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A NOTE ON HEMI-SLANT SUBMANIFOLDS OF PARA SASAKIAN MANIFOLDS

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Abstract. In this paper, we study hemi-slant submanifolds as a generalization of slant submanifolds of para contact manifolds. We particularly work out on hemi-slant submanifolds of para Sasakian manifold. Further, we obtain necessary and sufficient conditions for integrability of distributions which are involved in the definition of hemi-slant submanifolds of para Sasakian manifolds. Finally, we obtain the necessary and sufficient condition for a hemi-slant submanifold to be hemi-slant product and also provide some examples of such submanifolds.

Keywords: hemi-slant submanifolds; para Sasakian manifolds; totally geodesic foliations.
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1. Introduction

In the past two decades, almost contact geometry and related topics have been a rich research field for geometers due to its application in wide areas of physics as well as in pure mathematics. The notion of geometry of submanifolds begin with the idea of the extrinsic geometry of surface and it is developed for ambient space in the course of time. Nowadays this theory plays a key role in computer design, image processing, economic modelling as well as in mathematical

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physics and in mechanics. Immersions and submersions are special tool in the study of the theory of submanifolds.

As a natural generalization of both holomorphic immersions and totally real immersions, B. Y. Chen defined and study slant immersion in 1990 and consequent result on slant submanifolds were collected in his book [7]. Since then this interesting subject have been studied broadly by several geometers during last two decades [2], [10], [18], [19], [20] [21], [22], [25]. In 1996, A. Lotta [16] introduced the notion of slant immersion of a Riemannian manifold into almost contact metric manifold. Further slant submanifold was generalized as semi-slant submanifold, Pseudo-slant submanifold, bi-slant submanifold and hemi-slant submanifold etc., respectively, in different kinds of differentiable manifolds [17], [24], [25].

After the very remarkable work by [11], Cosymplectic manifold has become of great interest in last years. In nowadays, the importance of this manifold for the geometric description of time-dependent mechanics [see [1], [12]] is widely recognized. (Specially in the formulations of time dependent mechanics Cosymplectic manifold do play a major role.)

Motivated from above studies, we introduces hemi-slant submanifolds of para Sasakian manifolds so that both semi-slant and hemi-slant appears as particular cases of this introduced notion. The present paper is organized as follows: In section 2, we mention basic definition and some properties of para Sasakian manifolds. In section 3, we define hemi-slant submanifolds and some basic properties of submanifolds. Section 4 deals with necessary and sufficient conditions for integrability of distributions. In the last section, we provide some examples of such submanifolds.

2. PRELIMINARIES

We consider $\mathscr{M}$ is a $(2n+1)$-dimensional almost contact manifold [13] which carries a tensor field ϕ of the tangent space, $1-$form η and characteristic vector field ξ satisfying

$$\phi^2 = I - \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{1}$$

where $I : T\mathscr{M} \longrightarrow T\mathscr{M}$ is the identity map. We have from definition $\phi\xi = 0, \eta \circ \phi = 0$ and $rank(\phi) = 2n$.

Since any almost contact manifold $(\mathscr{M}, \phi, \xi, \eta)$ admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2}$$

for any vector fields $X, Y \in \Gamma(T\mathscr{M})$, where $\Gamma(T\mathscr{M})$ represents the Lie algebra of vector fields on $\mathscr{M}$. The manifold $\mathscr{M}$ together with the structure (ϕ, ξ, η, g) is called an almost contact metric manifold.

The immediate consequence of (2), we have

$$\eta(X) = g(X,\xi) \quad \text{and} \quad g(\phi X, Y) + g(X, \phi Y) = 0, \tag{3}$$

for all vector fields $X, Y \in \Gamma(T\mathscr{M})$.

An almost contact structure (ϕ, ξ, η) is said to be normal [4] if the almost complex structure J on the product manifold $\mathscr{M} \times R$ is given by

$$J(U,\, f\frac{d}{dt}) = (\phi U - f\xi,\, \eta(U)\frac{d}{dt}), \tag{4}$$

where $J^2 = -I$ and f is the differentiable function on $\mathscr{M} \times R$ has no torsion i.e., J is integrable. The condition for normality in terms of ϕ, ξ and η is $[\phi,\phi] + 2d\eta \otimes \xi = 0$ on $\mathscr{M}$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ.

An almost contact metric manifold is called a para Sasakian manifold [5], [23] if

$$(\widehat{\nabla}_X \phi) = 0, \tag{5}$$

$$\widehat{\nabla}_X \xi = 0, \tag{6}$$

where $\widehat{\nabla}$ represents the Levi-Civita connection of $(\widehat{M}, g)$.

The covariant derivative of ϕ is defined as

$$(\widehat{\nabla}_X \phi)Y = \widehat{\nabla}_X \phi Y - \phi \widehat{\nabla}_X Y,$$

using M is a para Sasakian manifold, we have

$$\phi \widehat{\nabla}_X Y = \widehat{\nabla}_X \phi Y. \tag{7}$$

Let M be a Riemannian manifold isometrically immersed in $\widehat{M}$ and the induced Riemannian metric on M is denoted by the same symbol g throughout in this paper. Let A and h denote the shape operator and second fundamental form, respectively, of immersion of M into $\widehat{M}$. The Gauss and Weingarten formulas of M into $\widehat{M}$ are given by [8]

$$\widehat{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{8}$$

and

$$\widehat{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{9}$$

for any vector fields $X,Y \in \Gamma(T\widehat{M})$ and V on $\Gamma(T^{\perp}M)$, where ∇ is the induced connection on M and $\nabla^{\perp}$ represents the connection on the normal bundle $T^{\perp}M$ of M and A_V is the shape operator of M with respect to normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, A_V and the second fundamental form $h : TM \otimes TM \longrightarrow T^{\perp}M$ of M into $\widehat{M}$ are related by

$$g(h(X,Y),V) = g(A_V X,Y), \tag{10}$$

for any vector fields $X,Y \in \Gamma(TM)$ and V on $\Gamma(T^{\perp}M)$.

The mean curvature vector is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i), \tag{11}$$

where n denotes the dimension of submanifold M and $\{e_1,e_2,....,e_n\}$ is the local orthonormal basis of tangent space at each point of M.

For any $X \in \Gamma(TM)$, we can write

$$\phi X = TX + NX, \tag{12}$$

where TX and NX are the tangential and normal components of ϕX on M respectively. Similarly for any $V \in T^{\perp}M$, we have

$$\phi V = tV + nV, \tag{13}$$

where tV and nV are the tangential and normal components of ϕV on M respectively.

A submanifold M of a Riemannian manifold $\widehat{M}$ is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H, \tag{14}$$

where H is the mean curvature vector. If $h(X,Y) = 0$ for all $X,Y \in \Gamma(TM)$, then M is said to be totally geodesic [9] and if $H = 0$, then M is said to be a minimal submanifold.

The covariant derivative of tangential and normal components of (12) and (13) are given as

$$(\widehat{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y,$$

$$(\widehat{\nabla}_X N)Y = \nabla_X^{\perp} NY - N\nabla_X Y,$$

$$(\widehat{\nabla}_X t)V = \nabla_X tV - t\nabla_X^{\perp} V$$

and

$$(\widehat{\nabla}_X n)V = \nabla_X^{\perp} nV - n\nabla_X^{\perp} V$$

for any $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Definition 2.1. *Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\widehat{M}$. A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be invariant* [3] *if $\phi\,(T_xM) \subseteq T_xM$, for every point $x \in M$.*

Definition 2.2. *A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be anti-invariant* [15] *if $\phi\,(T_xM) \subseteq T_x^{\perp}M$, for every point $x \in M$.*

Definition 2.3. *A submanifold M of an almost contact metric manifold $\widehat{M}$ is said to be slant* [6], *if for each non-zero vector X tangent to M at $x \in M$, the angle $\theta(X)$ between ϕX and T_xM is constant, i.e., it does not depend on the choice of the point $x \in M$ and $X \in T_xM$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold M is called proper slant submanifold if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$.*

We note that on a slant submanifold M if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. This means slant submanifold is a generalization of invariant and anti-invariant submanifolds.

3. Hemi-Slant Submanifolds of para Sasakian Manifolds

In the present section of the paper, we introduce the definition of hemi-slant submanifolds of para Sasakian manifolds and obtain some related results for later use.

Definition 3.1. *Hemi-slant submanifold M of para Sasakian manifold $\widehat{M}$ is a submanifold that admits three orthogonal complementary distributions D, D_θ and $D^\perp$ such that*
(i) TM admits the orthogonal direct decomposition

$$TM = D_\theta \oplus D^\perp \oplus <\xi> .$$

(ii) The distribution D_θ is slant with constant angle θ between ϕD_θ and D_θ. The angle θ is called slant angle.
(iii) The distribution $D^\perp$ is ϕ anti-invariant, i.e. $\phi D^\perp \subseteq T^\perp M$.

In this case, we call θ the hemi-slant angle of M. Suppose the dimension of distributions D_θ and $D^\perp$ are n_1 and n_2 respectively. Then we easily see the following particular cases:

We say that the hemi-slant submanifold M is proper if $D \neq 0$, $D^\perp \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.

This means hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant, semi-slant submanifolds and also they are the examples of hemi-slant submanifolds.

Let M be a hemi-slant submanifold of a para Sasakian manifold $\widehat{M}$. We denote the projections of $X \in \Gamma(TM)$ on the distributions D, D_θ and $D^\perp$ by P, Q and R respectively. Then we can write for any $X \in \Gamma(TM)$

$$X = PX + QX + RX + \eta(X)\xi. \tag{15}$$

Now, put

$$\phi X = TX + NX, \tag{16}$$

where TX and NX are tangential and normal components of ϕX on M.
Using (15) and (16), we obtain

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX. \tag{17}$$

Since $\phi D = D$ and $\phi D^{\perp} \subseteq T^{\perp}M$, we have $NPX = 0$ and $TRX = 0$. Therefore, we get

$$\phi X = TPX + TQX + NQX + NRX. \tag{18}$$

Then for any $X \in \Gamma(TM)$, it is easy to see that

$$TX = TPX + TQX$$

and

$$NX = NQX + NRX.$$

Thus from (18), we have the following decomposition

$$\phi(TM) = D \oplus TD_\theta \oplus ND_\theta \oplus ND^{\perp}. \tag{19}$$

where '$\oplus$' denotes orthogonal direct sum.

Since $ND_\theta \in \Gamma(T^{\perp}M)$ and $ND^{\perp} \in \Gamma(T^{\perp}M)$, we have

$$T^{\perp}M = ND_\theta \oplus ND^{\perp} \oplus \mu, \tag{20}$$

where μ is the orthogonal complement of $ND_\theta \oplus ND^{\perp}$ in $\Gamma(T^{\perp}M)$ and it is invariant with respect to ϕ.
For any non-zero vector field $V \in \Gamma(T^{\perp}M)$, we have

$$\phi V = tV + nV, \tag{21}$$

where $tV \in (D_\theta \oplus D^{\perp})$ and $nV \in \Gamma(\mu)$.

Using $\widehat{M}$ is a para Sasakian manifold, equations (8), (9), (12) and (13), and then on comparing tangential and normal components, we have following:

Lemma 3.1. *Let M be a submanifold of para Sasakian manifold $\widehat{M}$, then for any $X,Y \in \Gamma(TM)$, we have*

$$\nabla_X TY - A_{NY}X - T\nabla_X Y - th(X,Y) = 0$$

and

$$h(X,TY) + \nabla^{\perp}_X NY - N(\nabla_X Y) - nh(X,Y) = 0.$$

Lemma 3.2. *Let M be a hemi-slant submanifold of para Sasakian manifold $\widehat{M}$. Then we obtain*

$$TD_\theta = D_\theta,\ TD^{\perp} = \{0\},\ tND_\theta = D_\theta,\ tND^{\perp} = D^{\perp}. \tag{22}$$

Now, using (16) and (21) and using the fact that $\phi^2 = -I + \eta \otimes \xi$, then on comparing the tangential and normal components, we have the following:

Lemma 3.3. *Let M be a hemi-slant submanifolds of para Sasakian manifolds $\widehat{M}$. Then the endomorphism T and N, t and n in the tangent bundle of M, satisfies the following identities:*
(i) $T^2 + tN = -I + \eta \otimes \xi$ *on* TM,
(ii) $NT + nN = 0$ *on* TM,
(iii) $Nt + n^2 = -I$ *on* $(T^{\perp}M)$,
(iv) $Tt + tn = 0$ *on* $(T^{\perp}M)$,
where I is the identity operator.

Lemma 3.4. *Let M be a -hemi-slant submanifold of a para Sasakian manifold $\widehat{M}$. Then*

(1) $T^2X = -(\cos^2\theta)X$,
(2) $g(TX,TY) = (\cos^2\theta)g(X,Y)$,
(3) $g(NX,NY) = (\sin^2\theta)g(X,Y)$,

for any $X,Y \in D_\theta$.

Proof. The proof is the same as one found in [19]. □

Lemma 3.5. *Let M be a -hemi-slant submanifold of a para Sasakian manifold $\widehat{M}$. Then*

$$(\widehat{\nabla}_X T)Y = A_{NY}X + th(X,Y),$$

$$(\widehat{\nabla}_X N)Y = nh(X,Y) - h(X,TY),$$

$$(\widehat{\nabla}_X t)V = A_{nV}X - TA_V X$$

and

$$(\widehat{\nabla}_X n)V = -h(X,tV) - NA_V X$$

for any $X,Y \in \Gamma(TM)$ *and* $V \in \Gamma(T^{\perp}M)$.

Lemma 3.6. *Let M be a -hemi-slant submanifold of a para Sasakian manifold* $\widehat{M}$, *then*

$$A_{\phi Z}W = A_{\phi W}Z$$

for all $Z,W \in D^{\perp}$.

Proof. The proof is same as one found in [14]. □

Lemma 3.7. *Let M be a -hemi-slant submanifold of a para Sasakian manifold* $\widehat{M}$, *then*

(i) $g([X,Y],\xi) = 0$,

(ii) $g\left(\widehat{\nabla}_X Y, \xi\right) = 0$

for all $X,Y \in \Gamma\left(D_\theta \oplus D^{\perp}\right)$.

4. Integrability of Distributions and Decomposition Theorems

We now examine the integrability conditions for slant distribution D_θ and anti-invariant distribution $D^{\perp}$.

Theorem 4.1. *Let* (M,g,ϕ) *be a proper hemi-slant submanifold of a para Sasakian manifold* $(\widehat{M},g,\phi)$. *Then, the slant distribution* D_θ *is integrable if and only if*

$$g(A_{NW}Z - A_{NZ}W, TPX) = g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^{\perp}_Z NW - \nabla^{\perp}_W NZ, NRX), \tag{23}$$

for any $Z,W \in \Gamma(D_\theta)$ *and* $X \in \Gamma(D \oplus D^{\perp})$.

Proof. For any $Z,W \in \Gamma(D_\theta)$ and $X = PX + RX \in \Gamma(D \oplus D^\perp)$, using (2), (7) and (16) , we obtain

$$g([Z,W],X) = g(\widehat{\nabla}_Z NW, \phi X) - g(\widehat{\nabla}_Z \phi TW, X) - g(\widehat{\nabla}_W NZ, \phi X) + g(\widehat{\nabla}_W \phi TZ, X).$$

Then from (9), (16) and Lemma 3.4, we have

$$g([Z,W],X) = -g(A_{NW}Z - A_{NZ}W, \phi X) + \cos^2\theta g([Z,W],X) + g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^\perp_Z NW - \nabla^\perp_W NZ, \phi X),$$

which leads to

$$\sin^2\theta g([Z,W],X) = g(A_{NTW}Z - A_{NTZ}W, X) + g(\nabla^\perp_Z NW - \nabla^\perp_W NZ, NRX) - g(A_{NW}Z - A_{NZ}W, TPX),$$

hence the proof. □

From above theorem we have the following sufficient conditions for the slant distribution to be integrable:

Theorem 4.2. *Let M be a proper hemi-slant submanifold of a para Sasakian manifold $\widehat{M}$ if*

(24)
$$\nabla^\perp_Z NW - \nabla^\perp_W NZ \in ND_\theta \oplus \mu,$$
$$A_{NTW}Z - A_{NTZ}W \in D_\theta \quad \textit{and}$$
$$A_{NW}Z - A_{NZ}W \in D^\perp \oplus D_\theta,$$

for any $Z,W \in \Gamma(D_\theta)$, then the slant distribution D_θ is integrable.

Theorem 4.3. *Let M be a hemi-slant submanifold of a para Sasakian manifold $\widehat{M}$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$\nabla^{\perp}_{Z}NW-\nabla^{\perp}_{W}NZ\in ND^{\perp}\oplus\mu$$

for any $Z,W\in\Gamma(D^{\perp})$ and $X\in\Gamma(D\oplus D_{\theta})$.

Proof. For any $Z,W\in\Gamma(D^{\perp})$ and $X=PX+QX\in\Gamma(D\oplus D_{\theta})$, using (2), (7), (9), (16) and Lemma 3.7, we obtain

$$\begin{aligned}
g([Z,W],X) &= g(\widehat{\nabla}_{Z}\,\phi W,\phi X)-g(\widehat{\nabla}_{W}\,\phi Z,\phi X)\\
&= g(A_{\phi Z}W-A_{\phi W}Z,TPX)+g(\nabla^{\perp}_{Z}\phi W-\nabla^{\perp}_{W}\phi Z,NQX)\\
&= g(\nabla^{\perp}_{Z}NW-\nabla^{\perp}_{W}NZ,NQX)
\end{aligned}$$

hence the proof. □

We now investigate the geometry of leaves of anti-invariant and slant distribution.

Theorem 4.4. *Let M be a proper hemi-slant submanifold of a para Sasakian manifold $\widehat{M}$. Then, the slant distribution D_{θ} defines a totally geodesic foliation on M if and only if*

$$g(\nabla^{\perp}_{X}NY,NRZ) = g(A_{NY}X,TPZ)-g(A_{NTY}X,Z)\quad \textit{and} \tag{25}$$

$$g(A_{NY}X,tV) = g(\nabla^{\perp}_{X}NY,nV)-g(\nabla^{\perp}_{X}NTY,V), \tag{26}$$

for any $X,Y\in\Gamma(D_{\theta})$, $Z\in\Gamma(D\oplus D^{\perp})$ and $V\in\Gamma\left(T^{\perp}M\right)$.

Proof. For any $X,Y\in\Gamma(D_{\theta})$, $Z=PZ+RZ\in\Gamma(D\oplus D^{\perp})$ and using (2), (7) and (16), we have

$$\begin{aligned}
g(\widehat{\nabla}_{X}Y,Z) &= g(\widehat{\nabla}_{X}\phi Y,\phi Z)=g(\widehat{\nabla}_{X}TY,\phi Z)+g(\widehat{\nabla}_{X}NY,\phi Z)\\
&= -g(\widehat{\nabla}_{X}T^{2}Y,Z)-g(\widehat{\nabla}_{X}NTY,Z)+g(\widehat{\nabla}_{X}NY,TPZ+NRZ)
\end{aligned}$$

Then using (9), (16) and Lemma 3.4, and the fact that $NPZ=0$. We have

$$\begin{aligned} g(\widehat{\nabla}_X Y, Z) &= \cos^2\theta g(\widehat{\nabla}_X Y, Z) + g(A_{NTY}X, Z) \\ &\quad -g(A_{NY}X, TPZ) + g(\nabla^{\perp}_X NY, NRZ), \\ \sin^2\theta g(\widehat{\nabla}_X Y, Z) &= g(A_{NTY}X, Z) \\ &\quad -g(A_{NY}X, TPZ) + g(\nabla^{\perp}_X NY, NRZ). \end{aligned} \tag{27}$$

Similarly, we get

$$\sin^2\theta g(\widehat{\nabla}_X Y, V) = -g(\nabla^{\perp}_X NTY, V) - g(A_{NY}X, tV) + g(\nabla^{\perp}_X NY, nV). \tag{28}$$

Thus from (27) and (28), we have the assertions. □

Theorem 4.5. *Let M be a proper hemi-slant submanifold of a para Sasakian manifold $\widehat{M}$. Then, the anti-invariant distribution $D^{\perp}$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g(h(X,Y), NTQZ) &= g(\nabla^{\perp}_X NY, NQZ) \quad \textit{and} \\ g(A_{NY}X, tV) &= g(\nabla^{\perp}_X NY, nV), \end{aligned} \tag{29}$$

for any X, $Y \in \Gamma(D^{\perp})$, $Z \in \Gamma(D \oplus D_\theta)$ and $V \in \Gamma\left(T^{\perp}M\right)$.

Proof. Since X, $Y \in \Gamma(D^{\perp})$, $Z = PZ + QZ \in \Gamma(D \oplus D_\theta)$ and using (2), (7), (16) and the fact that M is a para Sasakian manifold, we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, Z) &= g(\widehat{\nabla}_X \phi Y, \phi Z) \\ &= g(\widehat{\nabla}_X \phi Y, \phi PZ) + g(\widehat{\nabla}_X \phi Y, TQZ) + g(\widehat{\nabla}_X \phi Y, NQZ) \\ &= g(\widehat{\nabla}_X Y, PZ) - g(\widehat{\nabla}_X Y, T^2 QZ) - g(\widehat{\nabla}_X Y, NTQZ) \\ &\quad +g(\widehat{\nabla}_X NY, NQZ). \end{aligned} \tag{30}$$

The left side of the equation (30) gives $g(\widehat{\nabla}_X Y, Z) = g(\widehat{\nabla}_X Y, PZ) + g(\widehat{\nabla}_X Y, QZ)$, using (8), (9) and Lemma 3.4, the equation (30) became

$$g(\widehat{\nabla}_X Y, \sin^2\theta QZ) = -g(h(X,Y), NTQZ) + g(\nabla^{\perp}_X NY, NQZ) \tag{31}$$

which gives first section of (29).

Now for any $X, Y \in \Gamma(D^{\perp})$ and $V \in \Gamma\left(T^{\perp}M\right)$, and using (2), (7), (9), (13), we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, V) &= g(\widehat{\nabla}_X \phi Y, \phi V) = g(\widehat{\nabla}_X NY, tV) + g(\widehat{\nabla}_X NY, nV), \\ &= -g(A_{NY}X, tV) + g(\nabla^{\perp}_X NY, nV), \end{aligned} \tag{32}$$

which gives second part of (29). □

From Theorems 4.4 and 4.5, we have following decomposition theorem:

Theorem 4.6. *Let M be a proper hemi-slant submanifold of a para Sasakian manifold $\widehat{M}$. Then the fiber of M is a local product Riemannian manifold of the form $M_{D_\theta} \times M_{D^{\perp}}$, where M_{D_θ} and $M_{D^{\perp}}$ are leaves of D_θ and $D^{\perp}$ respectively, if and only if the conditions* (29), (25) *and* (26) *holds.*

5. Examples

Example 5.1. *Consider a* $15-$*dimensional differentiable manifold*

$$\overline{M} = \{(x_1, x_2, ..., x_7, y_1, y_2, ..., y_7, z) \in \mathrm{R}^{15}\}.$$

We choose the vector fields

$$E_i = \frac{\partial}{\partial y_i},\ \ E_{7+i} = \frac{\partial}{\partial x_i},\ \ E_{15} = \xi = \frac{\partial}{\partial z},\ \ for\ \ i = 1, 2, ..., 7.$$

Let g be a Riemannian metric defined by

$$g = (dx_1)^2 + (dx_2)^2 + ... + (dx_7)^2 + (dy_1)^2 + (dy_2)^2 + ... + (dy_7)^2 + (dz)^2.$$

Then we find that $g(E_i, E_i) = 1$ and $g\left(E_i, E_j\right) = 0$, for $1 \leq i \neq j \leq 15$.

Hence $\{E_1, E_2, ..., E_{15}\}$ forms an orthonormal basis at each point of tangent space of $\overline{M}$. Thus $1-$form $\eta = dz$ is defined by $\eta(E) = g(E, \xi)$, for any $E \in \Gamma(T\overline{M})$.

We define $(1,1)$*-tensor field* ϕ *as*

$$\phi\left(\frac{\partial}{\partial x_i}\right)=\frac{\partial}{\partial y_i},\ \ \phi\left(\frac{\partial}{\partial y_j}\right)=-\frac{\partial}{\partial x_j},\ \ \phi\left(\frac{\partial}{\partial z}\right)=0\ \ \forall\, i,j=1,2,...,7.$$

By using linearity of ϕ *and g, we have*

$$\phi^2=-I+\eta\otimes\xi,\ \ \phi\xi=0,\ \ \eta(\xi)=1,$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\ \ \textit{for any } X,Y\in\Gamma(T\overline{M})$$

Hence $(\overline{M},\phi,\xi,\eta,g)$ *is an almost contact metric manifold. Also, we can easily shown that* $(\overline{M},\phi,\xi,\eta,g)$ *is a para Sasakian manifold of dimension* 15.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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