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ON SQUARE SUM DIFFERENCE COLORING OF GRAPHS

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Abstract. Let G be a graph with p vertices. A bijection $f : V(G) \to \{0.1, 2, \ldots, p-1\}$ is called a Square Sum Difference (SSD) coloring of G if the induced function. $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = [f(u)]^2 + [f(v)]^2 - f(u)f(v)$ is injective for all edges $uv \in E(G)$, A graph G is called an SSD colorable if G admits an SSD coloring. Further, an SSD coloring is called an odd square sum difference (OSSD) coloring, if $f^*(E)$ contains only odd integers. A graph G is called an OSSD colorable, if G admits an OSSD coloring.

Keywords: graph labeling; square sum difference coloring; SSD colorable graphs; odd square sum difference coloring; OSSD colorable graphs.

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1. INTRODUCTION

By a graph G, we mean a finite undirected simple graph. Graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions and a lot of different types of labeling were investigated [2]. The concept of a Square Sum labeling was introduced by Ajitha et. al [1] in 2009. S.G. Sonchhatra, G. V Ghodasaara [5] introduced a closely related concept namely, Sum Perfect Square labeling. Shaima [4] introduced Square Difference graphs and showed the existence of several square difference graphs. These ideas motivated us to

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introduce a new type of graph coloring, called square sum difference coloring (SSD coloring) and related type of graphs, called SSD graphs.

In this paper, we initiate a study of the square sum difference graphs. We prove that several types of graphs such as Trees, Paths, Cycles, Stars, Bi-stars, Wheel graphs,Complete graphs for $n < 4$, Helm graphs, Friendship graph, Gear graphs, Crown graphs, Double Crown graphs, Flower graphs, Ladder graphs, Coconut trees and Comb graphs are SSD graphs. We also prove that Paths, Bi-stars, Helm graphs, Ladder graphs and Comb graphs are odd SSD graphs as well. Terms not specified separately in this paper, we refer Harary [3].

2. Main Results

Definition 2.1. A bijection $f : V(G) \to \{0,1,2,\ldots p-1\}$ is called a Square Sum Difference (SSD) coloring if the induced function $f^* : E(G) \to \mathbb{N}$ given by $f^*(uv) = [f(u)]^2 + [f(v)]^2 - f(u)f(v)$ is injective, $\forall$ edges $uv \in E(G)$. A graph G is called an *SSD* graph if G admits SSD coloring.

Definition 2.2. An SSD coloring is called an odd square sum difference (OSSD) coloring, if $f^*(E)$ contains only odd integers. A graph G is called OSSD graph, if G admits OSSD coloring.

For $uv \in E(G)$, we can observe the following from the definition of SSD coloring: if $f(u) = 0$ then $f^*(uv) = v^2$, a perfect square. If $uv \in E(G)$ and $f(u) = 1$ then $f^*(uv = i) = i(i-1)+1; i = 2,3,\ldots)$. If $f(u) = m$ and $f(v) = m+1$ then $f^*(uv) = m(m+1)+1$. If u and v are odd integers or one of them is an odd integer; then $f^*(uv)$ is always an odd integer. If u and v are both even integers, then $f^*(uv)$ is always even.

Theorem 2.3. *Every tree is a SSD graph*

Proof: Let T be a tree with v_0 as the root where degree of v_0 is Δ. Let $\{v_1, v_2, \ldots v_n\}$ be the vertices of the tree T. Let $\{v_1, v_2, \ldots, v_k\}$ be the vertices at a distance 1 from v_0 and take it as level-I. Let $\{v_{k+1}, v_{k+2}, \ldots, v_t\}$ be vertices at a distance 2 from v_0 and take it as level-II, $k+1 \le t \le n-1$ and so on. Define $f : V(T) \to \{0,1,2,\ldots n-1\}$ by $f(v_i) = i$, $0 \le i \le n-1$.

The range of f is $\{0,1,2,\ldots,n-1\}$, which is same as the co-domain. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is bijection.

Using this coloring the edge colors at level-I are $1^2, 2^2, 3^2, \dots k^2$ by the definition $(f(u))^2 + (f(v))^2 - f(u)f(v)$ where $uv \in E(G)$. The edge labels in level-II are distinct from that in level-I.

At each consecutive level the vertex labels $f(v_i) < f(v_j) \ \ \forall\, i \& j$ where v_i is at level-k and v_i is at level-k+1. So corresponding edge labels are also in the increasing order. $f^*(uv) : E(G) \rightarrow \mathbb{N}$ is injective and hence trees are SSD graphs.

Theorem 2.4. *The path P_n are SSD graphs. Also P_n are OSSD graphs*

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of the path P_n of length $n-1$. Let $\{v_i v_{i+1};\ 1 \leq i \leq n-1\}$ be the edge set. $p = n$ and $q = n-1$.
Define $f : V(P_n) \rightarrow \{0, 1, 2, \dots, n-1\}$ by $f(v_i) = i-1;\ 1 \leq i \leq n.$ The range of f is $\{0, 1, 2, \dots, n-1\}$, which is same as the co-domain. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection.
Using the definition of edge coloring $f^*(uv) = (f(u))^2 + (f(v))^2 - f(u)f(v)\ \forall \quad u, v \in V(G)$. We have $f^*(v_i v_{i+1}) = i(i-1)+1;\ 1 \leq i \leq n-1$. Elements in the edge set are in the increasing order and all the edge colors are distinct. Also, all the edge colors are odd integers. So it admits SSD and also odd square sum difference coloring.

Theorem 2.5. *Cycle C_n are SSD graph, $n \geq 3$, $n \in \mathbb{N}$*

Proof: Let $\{v_1, v_2, \dots v_n\}$ be the vertices of the cycle C_n. Let $\{v_i v_{i+1}; \leq i \leq n-1\} \cup \{v_n v_1\}$ be the edge set. $p = n$ and $q = n$. Define $f : V(C_n) \rightarrow \{0, 1, 2, \dots n-1\}$ by $f(v_i) = i-1$ for $1 \leq i \leq n$. The range of f is $\{0, 1, 2, \dots, n-1\}$, which is same as the co-domain. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection. Using the definition of edge coloring function defined by , $f^*(uv) = (f(u))^2 + (f(v))^2 - f(u)f(v)$ where $u, v \in V(G)$, we have $f^*(v_i v_{i+1} = (i-1)i + 1\ ;\ 1 \leq i \leq n-1$.
$f^*(v_n v_1) = (n-1)^2$. Elements of edge sets are in the increasing order and all the edge colors are distinct. So the C_n is a SSD graph.

Theorem 2.6. *The stars K_{1n} are SSD graphs.*

Proof: Let v be the apex vertex and $v_1, v_2, \dots, v_n$ be the penedent vertices of the star K_{1n}. Let $\{vv_i; 1 \leq i \leq n\}$ be the edge set, $p = n+1$, $q = n$. Define $f : V(K_{1n}) \rightarrow \{0, 1, 2, \dots n\}$ as

$f(v_i) = i$, $1 \leq i \leq n$ and $f(v) = 0$. The range of f is $\{1,2,\ldots,n\} \cup \{0\}$ which is same as the co-domain. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is bijection. It can be deduced that the edge colors are $f^*(vv_i) = i^2$; $1 \leq i \leq n$. The elemests of the edge sets are in the increaing order and are all distinct, so star K_{1n} admits square sum difference coloring.

Theorem 2.7. *The bistars $B_{n\,n}$ are SSD graphs. Also $B_{n\,n}$ are OSSD graphs.*

Proof: Let u and v be the apex vertices of the bistar B_{nn}. Let $u_1, u_2, \ldots u_n, v_1, v_2, \ldots v_n$ be the pendent vertices. Let the edge set $\{uu_i, i \leq i \leq n\} \cup \{vv_i, 1 \leq i \leq n\} \cup \{uv\}$. Here $p = 2n+2$, $q = 2n+1$. Define $f : V(B_{nn}) \to \{0,1,2,\ldots 2n+1\}$ by $f(u) = 0, f(v) = 1$
$f(u_i) = 2i+1; 1 \leq i \leq n$, $f(v_i) = 2i, 1 \leq i \leq n$.
The range of f is $\{0\} \cup \{1\} \cup \{3,5,7,\ldots,2n+1\} \cup \{2,4,6,\ldots,2n\}$, which is same as the co-domain $\{0,1,2,\ldots,2n+1\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection. Using the above defined vertex coloring, the edge colors are obtained by $f^*(uv) = 1$, $f^*(uu_i) = (2i+1)^2; 1 \leq i \leq n$,$f^*(vv_i) = (2i-1)2i+1$; $1 \leq i \leq n$. All the edge colors are distinct by the definition of edge colors. So every bistar B_{nn} are square sum difference graphs. Moreover all the edge colors are odd integers, it posses odd square sum difference graphs.

Theorem 2.8. *The wheel graph W_n is SSD graph for $n \geq 4$*

Proof: Let V be the apex vertex of the wheel and $v_1, v_2, \ldots v_n$ be the rim vertices.
Let $\{vv_i, 1 \leq i \leq n\}$ be the verte set and $\{vv_i, 1 \leq i \leq n\} \cup \{v_iv_{i+1}, 1 \leq i \leq n\}$ be edge set.
$p = n+1, q = 2n$. Define $f : V(W_n) \to \{0,1,2,\ldots n\}$ as $f(v) = 0, f(v_i) = i; 1 \leq i \leq n-2, f(v_{n-1}) = n, f(v_n) = n-1$ The range of f is $\{0\} \cup \{1,2,\ldots,n-2\} \cup \{n\} \cup \{n-1\}$, which is same as the co-domain $\{0,1,\ldots,n\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection. In view of above coloring pattern edge colors are $f^*(vv_i) = i^2, 1 \leq i \leq n$,$f^*(v_iv_{i+1}) = (i+1)i+1, 1 \leq i \leq n-3$,$f^*(v_{n-2}v_n) = n^2+4$, $f^*(v_nv_{n-1}) = n(n-1)+1$,$f^*(v_nv_1) = n^2$. The elements of the edge set are in the increasing order and are all distinct. So W_n admits SSD.

Corollary 2.9. W_3 is not a square sum difference graph.

In W_3 all the edges are adjacent (graph is K_4). Using $0,1,2,3$ as vertex colors, edges with end vertices $1,3$ and $2,4$ takes the same value 7 as its edge colors. So it violates the condition of injection.

Theorem 2.10. *Complete graph K_n is square sum difference graph [SSD] for $n < 4$; $n \in \mathbb{N}$*

Proof: For $n = 1,2$, $K_1 and K_2$ are special case of tree. So using theorem 2.7, $K_1 and K_2$ are SSD graphs. For $n = 3$ the graph K_3 is same as cycle C_3, as per theorem 2.9, K_3 is SSD graph. Consider $n = 4$, when we use $0,1,2,3$ as vertex labels, edges with end vertices, $1,3$ and $2,3$ takes the color 7 as its edge colors. So it violates the the condition of edge injection. Hence K_4 is not SSD graphs. Again $K_4 \subset K_5 \subset K_6 \ldots \subset K_n$. Hence K_n is not SSD graphs $\forall\ n \geq 4$, $n \in \mathbb{N}$

Theorem 2.11. *Helm graphs H_n are SSD graphs* $\forall n \geq 4$. *Also H_n are OSSD graphs*

Proof: Let the central vertex v and $v_1, v_2, \ldots v_n$ be the successive vertices on the cycle and the pendent vertices be $w_1, w_2, \ldots w_n$ in the same order. Let $\{v, v_i, w_i; 1 \leq i \leq n\}$ be the vertex set and the $\{vv_i; 1 \leq i \leq n\} \cup \{v_iv_{i+1}; 1 \leq i \leq n-1\} \cup \{v_iw_i; 1 \leq i \leq n\} \cup \{v_nv_1\}$ be the edge set.
$p = 2n+1, q = 3n$. Define $f : V(H_n) \to \{0,1,2,\ldots,2n\}$ by $f(v) = 0$, $f(v_i) = 2i-1; 1 \leq i \leq n$, $f(w_i) = 2i; 1 \leq i \leq n$. The range of f is $\{0\} \cup \{1,3,\ldots,2n-1\} \cup \{2,4,6,\ldots,2n\}$, which is same as the co-domain $\{0,1,\ldots,2n\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection. Using the above vertex colors, edge colors are obtained as

$f^*(vv_i) = (2i-1)^2; 1 \leq i \leq n$

$f^*(v_iw_i) = 2i(21-1)+1; 1 \leq i \leq n$ $f^*(v_iv_{i+1}) = (2i-1)(2i+1)+4; 1 \leq i \leq n-1$

$f^*(v_nv_1) = 2n(2n-3)+3$

All the edge colors are distinct are all odd integers. So H_n are SSD and odd SSD graphs.

Theorem 2.12. *The friendship graph F_n are SSD graphs.*

Proof: Let F_n be the friendshio graph with n triangles and one apex vertex v. Let v_{i1} and v_{i2} , $1 \leq i \leq n$ be the vertices of the base edge of the triangle. Let $\{v, v_{i1}, v_{i2}; 1 \leq i \leq n\}$ be the vertex set and $\{vv_{i1}, vv_{i2},; 1 \leq i \leq\} \cup \{v_{i1}v_{i2}; 1 \leq i \leq n\}$ be edge set. $p = 2n+1, q = 3n$.

Define $f : V(F_n) \to \{0,1,2\ldots 2n\}$ by $f(v) = 0$

$f(v_{i1}) = 2i-1, 1 \leq i \leq n$

$f(v_{i2}) = 2i, 1 \leq i \leq n$.The range of f is $\{0\} \cup \{1,3,\ldots,2n-1\} \cup \{2,4,6,\ldots,2n\}$, which is same as the co-domain $\{0,1,\ldots,2n\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection. Using the above vertex colors, edge colors are

$f^*(vv_{i1}) = (2i-1)^2; 1 \leq i \leq n$

$f^*(vv_{i2}) = (2i)^2; 1 \leq i \leq n$

$f^*(v_{i1}v_{i2}) = (2i-1)(2i)+1; 1 \leq i \leq n$

It can be inferred that all the edge colors are distinct. So F_n admits square sum difference coloring.

Theorem 2.13. *Gear graphs G_n are SSD graphs.*

Proof: Let v be the apex vertex and $v_1, v_2, \ldots v_{2n}$ be vertices of the rim of the gear graph G. $\{v, v_i; 1 \leq i \leq 2n\}$ be vertex set and $\{vv_{2i-1}; 1 \leq i \leq n\} \cup \{v_iv_{i+1}; 1 \leq i \leq 2n-1\} \cup \{v_{2n}v_1\}$ be the edge set. $p = 2n+1, q = 3n$.

Define $f : V(G_n) \to \{0,1,2\ldots 2n\}$ as follows $f(v) = 0$

$f(v_i) = i, 1 \leq i \leq 2n-2$

$f(v_{2n}) = 2n-1$

$f(v_{2n-1}) = 2n$. The range of f is $\{0\} \cup \{1,2,\ldots,2n-2\} \cup \{2n-1\} \cup \{2n\}$, which is same as the co-domain $\{0,1,\ldots,2n\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection

It can be inferred from the vertex colors, the edge colors are $f^*(vv_{2i-1}) = (2i-1)^2; 1 \leq i \leq n-1$

$f^*(vv_{2n-1}) = (2n)^2$

$f^*(v_iv_{i+1}) = i(i+1)+1; 1 \leq i \leq 2n-3$

$f^*(v_{2n-2}v_{2n-1} = 4n(n-1)+4$

$f^*(v_{2n-1}v_{2n} = 2n(2n-1)+1$

$f^*(v_{2n}v_1 = 2n(2n-3)+3$

All the edge colors are distinct. So G_n admits SSD coloring.

Theorem 2.14. *Crown graphs C_n^+ are SSD graphs.*

Proof: Let $v_1, v_2, \dots v_n$ be vertices of C_n and $w_1, w_2, \dots w_n$ be the pendent vertices added in the verrtices of cycle C_n. *ie*; v_i to w_i are pendent edges $i = 1, 2, \dots n$. Let $\{v_i, w_i; 1 \leq i \leq n\}$ be vertex set and $\{v_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{v_i w_i; 1 \leq i \leq n\} \cup \{v_n v_1\}$ be the edge set. $p = 2n, q = 2n$.
Define $f : V(G) \to \{0, 1, 2 \dots 2n-1\}$ as follows $f(v_{i+1}) = 2i; 0 \leq i \leq n-1$
$f(w_i) = 2i-1, 1 \leq i \leq n$. The range of f is $\{0, 2, 4, \dots, 2n-2\} \cup \{1, 3, 5, \dots, 2n-1\}$, which is same as the co-domain $\{0, 1, 2 \dots, 2n-1\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection. Using the above vertex coloring, edge colors are $f^*(v_i v_{i+1}) = 2i(2i-2) + 4;\ 1 \leq i \leq n-1$
$f^*(v_i w_i) = (2n-2)(2i-1) + 1;\ 1 \leq i \leq n$
$f^*(v_n v_1) = (2n-2)^2$
In view of the above edge colors, all the edge colors are distinct. So C_n admits SSD coloring.

Theorem 2.15. *Double crown C_n^{++} are SSD graphs.*

Proof: Let $v_1, v_2, \dots v_n$ be vertices of cycle C_n. Let $v_{i1}, v_{i2}, i = 1, 2, \dots n$ be the pendent vertices attaching to v_i Let $\{v_i, v_{i1}, v_{i2} 1 \leq i \leq n\}$ be vertex set and $\{v_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i v_{i1}, 1 \leq i \leq n\} \cup \{v_i v_{i2}, 1 \leq i \leq n\}$ be the edge set. $p = 3n, q = 3n$.
Define $f : V(C_n^{++}) \to \{0, 1, 2 \dots 3n-1\}$ as follows $f(v_i) = 3(i-1); 1 \leq i \leq n$
$f(v_{i1}) = 3i-2, 1 \leq i \leq n$
$f(v_{i2}) = 3i-1, 1 \leq i \leq n$. The range of f is $\{0, 3, 6 \dots, 3n-3\} \cup \{1, 4, 7, \dots, 3n-2\} \cup \{2, 5, 8, \dots, 3n+1\}$, which is same as the co-domain $\{0, 1, 2, 3, \dots, 3n-1\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection Using the above vertex coloring, edge colors are obtained as
$f^*(v_i v_{i+1}) = 9[i(i-1) + 1]; 0 \leq i \leq n-1$
$f^*(v_i v_{i1}) = 3(i-1)(3i-2) + 1 \leq i \leq n$
$f^*(v_i v_{i2}) = 3(i-1)(3i-1) + 4, 1 \leq i \leq n$
$f^*(v_n v_1) = 3(n-1)^2$
In view of the above edge colors, all the edge colors are distinct. So C_n^{++} admits SSD coloring.

Theorem 2.16. *Flower graphs Fl_n are SSD graphs.*

Proof: Let the central vertex be v. Let $v_1, v_2, \ldots v_n$ be the successive vertices of cycle. Let $w_1, w_{i2}, \ldots w_n$ be the pendent vertices in the same order. Let $\{v, v_i, w_i, ; 1 \leq i \leq n\}$ be vertex set and $\{vv_i; 1 \leq i \leq n\} \cup \{v_i v_{1+1}; 1 \leq i \leq n-1\} \cup \{v_i w_i, 1 \leq i \leq n\} \cup \{v_n v_1\} \cup \{vw_i, 1 \leq i \leq n\}$ be the edge set. $p = 2n+1, q = 4n$.

Define $f : V(Fl_n) \to \{0, 1, 2 \ldots 2n\}$ as follows

$f(v) = 0)$

$f(v_i) = 2i-1; 1 \leq i \leq n$

$f(w_i) = 2i; 1 \leq i \leq n$. The range of f is $\{0\} \cup \{1, 3, 5, \ldots, 2n-1\} \cup \{2, 4, 6, \ldots, 2n\}$, which is same as the co-domain $\{0, 1, 2, \ldots, 2n\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection. Using the above vertex coloring, edge colors are

$f^*(vv_i) = 2i - 1^2; 1 \leq i \leq n$

$f^*(vw_i) = (2i)^2; 1 \leq i \leq n$

$f^*(v_i w_1) = (2i-1)2i + 1; 1 \leq i \leq n$

$f^*(v_i v_{i+1}) = (4i)^2 + 3; 1 \leq i \leq n-1$

$f^*(v_n v_1) = 2n(2n-3) + 3$

In view of the above edge colors, all the edge colors are distinct. So Fl_n admits SSD coloring.

Theorem 2.17. *Ladder graphs L_n are SSD graphs. Also ladder graphs admits OSSD graphs.*

Proof: Let $v_1, v_2, \ldots v_n$ are the vertices of one side of the ladder graph L_n and $w_1, w_2, \ldots w_n$ are the vertices of the ladder graph. Let $\{v_i, w_i, ; 1 \leq i \leq n\}$ be vertex set and $\{v_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{w_i w_{1+1}; 1 \leq i \leq n-1\} \cup \{v_i w_i, 1 \leq i \leq n\}$ be the edge set. $p = 2n, q = 3n-2$.

Define $f : V(L_n) \to \{0, 1, 2 \ldots 2n-1\}$ as follows

$f(v_{2i-1}) = 4i - 4; 1 \leq i \leq \lfloor (\frac{n+1}{2}) \rfloor$

$f(v_{2i}) = 4i - 1; 1 \leq i \leq \lceil (\frac{n-1}{2}) \rceil$

$f(w_{2i-1}) = 4i - 3; 1 \leq i \leq \lfloor (\frac{n+1}{2}) \rfloor$

$f(w_{2i}) = 4i - 2; 1 \leq i \leq \lceil (\frac{n-1}{2}) \rceil$

The range of f is $\{0, 4, 8, \ldots, 2n-4\} \cup \{3, 7, 11, \ldots, 2n-1\} \cup \{1, 5, 9, \ldots, 2n-3\} \cup \{2, 6, \ldots, 2n-2\}$, which is same as the co-domain $\{0, 1, 2, \ldots, 2n-1\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection Using the above

vertex coloring, edge colors are obtained as follows $f^*(v_iw_i) = (2i-1)(2i-2)+1; 1 \le i \le n$
$f^*(v_{2i-1}v_{2i}) = 4i(4i-5)+13; 1 \le i \le \lfloor \frac{n}{2} \rfloor$
$f^*(v_{2i}v_{2i+1}) = 4i(4i-1)+1; 1 \le i \le \lfloor \frac{n}{2} \rfloor$
$f^*(w_{2i-1}w_{2i}) = 4i(4i-5)+7; 1 \le i \le \lfloor \frac{n}{2} \rfloor$
$f^*(v_{2i}v_{2i+1}) = 4i(4i-1)+7; 1 \le i \le \lfloor \frac{n}{2} \rfloor$
In view of the above edge coloring pattern, all the edge colors are distinct. So L_n admits SSD coloring. All the edge colors are odd integers. So it admits odd SSD coloring.

Theorem 2.18. *The coconut trees are SSD graphs.*

Proof: Let $v_1, v_2, \dots v_k$ be the vertices of path having length $k-1$ and $v_{k+1}, v_{k+2}, \dots v_n$ be the pendent vertices being adajcent with v_1. Let G be the resulting graph. Let $\{v_1, v_i; 2 \le i \le n\}$ be vertex set and $\{v_iv_{i+1}; 1 \le i \le k-1\} \cup \{v_1v_{k+i}; 1 \le i \le n-k\}$ be the edge set. $p = n, q = n-1$.
Define $f : V(G) \to \{0, 1, 2 \dots n-1\}$ as follows $f(v_1) = 0)$
$f(v_j) = j-1; 1 \le j \le k-1$
$f(v_{k+i}) = (k+i-1); 1 \le i \le n-k$
The range of f is $\{0, 1, 2, \dots, k-1\} \cup \{k, k+1, \dots, n-1\}$, which is same as the co-domain $\{0, 1, 2, \dots, n-1\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \ne v_2 \implies f(v_1) \ne f(v_2)$. So f is a bijection In view of the vertex colors, edge colors are obtained as
$f^*(v_iv_{i+1}) = i(i-1)+1; 1 \le i \le k-1$
$f^*(v_1v_{i+1}) = (i)^2; k \le i \le n-1$
According to this edge coloring, all the edge colors are distinct. So coconut trees are square sum difference graphs.

Theorem 2.19. *The comb graph $P_n \bigodot K_1$ admits square sum difference coloring. Comb graph admits odd sum difference coloring.*

Proof: Let $v_1, v_2, \dots v_n$ be the path P_n of length $n-1$, $u_1, u_2, \dots u_n$ be the pendent vertices in the same order. Let G be the resulting graph. Let $\{v_i, u_i; 1 \le i \le n\}$ be vertex set and $\{v_iv_{i+1}; 1 \le i \le n-1\} \cup \{v_iu_i; 1 \le i \le n\}$ be the edge set. $p = 2n, q = 2n-1$.
Define $f : V(G) \to \{0, 1, 2 \dots 2n-1\}$ as follows $f(v_i) = i-1; 1 \le i \le n$
$f(u_i) = n+i-1; 1 \le i \le n$

The range of f is $\{0,1,2,\ldots,n-1\} \cup \{n,n+1,\ldots,2n-1\}$, which is same as the co-domain $\{0,1,2,\ldots,2n-1\}$. Also, for any two distinct vertices $v_1, v_2 \in V(G)$ and $v_1 \neq v_2 \implies f(v_1) \neq f(v_2)$. So f is a bijection Using this vertex coloring, edge colors are obtained as

$f^*(v_i v_{i+1}) = i(i-1)+1; 1 \leq i \leq n-1$

$f^*(v_i u_i) = (i-1)^2 + n(n-1) + ni; 1 \leq i \leq n$

It admits all the conditions of square sum difference graphs. All the edge colors are odd; so it admits OSSD coloring.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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