{\mathrm{b}}(\mathrm{Qx'}, \mathrm{Rx'}) + \delta\mathrm{m}_{\nu}^{\mathrm{b}}(\mathrm{Rx}, \mathrm{Qx}) + \eta\mathrm{m}_{\nu}^{\mathrm{b}}(\mathrm{Px'}, \mathrm{Rx'})$$
$$\leq \alpha\mathrm{m}_{\nu}^{\mathrm{b}}(l, l') + \beta\mathrm{m}_{\nu}^{\mathrm{b}}(l, l) + \gamma\mathrm{m}_{\nu}^{\mathrm{b}}(l', l') + \delta\mathrm{m}_{\nu}^{\mathrm{b}}(l, l) + \eta\mathrm{m}_{\nu}^{\mathrm{b}}(l', l')$$
$$\leq \alpha\mathrm{m}_{\nu}^{\mathrm{b}}(l, l') + \beta\mathrm{m}_{\nu}^{\mathrm{b}}(l, l') + \gamma\mathrm{m}_{\nu}^{\mathrm{b}}(l, l') + \delta\mathrm{m}_{\nu}^{\mathrm{b}}(l, l') + \eta\mathrm{m}_{\nu}^{\mathrm{b}}(l, l')$$
$$\leq \{\alpha + \beta + \gamma + \delta + \eta\}\, \mathrm{m}_{\nu}^{\mathrm{b}}(l, l') \quad \text{where} \quad (\alpha + \beta + \gamma + \delta + \eta < 1/s < 1)$$
$$< \mathrm{m}_{\nu}^{\mathrm{b}}(l', l') \quad \text{This is a contradiction.}$$

Hence, $l = l'$ and l is a unique point of coincidence of P, Q, and R.

Given that (P, R) and (Q, R) are weakly compatible, that is,

Pl = PRx = RPx = Rl, and let f1 = Pl, then => f1 = Pl = PRx = RPx = Rl

Ql = QRx = RQx = Rl, and let f2 = Ql, then => f2 = Ql = QRx = RQx = Rl

We get Rl = Pl = Ql = f1 = f2. From the uniqueness proof for the point of coincidence, we obtain f1 = f2 = l and Rl = Pl = Ql = l, where l is the common fixed point for P, Q, and R.

Hence the proof of the theorem is complete.

4. COROLLARIES

4.1 Corollary **Banach-type Contraction:**

Let (U, m_{ν}^{b}) be an $\mathrm{M}_{\nu}^{\mathrm{b}}$ −complete metric space with a real number s ≥ 1 and $\nu \in \mathbb{N}$.

Suppose P, Q, R : U → U are the self-mappings satisfying the Banach-type contraction principle

$$\mathrm{m}_{\nu}^{b}(\mathrm{Px}, \mathrm{Qy}) \leq \alpha\mathrm{m}_{\nu}^{b}(\mathrm{Rx}, \mathrm{Ry})$$

for all x, y ϵ U where α ϵ [0, 1), such that $\alpha < 1/s$.

If P (U) ⊆ R (U), Q (U) ⊆ R (U), and either P (U) union Q (U) or R (U) is a complete subspace of U, then P, Q, R have a unique point of coincidence in U. Moreover, if (P, R) and (Q, R) are weakly compatible, then P, Q, and R have a unique common fixed point.

Proof: Substituting $\beta = \gamma = \delta = \eta = 0$ in Theorem 3.2, we obtain the proof of this corollary.

4.2 Corollary **Kannan-type Contraction:**

Let (U, m_{ν}^{b}) be an $\mathrm{M}_{\nu}^{\mathrm{b}}$ −complete metric space with a real number s ≥ 1 and $\nu \in \mathbb{N}$.

Suppose P, Q, R : U → U are the self-mappings satisfying a Kannan-type contraction principle.

$$\mathrm{m}_{\nu}^{b}(\mathrm{Px}, \mathrm{Qy}) \leq \beta\mathrm{m}_{\nu}^{b}(\mathrm{Px}, \mathrm{Rx}) + \gamma\mathrm{m}_{\nu}^{b}(\mathrm{Qy}, \mathrm{Ry}).$$

for all x, y ϵ U where β, γ ϵ [0, 1) such that $\beta + \gamma < 1/s$.

If P (U) ⊆ R (U), Q (U) ⊆ R (U), and either P (U) union Q (U) or R (U) is a complete subspace of U, then P, Q, R have a unique point of coincidence in U. Moreover, if (P, R) and (Q, R) are weakly compatible, then P, Q, and R have a unique common fixed point.

Proof: Substituting $\alpha = \delta = \eta = 0$ in Theorem 3.2, we obtain the proof of this corollary.

4.3 Corollary **Reich-type Contraction:**

Let (U, m_ν^b) be an M_ν^b −complete metric space with a real number $s \geq 1$ and $\nu \in \mathbb{N}$.

Suppose P, Q, R : U → U are the self-mappings satisfying a Reich-type contraction principle.

$$m_\nu^b(Px, Qy) \leq \alpha m_\nu^b(Rx, Ry) + \beta m_\nu^b(Px, Rx) + \gamma m_\nu^b(Qy, Ry).$$

for all x, y ϵ U where $\alpha, \beta, \gamma \ \epsilon \ [0, 1)$ such that $\alpha + \beta + \gamma < 1/s$.

If P (U) ⊆ R (U), Q (U) ⊆ R (U), and either P (U) union Q (U) or R (U) is a complete subspace of U, then P, Q, R have a unique point of coincidence in U. Moreover, if (P, R) and (Q, R) are weakly compatible, then P, Q, and R have a unique common fixed point.

Proof: Substituting $\delta = \eta = 0$ in Theorem 3.2, we obtain the proof of this corollary.

5. EXAMPLES

5.1 problem to verify the Theorem 3.2

Let U = [0, 1] and the map m_ν^b : U → U be a M_ν^b-complete metric space with $m_\nu^b(x, y) = (\max\{x, y\})^2$ for s = 1 and $\nu = 2$ (it is a M_ν^b-complete metric space for any s and ν).

In addition, $m_\nu^b(x, x) = x^2$ (note $m_\nu^b(x, x) \neq 0$ distance function between same point is not zero)

Given P, Q, and R are self-maps defined on U with Px = x/8, Qx = x/4, and Rx = x/2.

Solution:

P(U) = [0,1/8]; Q(U) = [0, 1/4]; R(U) = [0, 1/2] implies P(U) and Q(U) ⊆ R(U).

PRx = Px/2 = x/16 and RPx = Rx/8 = x/16

QRx = Qx/2 = x/8 and RQx = Rx/4 = x/8 imply that (P, R) and (Q, R) are weakly compatible.

For some x < y {x/8 < y/8 < y/4 },

$m_\nu^b(Px, Qy) \leq \alpha m_\nu^b(Rx, Ry) + \beta m_\nu^b(Px, Rx) + \gamma m_\nu^b(Qy, Ry) + \delta m_\nu^b(Rx, Qx) + \eta m_\nu^b(Py, Ry)$.

$m_\nu^b(x/8, y/4) \leq \alpha m_\nu^b(x/2, y/2) + \beta m_\nu^b(x/8, x/2) + \gamma m_\nu^b(y/4, y/2) + \delta m_\nu^b(x/2, x/4) + \eta m_\nu^b(y/8, y/2)$

$$(y/4)^2 \leq \alpha (y/2)^2 + \beta (x/2)^2 + \gamma (y/2)^2 + \delta (x/2)^2 + \eta (y/2)^2 \leq (\alpha + \gamma + \eta)\, y^2/4$$

It is true for $\alpha = \gamma = \eta = \frac{1}{4} \quad and \quad \beta = \delta = 0$

Next, for x (same value x = y)

m_v^b (Px, Qx)$\leq$ αm_v^b (Rx, Rx) + βm_v^b (Px, Rx) + γm_v^b (Qx, Rx) + δm_v^b (Rx, Qx) + ηm_v^b (Px, Rx).

m_v^b (x/8, x/4)$\leq$ αm_v^b (x/2,x/2) + βm_v^b (x/8,x/2) + γm_v^b (x/4,x/2) + δm_v^b (x/2,x/4) + ηm_v^b (x/8, x/2)

$$(x/4)^2 \leq \alpha (x/2)^2 + \beta (x/2)^2 + \gamma (x/2)^2 + \delta (x/2)^2 + \eta (x/2)^2 \leq (\alpha+\beta+\gamma+\delta+\eta)\, x^2/4$$

It is true for $1 > (\alpha+\beta+\gamma+\delta+\eta) > \frac{1}{4}$ and in particular for $\alpha = \gamma = \eta = \frac{1}{4}$ and $\beta = \delta = 0$

This example satisfies all the properties of Theorem 3.2.

P, Q, and R have one common and unique fixed point “0”.

5.2 Problem

Verify the Theorem 3.2.

Let X = {a, b, c, d}and define the map m_v^b: X × X→ $\mathbb{R}^2$.

i) m_v^b (x, y) = (0, 0) for x = y.

ii) m_v^b (x, y) = m_v^b (y, x)

iii) m_v^b (a, b) = (3, 6)

iv) m_v^b (a, c) = m_v^b (b, c) = (1, 2)

v) m_v^b (a, d) = m_v^b (b, d) = m_v^b (c, d) = (2, 4).

And self-maps P, Q, R : X → X such that

1) P (x) = Px = c for all x

2) Q (x) = Qx = c for x ≠ d, and Q (x) = Qx = a for x = d,

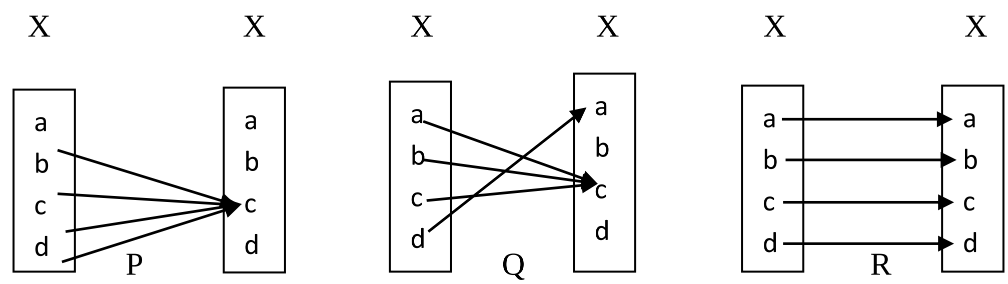
3) R (x) = Rx = x for all x

Solution: The map m_v^b: X × X → $\mathbb{R}^2$ is an M_v^b complete metric space for s = 1 and v = 2 and it's mapping is expressed in the Table 1.

m_v^b	a	b	c	d
a	(0, 0)	(3, 6)	(1, 2)	(2, 4)
b	(3, 6)	(0, 0)	(1, 2)	(2, 4)
c	(1, 2)	(1, 2)	(0, 0)	(2, 4)
d	(2, 4)	(2, 4)	(2, 4)	(0, 0)

Table 1.The mapping m_v^b

Three self maps are given below as



P (X) = c; Q (X) = {a, c}, and R (X) = {a, b, c, d} implies P (X), Q (X) $\subseteq$ R (X).

for all x $\in$ X, PRx = Px = c, and RPx = Rc = c implies that (P, R) is weakly compatible.

for all x $\in$ X, x $\neq$ d, QRx = Qx = c, and RQx = Rc = c.

for all x = d, QRd = Qd = a, and RQd = Ra = a implies that (Q, R) is weakly compatible.

$m_\nu^b(Px, Qy) \leq \alpha m_\nu^b(Rx, Ry) + \beta m_\nu^b(Px, Rx) + \gamma m_\nu^b(Qy, Ry) + \delta m_\nu^b(Rx, Qx) + \eta m_\nu^b(Py, Ry)$.

This inequality is true if $\alpha = \beta = \gamma = \eta = \frac{1}{5}$ and $\delta = 0$. The inequality can be verified for all possible values of x and y. Some of the values of x and y are given in the Table 2 and Table 3, for the reference. All the properties of Theorem 3.2 are satisfied. And it is clear that the common fixed point of P, Q, R is "c".

(x, y)	(Px, Qy)	(Rx, Ry)	(Px, Rx)	(Qy, Ry)	(Rx, Qx)	(Py, Ry)
(a, b)	(c, c)	(a, b)	(c, a)	(c, b)	(a, c)	(c, b)
(a, c)	(c, c)	(a, c)	(c, a)	(c, c)	(a, c)	(c, c)
(a, d)	(c, a)	(a, d)	(c, a)	(a, d)	(a, c)	(c, d)
(b, c)	(c, c)	(b, c)	(c, b)	(c, c)	(b, c)	(c, c)
(b, d)	(c, a)	(b, d)	(c, b)	(a, d)	(b, c)	(c, d)
(c, d)	(c, a)	(c, d)	(c, c)	(a, d)	(c, c)	(c, d)
(d, d)	(c, a)	(d, d)	(c, d)	(a, d)	(d, a)	(c, d)

Table 2. Different values of x and y.

(x, y)	m_v^b (Px, Qy)	m_v^b (Rx, Ry)	m_v^b (Px, Rx)	m_v^b (Qy, Ry)	m_v^b (Rx, Qx)	m_v^b (Py, Ry)
(a, b)	(0, 0)	(3, 6)	(1, 2)	(1, 2)	(1, 2)	(1, 2)
(a, c)	(0, 0)	(1, 2)	(1, 2)	(0, 0)	(1, 2)	(0, 0)
(a, d)	(1, 2)	(2, 4)	(1, 2)	(2, 4)	(1, 2)	(2, 4)
(b, c)	(0, 0)	(1, 2)	(1, 2)	(0, 0)	(1, 2)	(0, 0)
(b, d)	(1, 2)	(2, 4)	(1, 2)	(2, 4)	(1, 2)	(2, 4)
(c, d)	(1, 2)	(2, 4)	(0, 0)	(2, 4)	(0, 0)	(2, 4)
(d, d)	(1, 2)	(0, 0)	(2, 4)	(2, 4)	(2, 4)	(2, 4)

Table 3. For (x, y) corresponding m_v^b map values(ignored obvious cases such as (Px, Qy) = (c, c))

Note:

In Example 5.2, if Qd = d then Theorem 3.2, cannot be applied because the inequality

$m_v^b(Px, Qy) \leq \alpha m_v^b(Rx, Ry) + \beta m_v^b(Px, Rx) + \gamma m_v^b(Qy, Ry) + \delta m_v^b(Rx, Qx) + \eta m_v^b(Py, Ry)$.

is not true for all cases, even though a common fixed point exists.

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REFERENCES

[1] M. Gera, N. Nawander, N. Tharwani, P. Bhatia, Operations research in food delivery, Int. J. Adv. Res. Develop. 3 (2018), 73-78.

[2] A. Kumar, S. Rathee, N. Kumar, The point of coincidence and common fixed point for three mappings in cone metric spaces, J. Appl. Math. 2013 (2013), 146794.

[3] M. Joshi, A. Tomar, H.A. Nabwey, R. George, On unique and nonunique fixed points and fixed circles in M_v^b-metric space and application to cantilever beam problem, J. Funct. Spaces, 2021 (2021), 6681044.

[4] B.E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257–290.

[5] F. Vetro, F-contractions of Hardy–Rogers-type and application to multistage decision, Nonlinear Anal.: Model. Control. 21 (2016), 531–546.

[6] V. Kiran, K. Rajani Devi, J. Niranjan Goud, Common fixed point theorems for three self maps of a complete S-metric space, Malaya J. Mat. 8 (2020), 363–368.

[7] K. Prudhvi, M. Rangamma, Common fixed points for three maps in cone metric spaces, Math. Theory Model. 2 (2012), 18-22.

[8] F. Gu, Y. Yin, Common fixed point for three pairs of self maps satisfying common (E.A) property in generalized metric spaces, Abstr. Appl. Anal. 2013 (2013), 808092.