Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:175
https://doi.org/10.28919/jmcs/7453
ISSN: 1927-5307

AN OPTIMAL CONTROL ON THE DYNAMICS OF WILDEBEEST, ZEBRA AND LION PREY PREDATOR INTERACTIONS IN SERENGETI ECOSYSTEM

RAYMOND CHARLES[1,*], OLUWOLE DANIEL MAKINDE[2], MONICA KUNG'ARO[1]

[1]Department of Mathematics and Statistics, The University of Dodoma, P.O.Box 338, Tanzania

[2]Faculty of Military Science, Stellenbosch University, Private Bag X2, Saldanha 7395, South Africa

Abstract. In this paper we present an application of optimal control theory to assess the effectiveness of control measures of the dynamics of wildebeest, zebra and lion prey-predator system of Serengeti ecosystem. This is done by proposing control variables such as education (for retaliatory killing), construction of dams (for drought) and treatment (for infections). The main goal is to maximize the population density. For this aim, the Pontryagin's maximum principle has been applied. The optimal control is characterized in terms of optimality system and solved numerically for several scenarios. Results shows that multiple optimal control measures is the most effective strategy in management of wildlife populations.

Keywords: prey; predator; optimal control; serengeti ecosystem.

2010 AMS Subject Classification: 49L20.

1. INTODUCTION

The dynamics and interaction among species in the ecosystem with their complex behaviour is the principal part of mathematical ecology and is a concern of many biological and ecological processes. All organism in the ecosystem are interdependent [1], hence studying their behaviour and dynamics is vital for scientific management of the ecosystem [9]. Serengeti ecosystem is

*Corresponding author
E-mail address: raymond.charles19@gmail.com

Received April 25, 2022

extraordinary in species and it is one of the most popular wildlife reserve in the world [7]. However many biological species in Serengeti ecosystem has been driven to extinction due to external forces such as predation, over exploitation, disease, droughts and mismanagement of the ecosystem [11].

The ecosystem also has been the subject of various hazards such as pollution, fire, drought and catastrophes that lead to perturbations of the ecosystem. Some of the strong studied perturbation of Serengeti ecosystem are the dry season drought of 1993 [11] and the eruption of rinderpest disease in wildebeest as well as canine dispenser virus of 1994 [12]. Poaching also is considered by many to threaten the population viability of prey-predator species in Serengeti ecosystem. Poaching and trophy hunting have always contributed to the off-take of many species of the ecosystem. [5] mention 55% of household around the ecosystem to use bush meat atleast once a year. Also lion killing due to cultural practices [5] by Maasai, has been threatening the population of lion.

Hence if Serengeti ecosystem is to retain its diverse and abundant fauna, more efficient long-term conservation is needed. Based on this understanding, formulating a mathematical model to study how the ecosystem behave may well provide a way to achieve this. Prey-predator model has been widely studied in literature. However very few studies are on optimal control of prey-predator systems. Recently for example [11] studied the threat to lion-wildebeest prey-predator dynamics with optimal control in Serengeti ecosystem. Other studies are [2], [3], [1] and [5]. But none of these has considered the aspect of disease, drought and retaliatory killing as the threat to be controlled for the survival of prey-predator system particularly wildebeest, zebra and lion in the Serengeti ecosystem. This study intends to apply optimal control theory to maximize wildebeest, zebra and lion which the study regard as keystone species of Serengeti ecosystem.

2. Model Formulation

It will be assumed that lion depends completely on wildebeest and zebra as the source of food where wildebeest and zebra has unlimited sources of food. The dynamic therefore will follow the Holling type II function response. In this case $x(t)$, $y(t)$ and $z(t)$ represents the population of wildebeest, zebra and lion respectively. In the absence of predator, droughts and disease, prey species are assumed to grow logistically with carrying capacities k and l respectively. However the inter-specific competition among wildebeest and zebra is exploitative. From the above assumptions we formulate the system of model equation;

$$
\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k}) - b_{12}xy - \frac{b_{13}xz}{1+ax} - h_1x - w_1x \\
\frac{dy}{dt} &= sy(1-\frac{s}{l}) - b_{21}xy - \frac{b_{23}yz}{1+dy} - h_2y - w_2y \\
\frac{dz}{dt} &= -cz + \frac{b_{31}b_{13}xz}{1+ax} + \frac{b_{32}b_{23}yz}{1+dy} - ez - w_3z
\end{aligned}
\tag{1}
$$

Where r and s are the intrinsic growth rates of wildebeest and zebra respectively. k and l are the environmental carrying capacities of wildebeest and zebra respectively, b_{12} and b_{21} are the coefficient of inter-specific competition among wildebeest and zebra, h_1 and h_2 are the wildebeest and zebra death rates due to droughts respectively, e is the lion death rate due to retaliatory killing, w_1, w_2 and w_3 are wildebeest, zebra and lion death rates due to infections,b_{13} and b_{23} are predation coefficients, b_{31} and b_{32} are the conversion parameters of lion to wildebeest and zebra respectively, c is the natural mortality rate of lion, a and d are the half saturation constants of lion to wildebeest and zebra respectively. Introducing control variables to the system (1) the model become;

$$
\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k}) - b_{12}xy - \frac{b_{13}xz}{1+ax} - (1-u_1(t))h_1x - (1-u_3)(t)w_1x \\
\frac{dy}{dt} &= sy(1-\frac{y}{l}) - b_{21}xy - \frac{b_{23}yz}{1+dy} - (1-u_1(t))h_2y - (1-u_3(t))w_2y \\
\frac{dz}{dt} &= -cz + \frac{b_{31}b_{13}xz}{1+ax} + \frac{b_{32}b_{23}yz}{1+dy} - (1-u_2(t)ez - (1-u_3(t))w_3z
\end{aligned}
\tag{2}
$$

Where, $u_1(t)$, $u_2(t)$ and $u_3(t)$ are control variables, $u_1(t)$ is the construction of dams for drought, $u_2(t)$ is treatment for infections and $u_3(t)$ is education for retaliatory killing.

3. Analysis of Optimal Control

An objective function J is formulated and maximized subject to the number to the number of affected species;

$$J = max\left[(A_1x(T)+A_2y(T)+A_3z(T)) - \int_0^T (B_1\frac{u_1^2}{2}+B_2\frac{u_2^2}{2}+B_3\frac{u_3^2}{2})dt\right] \tag{3}$$

Where A_1, A_2, A_3 are positive constant weights for wildebeest, zebra and lion respectively. B_1, B_2, B_3 are positive constant weights balancing the cost elements attached to the control parameters u_1, u_2, u_3. The weights used here are intended only for theoretical purpose to investigate the effect of various control practices. Importantly the cost associated to any control scenario is presumed to be non-linear and takes a quadratic form which is $\frac{B_1u_1^2}{2}$ refers to the cost of control efforts on construction of dam to reduce the effects of drought, $\frac{B_2u_2^2}{2}$ refers to the cost of control efforts of treatment to reduce the effects of infections, $\frac{B_3u_3}{2}$ is the cost of education for mitigating the effects of retaliatory killings.

In the light of the objective function $J(u_1,u_2,u_3)$, the intention is to maximize J. Therefore it is required to find numerically the optimal control u_1^*, u_2^*, u_3^* such that;

$$J(u_1^*,u_2^*,u_3^*) = \max_{u_1^*,u_2^*,u_3^*\in U} J(u_1,u_2,u_3) \tag{4}$$

for $U=(u_1^*,u_2^*,u_3^*)$ such that u_1^*, u_2^*, u_3^* are measurable with $0 \leq u_1 \leq 1$, $0 \leq u_2 \leq 1$ and $0 \leq u_3 \leq 1$, for $t \in [0,T]$.

3.1. Existence of Optimal control.

Theorem 1. *An optimal control set $(u_1^*,u_2^*,u_3^*) \in U$ with corresponding non-negativity states (x,y,z) that maximize the objective function $J(u_1(t),u_2(t),u_3(t))$ exists.*

Proof: The positiveness and consistent boundness of the state variables alongside the controls on $[0,T]$ suggests that the existence of a maximizing sequence $J(u_1^n(t),u_2^n(t),u_3^n(t))$ such that;

$$\lim_{n\to\infty} J(u_1^n(t),u_2^n(t),u_3^n(t)) = \inf_{(u_1^n(t),u_2^n(t),u_3^n(t))\in U} J(u_1^n(t),u_2^n(t),u_3^n(t)) \tag{5}$$

The boundness of all the state and control parameters insinuates that all derivatives of the state variables are bounded as well. Supposing the respective sequence of the state variables denoted by (x,y,z), subsequently, all the state variable are Lipschitz continous with the same Lipschitz constant. This means that, the sequence (x,y,z) is consistently equicontinous in $[0,T]$. In accordance with the method by [8], the state state sequence that converges steadly to (x,y,z) in $[0,T]$. Also, it can be taken that, the control sequence $u_n^n = (x^n, y^n, z^n)$ has sequence that weakly converges in $L^2(0,T)$. Let $(u_1^*, u_2^*, u_3^*) \in U$ be in the form of $u_i^n \to u_i^*$ weakly in $L^2(0,T)$ for $i = 1,2,3,....$ Implementing the lower semi-continuity of norms in weak L^2, we have;

$$\| u_i^* \|^2 L^2 \leq \lim_{n\to\infty} \inf \| u_i^n(t) \|^2 L^2, i = 1,2,3... \tag{6}$$

This means that;

$$J(u_1^*, u_2^*, u_3^*) \geq \lim_{n\to\infty} \int_0^T (A_1x(T) + A_2y(T) + A_3z(T) - \int_0^T (\frac{B_1u_1^2}{2} + \frac{B_2u_2^2}{2} + \frac{B_3u_3^2}{2}) \tag{7}$$

Therefore; the control set (u_1^*, u_2^*, u_3^*) that maximize the objective function $J(u_1, u_2, u_3)$ exists.

3.2. Characterization of optimal control. In this section, we derive conditions required for optimal control, characterizing optimal control using upper and lower bound technique and formulating optimality system that characterize the optimal control. The asential requirement is that; the optimal pair should satisfy the necessary conditions that come from Pontryagin Maximum Principle [8] and which are also discussed in [4]. This principle converts state (2), objective function (3) and control (4) into minimal value of Lagrangian of optimal problem. The Lagrangian of the optimal problem is given by;

$$\Gamma = (A_1x(T) + A_2y(T) + A_3z(T) - (\frac{B_1u_1^2}{2} + \frac{B_2u_2^2}{2} + \frac{B_3u_3^2}{2}) \tag{8}$$

In order to seek for maximum Lagrangian of optimal problem, we define the Hamilitonian H for the control problem with respect to u_1 and u_2 as;

$$H = (A_1x(T) + A_2y(T) + A_3z(T) - (\frac{B_1u_1^2}{2} + \frac{B_2u_2^2}{2} + \frac{B_3u_3^2}{2}) + \sum_{i=1}^{3} \lambda_i f_i \tag{9}$$

where f_i is the right hand side of the differential equation of ith state variable in system (2) and λ_i for $i = 1,2,3$ is the set of adjoint functions. That means the Hamiltonian consists of integrand of objective functional and the inner product of right hand side of state equations and corresponding adjoint variables L_1, L_2, L_3. The expanded expression form of Hamiltonian H in (9) is given by;

$H = A_1x(T) + A_2y(T) + A_3z(T) - (\frac{B_1u_1^2}{2} + \frac{B_2u_2^2}{2} + \frac{B_3u_3^2}{2})$

$+L_1\left[\frac{dx}{dt}=rx(1-\frac{x}{l})-b_{12}xy-\frac{b_{13}xz}{1+ax}-(1-u_1(t))h_1x-(1-u_3)(t)w_1x\right]$
$+L_2\left[\frac{dy}{dt}=sy(1-\frac{y}{l})-b_{21}xy-\frac{b_{23}yz}{1+dy}-(1-u_1(t))h_2y-(1-u_3(t))w_2y\right]$
$+L_3\left[\frac{dz}{dt}=-cz+\frac{b_{31}b_{13}xz}{1+ax}+\frac{b_{32}b_{23}yz}{1+dy}-(1-u_2(t)ez-(1-u_3(t))w_3z\right]$

Where L_1, L_2 and L_3 are adjoint co-state variables. Applying pontryagin maximum principle [10] and the existence results for the optimal control from (Fleming and Rishel, 1975),the following preposition is obtained.

Theorem 2. *For the optimal control u_1^*, u_2^* and u_3^* that maximizes $j(u_1,u_2,u_3)$ over U, there exists adjoint variables L_1, L_2, L_3 satisfying;*
$\frac{-\partial H}{\partial x}=\frac{\partial L_1}{dt}=-A_1-L_1r+\frac{2L_1rx}{k}+L_1b_{12}y+L_2b_{21}y+\frac{L_1b_{13}z}{(1+ax)^2}$
$+L_1(1-u_1)h_1+L_1(1-u_2)w_1-\frac{L_3b_{31}b_{13}z}{(1+ax)^2}$
$\frac{-\partial H}{\partial y}=\frac{\partial L_2}{dt}=-A_2-L_2s+\frac{2L_2sy}{l}+L_1b_{12}x+b_{21}L_2x+\frac{L_2b_{23}z}{(1+dy)^2}$
$+L_2(1-u_1)h_2+L_2(1-u_2)w_2-\frac{L_3b_{32}b_{23}z}{(1+ax)^2}$
$\frac{-\partial H}{\partial z}=\frac{\partial L_3}{dt}=-A_3+\frac{L_1b_{13}x}{(1+ax)^2}+\frac{L_2b_{23}y}{(1+dy)^2}-L_3c-\frac{L_3b_{31}b_{13}x}{(1+ax)^2}$
$-\frac{L_3b_{32}b_{23}y}{(1+ax)^2}-L_3(1-u_3)e+L_3(1-u_2)w_3$ *with transversality conditions $L_1(T)=A_1$, $L_2(T)=A_2$ and $L_3(T)=A_3$. The following characterization holds on the interior of the control set;*
$u_1^*=\min\left\{1,\max\left(0,\frac{L_1h_1x+L_2h_2y}{A_1}\right)\right\}$, $u_2^*=\min\left\{1,\max\left(0,\frac{L_2h_2y+L_3ez}{B_2}\right)\right\}$
$u_3^*=\min\left\{1,\max\left(0,\frac{L_1w_1x+L_2w_2y+L_3w_3z}{B_3}\right)\right\}$ *where L_1, L_2, L_3 are solutions of ().*

Proof: To prove this, the function (9) is differentiated partially w.r.t its state variables which gives the adjoint system. With the Pontyagin's Maximum Principle, we get the following adjoint system evaluated at the optimal control pair corresponding to the state variables.
$H=A_1x+A_2y+A_3z-\frac{B_1u_1^2}{2}-\frac{B_2u_2^2}{2}-\frac{B_3u_3^2}{2}+L_1rx-L_1x^2-\alpha_1L_1xy-\frac{\beta_1L_1(1-m)xz}{1+\lambda_1(1-m)x}$
$-L_1(1-u_1(t))P_1x-L_1(1-u_3(t))w_1x+L_2sy-L_2s^2-\alpha_2L_2xy-\frac{\beta_2L_2(1-m)yz}{1+\lambda_2(1-m)y}$
$-L_2(1-u_1(t))P_2y-L_2(1-u_3(t))w_2y-L_3zc+\frac{L_3\mu_1(1-m)xz}{1+\lambda_1(1-m)x}+\frac{L_3\mu_2(1-m)yz}{1+\lambda_2(1-m)y}$
$+(1-u_2(t))L_3qz-(1-u_3(t))w_3L_3z$
$\frac{-\partial H}{\partial x}=\frac{\partial L_1}{dt}=-A_1+L_1r_1+2L_1x+\alpha_1L_1y+\frac{\beta_1L_1(1-m)z}{\lambda_1^2(1-m)^2}$
$+(1-u_1(t))L_1P_1+(1-u_3(t))L_1w_1-\frac{\mu_1L_3(1-m)z}{\lambda_1^2(1-m)^2}$
$\frac{-\partial H}{\partial y}=\frac{\partial L_2}{dt}=-A_2-L_2s+2L_2s+\alpha_2L_2x+\frac{\beta_2L_2(1-m)z}{\lambda_2^2(1-m)^2}$
$+(1-u_1(t))L_2P_2+(1-u_3(t))L_2w_2-\frac{\mu_2L_3(1-m)z}{\lambda_2^2(1-m)^2}$

$\frac{-\partial H}{\partial z} = \frac{\partial L_3}{dt} = -A_3 + \frac{\beta_1 L_1(1-m)x}{1+\lambda_1(1-m)x} + \frac{\beta_2 L_2(1-m)y}{1+\lambda_2(1-m)y} + L_3 c - \frac{\mu_1 L_3(1-m)x}{1+\lambda_1(1-m)x}$
$+ \frac{\mu_2 L_3(1-m)y}{1+\lambda_2(1-m)y} - (1-u_2(t))L_3 q - (1-u_3(t))L_3 w_3$

Now to obtain the optimal control solution $(u_i, i = 1,2,3...)$ of our Lagrangian we differentiate partially the Lagrangian L, with respect to u_1, u_2, u_3 and set it to zero as follows;

$\frac{\partial H}{\partial u_1} = -B_1 u_1 + L_1 h_1 x + L_2 h_2 y$

$\frac{\partial H}{\partial u_2} = -B_2 u_2 + L_1 w_1 x + L_2 w_2 y - L_3 w_3 z$

$\frac{\partial H}{\partial u_3} = -B_3 u_3 + L_3 e z$

Setting $\frac{\partial L}{\partial u_i} = 0$ for $i = 1,2,3$ and solving for the optimal control, we obtain;

$$u_1^* = \frac{L_1 h_1 x + L_2 h_2 y}{B_1},\ u_2^* = \frac{L_1 w_1 x + L_2 w_2 y + L_3 w_3 z}{B_2},\ u_3^* = \frac{L_3 e z}{B_3}$$

4. Numerical Simulation

In this section, we study numerically an optimal control of prey-predator system of Serengeti ecosystem. By the virtue of the fact that, the Serengeti ecosystem is complex and extraordinary and that a single control measure can not present all the threats, an investigation on the impacts of merging a minimum of three control parameters over eight years period is done. Further more, estimation of the objective functions real weight is extremely demanding and requires a bunch of information. In that regard, the objective weights are chosen on theoretical basis as $A_1 = 60$, $A_2 = 10$, $A_3 = 90$, $B_1 = 100$, $B_2 = 150$, $B_3 = 80$ just to grant the control parameters proposed in the paper and the initial state variable are chosen as; $x(0) = 40$, $y(0) = 30$ and $z(0) = 20$. Other parameters are $r = 2.09$, $s = 2.09$, $k = 200$, $l = 100$, $b_{12} = 0.001$, $b_{21} = 0.002$, $b_{13} = 0.02$, $b_{23} = 0.03$, $c = 1$, $b_{31} = 1.5$, $a = 0.1$, $d = 0.2$, $h_1 = 0.15$, $h_2 = 0.1$, $e = 0.05$, $w_1 = 0.05$, $w_2 = 0.05$, $w_3 = 0.25$. Next we investigate the of the following optimal control strategies on the wildebeest, zebra and lion prey-predator population under threats;

4.1. Strategy A: Application of construction of dams to control drought.

In this control scenario, the construction of dams u_1 is utilized to optimize the objective function J while treatment u_2 and education u_3 are not practiced. Figure 1 indicates the significant difference in the prey and predator populations with optimal control strategy compared to prey and predator populations without control.

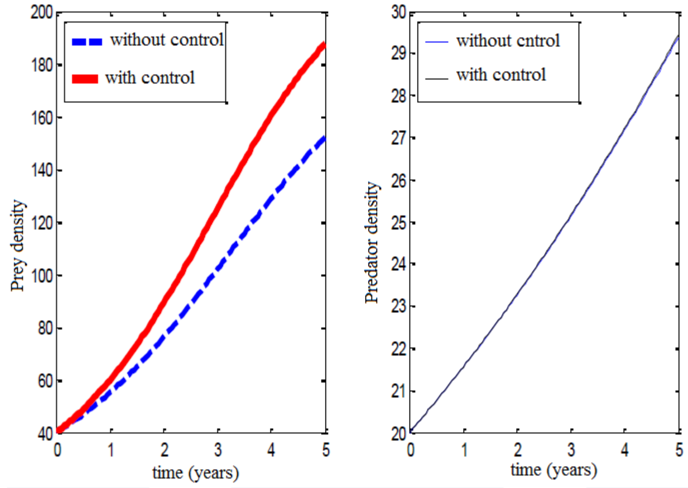

FIGURE 1. *Simulation of the system (2) showing the effect of optimal application of construction of dams*

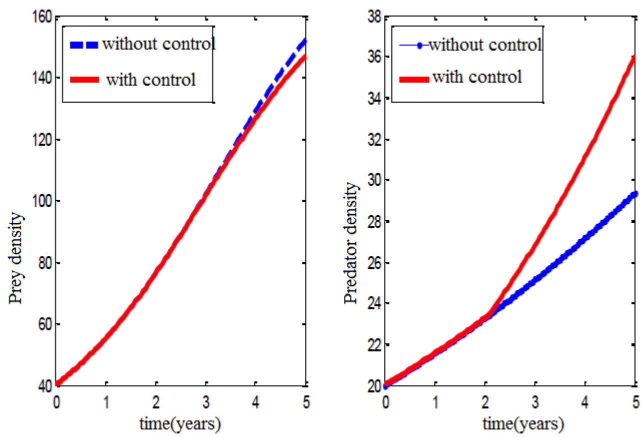

FIGURE 2. *Simulation of the system (2) showing the effect of optimal application of treatment*

4.2. Strategy B: Treatment to Control Infections. Under this scheme, the application of treatment u_2 is utilized to optimize the objective function J while construction of dams u_1 and education u_3 are not implemented. Results in figure 2 shows a significant difference in the prey and predator populations with optimal strategy compared to those without control.

4.3. Strategy C: Education to Control retaliatory killing. In this strategy, education, u_3 is used to optimize the objective function J while the set of application of treatment u_2 and construction of dams u_1 are not taken into account. In figure 3, the results show a significant

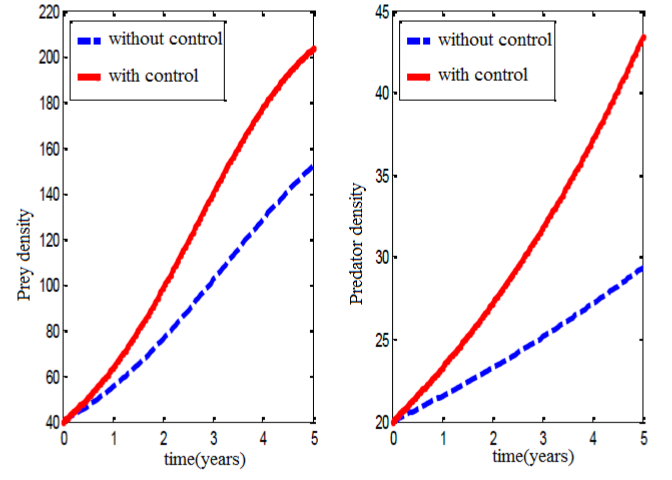

FIGURE 3. *Simulation of the system (2) showing the effect of optimal application of education*

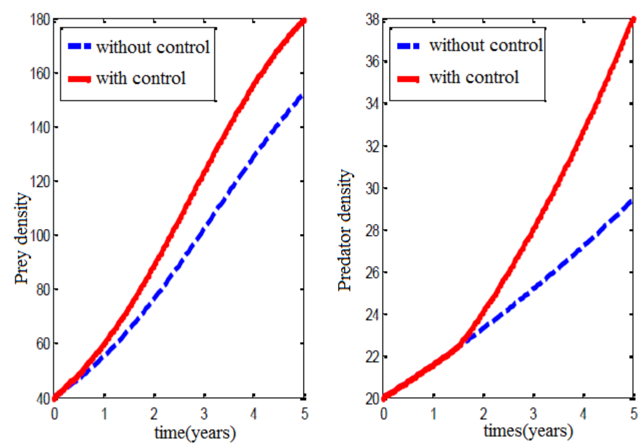

FIGURE 4. *Simulation of the system (2) showing the effect of optimal application of construction of dams and treatment*

difference in the predator population with optimal strategy compared to predator without control.

4.4. Strategy D: Combination of contruction of dams and treatment. In this aspect, the application of construction of dams u_1 and treatment, u_2 are utilized in optimization of the objective function whilst education u_3 is not taken into account. Figure 4 shows the significant difference between prey and predator before and after control.

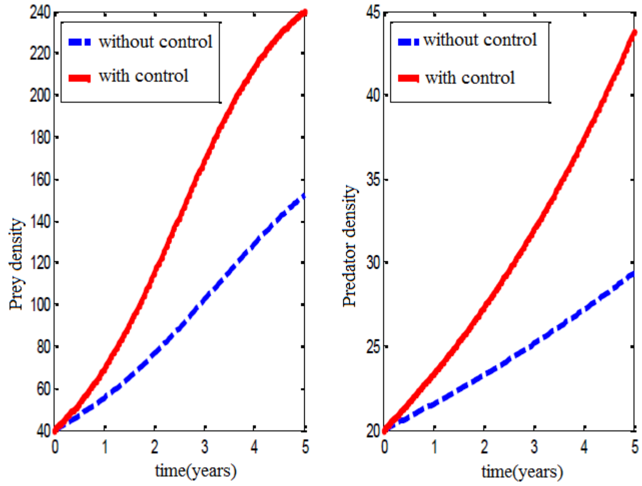

FIGURE 5. *Simulation of the system (2) showing the effect of optimal application of construction of dams and education*

4.5. Strategy E: Combination of Construction of dams and Education. The application of construction of dams u_1 and education u_3 are implemented in optimization of the objective function whilst we set treatment u_2 to be zero. In figure 5, the results shows significant difference between prey and predator population before and after control.

4.6. Strategy F: Combination of Treatment and Education. The application of treatment u_2 and education u_3 are used in optimization of the objective function J while the construction of dams is set to zero. In figure 6, the results shows a significant difference in prey and predator populations before and after control.

4.7. Strategy G: Combination of construction of dams, treatment and education. Here, the combination of all controls; construction of dams u_1, treatment, u_2 and education, u_3 are utilized in optimization of the objective function J. In figure 7, the results show significant difference in prey and predator populations before and after the control.

5. DISCUSSION AND CONCLUSION

In this paper we presented a prey-predator model with optimal control and control variables was introduced in the model where construction of dam was introduced to control drought,

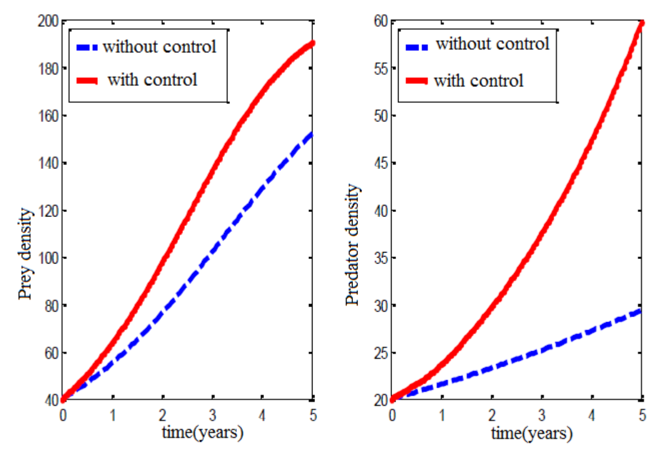

FIGURE 6. *Simulation of the system (2) showing the effect of optimal application of treatment and education*

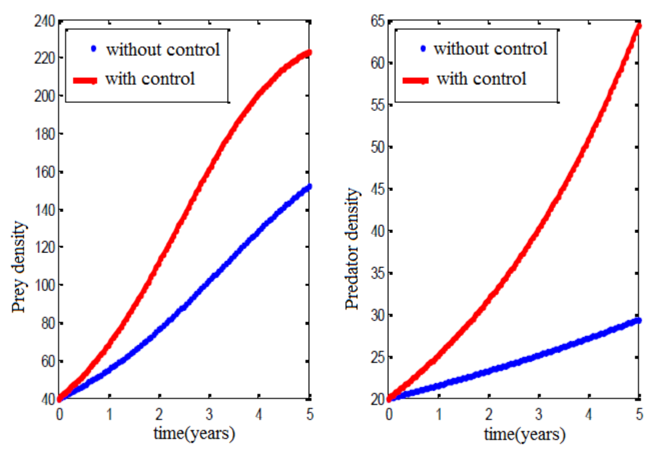

FIGURE 7. *Simulation of the system (2) showing the effect of optimal application of construction of dams, treatment and education*

treatment was introduced to control infections and education was introduced to control retaliatory killing in Serengeti ecosystem. In investigating the use of optimal control, we use one control at a time, combination of two at a time while setting others to zero to compare the effect of optimal strategies on elimination those threats of the ecosystem. The use of all control were also considered. Numerical results suggests that, the use of all control has the greatest impact in the control of the ecosystem.

ACKNOWLEDGEMENT

We acknowledge the Dar es salaam Institute of Technology for its support.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A.B. Ashine, D.M. Gebru, Mathematical modeling of a predator-prey model with modified Leslie-Gower and Holling-type II schemes, Glob. J. Sci. Front. Res.: F. Math. Dec. Sci. 17 (2007), 1-16.

[2] K. Chakraborty, T.K. Kar, Optimal Control of harvest and bifurcation of a prey-predator model with stage structure, Appl. Math. Comput. 217 (2017), 8778-8792.

[3] K.S. Chaudhuri, T.K. Kar, Harvesting in two prey-one predator fishery: Abioeconomic model, ANZIAM J. 45 (2004), 443-456.

[4] W. Fleming, R. Rishel, Deterministic and stochastic optimal control, Springer Science and Business Media, (2012).

[5] G. Haris, S. Thirgood, J. Hopcraft, et al. Global decline in aggregated migrations of mammals, Endang. Spec. Res. 7 (2009), 55-76.

[6] C.S. Holling, The function response of predators to prey density and its role in mimicry and population regulation, Mem. Entomol. Soc. Canada, 97 (1965), 1-60.

[7] J. Hopcraft, A. Sinclair, R. Holdo, et al. Why wildebeest the most abundant herbivore in the Serengeti ecosystem? Serengeti IV: Sustaining biodiversity in a coupled human natural system, University of Chicago Press, Chicago, 2013.

[8] D. Lukes, Differential equations: classical to controlled, Mathematics in Science and Engineering, Academic Press, New York, 1982.

[9] O. D, Makinde, Solving ratio-dependent predator-prey systems with constant effort harvesting using Adomian decomposition method, Appl. Math. Comput. 186 (2007), 17-22.

[10] L.S. Pontryagin, The Mathematical Theory of Optimal Processes, CRC Press, (1962).

[11] T. Sagamiko, N. Shaban, C.L. Nahonyo et al. Optimal control of threatened wildebeest-lion prey-predator system in the serengeti ecosystem, J. Ecol. 5 (2015), 110-119.

[12] V.J. Bennett, Computer modelling the Serengeti-Mara ecosystem. PhD thesis, University of Leeds, (2003).