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THE v-ANALOGUE OF SOME INEQUALITIES FOR THE GAMMA DISTRIBUTION

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Abstract. In this paper, we present some inequalities involving the gamma v-random variables via some classical inequalities such as Hölder's integral inequality and Chebychev's inequality for synchronous (or asynchronous) mappings.

Keywords: gamma v-distribution; random variable; mean; variance; Hölder's inequality; Chebychev's inequality.

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1. Introduction

A continuous random variable X is said to have a gamma distribution with parameters $m > 0$ and $\alpha > 0$, if its probability distribution function is defined [2] by

$$f(x;m,\alpha) = \begin{cases} \frac{1}{\Gamma(m)}\alpha^m x^{m-1}e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{1}$$

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The function $\Gamma(m)$ is the classical gamma function and is defined [1] as

$$\Gamma(m) = \int_0^\infty x^{m-1}e^{-x}dx. \tag{2}$$

When $\alpha = 1$, we refer to (1) as a one-parameter gamma distribution function. This is defined as

$$f(x;m) = \begin{cases} \frac{1}{\Gamma(m)}x^{m-1}e^{-x}, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{3}$$

If X is a random variable that is gamma distributed with parameters m and α, then we simply write $X \sim \Gamma(x;m,\alpha)$ or $X \sim \Gamma(m,\alpha)$ where x is a variable.

Let $X \sim \Gamma(x;m,\alpha)$. Then the Expectation or mean of X is defined as

$$E(X) = \int_0^\infty xf(x;m,\alpha)dx = \int_0^\infty \frac{1}{\Gamma(m)}\alpha^m x^m e^{-\alpha x}dx \tag{4}$$

provided the improper integral exists. The expectation is said to be finite when $E(X) < \infty$.

Also, the variance of X is defined as

$$Var(X) = E(X^2) - [E(X)]^2 \tag{5}$$

where $E(X^2) = \displaystyle\int_0^\infty x^2 f(x;m,\alpha)dx$.

The moment generating function of X is defined for $t \in \mathbb{R}$ as

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}f(x;m,\alpha)dx \tag{6}$$

provided the operator $E(.) < \infty$.

The k^{th} moment about $x = 0$ of X is defined by

$$E(X^k) = \int_0^\infty x^k f(x;m,\alpha)dx,\ k = 1,2,\cdots. \tag{7}$$

For more details on the gamma distribution and its properties, see [7].

In recent years many researchers have established some inequalities involving gamma random variables. In [4], the authors presented mathematical inequalities with applications to the beta and gamma mappings. They also established some integral inequalities, relations and identities for the gamma and beta distributions. Certain inequalities for the gamma and beta functions via some classical integral inequalities have also been established, see [5], [6] and

[9]. In [10] and [11], the authors introduced the k-analogue of the gamma and beta distribution functions and established some properties of the k-gamma variable. Also in [12], the authors gave new inequalities for the k-analog random variables.

The v-Gamma Function. The v-gamma function is a one-parameter deformation of the classical gamma function. It was introduced in [3] for $s>0$ and $v>0$ as

$$\Gamma_v(s)=\int_0^\infty \left(\frac{t}{v}\right)^{\frac{s}{v}-1} e^{-t}dt. \tag{8}$$

Alternatively, it is defined as

$$\Gamma_v(s)=\lim_{n\to\infty}\frac{n!\left(\frac{n}{v}\right)^{\frac{s}{v}} v^{n+1}}{(s)_{n,v}},\ s>0,\ v>0,\ n\in\mathbb{N}, \tag{9}$$

where $(s)_{n,v}=(s)(s+v)(s+2v)\cdots(s+(n-1)v$ is the v-Pochammer symbol.

When $v=1$, $\Gamma_v(s)=\Gamma(s)$ and $(s)_{n,v}$ tends to the usual Pochammer symbol $(s)_n$ which is defined as

$$(s)_n=\begin{cases}(s)(s+1)(s+2)\cdots(s+(n-1), & n\geq 1\\ 1, & n=0.\end{cases}$$

The authors also established the following functional properties:

$$\begin{aligned}\Gamma_v(v)&=1,\\ \Gamma_v(s+v)&=\frac{s}{v^2}\Gamma_v(s),\\ \Gamma_v(s+nv)&=\frac{(s)_{n,v}}{v^{2n}}\Gamma_v(s).\end{aligned} \tag{10}$$

where $(s)_{n,v}$ is the v-Pochammer symbol. Clearly when $v=1$, the above properties hold for the classical gamma function.

2. Preliminaries

In order to obtain our main results, we need the following Lemmas.

Lemma 2.1. *(Chebychev's integral inequality* [8]*) Let $f,h,g:I\to\mathbb{R}$ be mappings such that $h(x)\geq 0$, $h(x)f(x)g(x)$, $h(x)f(x)$ and $h(x)g(x)$ are integrable on I. If $f(x)$ and $g(x)$ are synchronous (asynchronous) on I. That is $[f(x)-f(y)]\,[g(x)-g(y)]\geq(\leq)0$ for all $x,y\in\mathbb{I}$. Then*

$$\int_I h(x)dx \int_I h(x)f(x)g(x)dx \geq (\leq) \int_I h(x)f(x)dx \int_I h(x)g(x)dx. \tag{11}$$

Lemma 2.2. *(Weighted Hölder's inequality* [13]*) Let $p > 0$ and $q > 0$ such that $p+q = 1$. Let f, g and h be continuous functions on $(0,\infty)$. Then the weighted Hölder's inequality for integrals is given by*

$$\int_0^\infty f(x)g(x)h(x)dx \leq \left(\int_0^\infty |f(x)|^{\frac{1}{p}} h(x)dx\right)^p \left(\int_0^\infty |g(x)|^{\frac{1}{q}} h(x)dx\right)^q. \tag{12}$$

3. Main Results

Now we state and prove the results of the paper. We begin with a definition.

Definition 3.1. A continuous random variable S is said to have a gamma v-distribution with parameters $m > 0$ and $\alpha > 0$, if its probability distribution function is of the form

$$f_v(s; m, \alpha) = \begin{cases} \frac{1}{\Gamma_v(m)} \alpha^{\frac{m}{v}} \left(\frac{s}{v}\right)^{\frac{m}{v}-1} e^{-\alpha s}, & s \geq 0 \\ 0, & s < 0 \end{cases} \tag{13}$$

where $v > 0$ and $\Gamma_v(m)$ is the v-gamma function defined in (8).

Proposition 3.2. *The gamma v-distribution is a probability distribution function.*

Proof. Let $u = \alpha s$ in (13). Then by change of variable, we have

$$\begin{aligned} \int_0^\infty f_v(s; m, \alpha)ds &= \int_0^\infty \frac{1}{\Gamma_v(m)} \alpha^{\frac{m}{v}} \left(\frac{s}{v}\right)^{\frac{m}{v}-1} e^{-\alpha s} ds \\ &= \frac{1}{\Gamma_v(m)} \int_0^\infty \left(\frac{u}{v}\right)^{\frac{m}{v}-1} e^{-u} du \\ &= \frac{1}{\Gamma_v(m)} \Gamma_v(m) \\ &= 1. \end{aligned}$$

Proposition 3.3. *The following properties are satisfied.*

(i) Expectation of the gamma v-distribution is equal to $\frac{m}{\alpha v}$.

(ii) Variance of the gamma v-distribution is equal to $\frac{m}{\alpha^2 v}$.

(iii) Moment generating function of the gamma v-distribution is equal to

$$\left(\frac{\alpha}{\alpha - t}\right)^{\frac{m}{v}}, \ \alpha > t > 0.$$

Proof. (i) By using (4) and (10), we have

$$\begin{aligned} E_v[S] &= \int_0^\infty s f_v(s; m, \alpha) ds \\ &= \int_0^\infty \frac{s}{\Gamma_v(m)} \alpha^{\frac{m}{v}} \left(\frac{s}{v}\right)^{\frac{m}{v}-1} e^{-\alpha s} ds \\ &= \frac{v}{\Gamma_v(m)} \int_0^\infty \alpha^{\frac{m}{v}} \left(\frac{s}{v}\right)^{\frac{m}{v}} e^{-\alpha s} ds. \end{aligned}$$

Let $u = \alpha s$. Then by change of variable, we have

$$\begin{aligned} E_v[S] &= \frac{v}{\alpha \Gamma_v(m)} \int_0^\infty \left(\frac{u}{v}\right)^{\frac{m}{v}} e^{-u} du \\ &= \frac{v \Gamma_v(m+v)}{\alpha \Gamma_v(m)} \\ &= \frac{m}{\alpha v}. \end{aligned}$$

(ii) We first compute $E_v[S^2]$ as follows:

$$\begin{aligned} E_v[S^2] &= \int_0^\infty \frac{s^2}{\Gamma_v(m)} \alpha^{\frac{m}{v}} \left(\frac{s}{v}\right)^{\frac{m}{v}-1} e^{-\alpha s} ds \\ &= \frac{v^2}{\Gamma_v(m)} \int_0^\infty \alpha^{\frac{m}{v}} \left(\frac{s}{v}\right)^{\frac{m}{v}+1} e^{-\alpha s} ds. \end{aligned}$$

Let $u = \alpha s$. Then by change of variable, we have

$$E_v[S^2] = \frac{v^2}{\alpha^2 \Gamma_v(m)} \int_0^\infty \left(\frac{u}{v}\right)^{\frac{m}{v}+1} e^{-u} du$$

as a result of (10).

Thus variance of S is given by

$$(Var)_v[S] = \frac{m(m+v)}{\alpha^2 v^2} - \left(\frac{m}{\alpha v}\right)^2 = \frac{m}{\alpha^2 v}. \tag{14}$$

(iii) From (6), we define the moment generating function of the gamma v-distribution as

$$M_{0,v}(t) = E_v(e^{ts}) = \int_0^\infty \frac{\alpha^{\frac{m}{v}}}{\Gamma_v(m)} \left(\frac{s}{v}\right)^{\frac{m}{v}-1} e^{-(\alpha - t)s} ds. \tag{15}$$

Let $u=(\alpha-t)s$ in (15). Then by change of variable, we have

$$\begin{aligned}M_{0,v}(t) &= \frac{1}{\Gamma_v(m)}\int_0^\infty \alpha^{\frac{m}{v}}\left(\frac{u}{v(\alpha-t)}\right)^{\frac{m}{v}-1} e^{-u}\frac{du}{\alpha-t} \\ &= \left(\frac{\alpha}{\alpha-t}\right)^{\frac{m}{v}}\int_0^\infty \frac{1}{\Gamma_v(m)}e^{-u}\left(\frac{u}{v}\right)^{\frac{m}{v}-1}du \\ &= \left(\frac{\alpha}{\alpha-t}\right)^{\frac{m}{v}}, \ \alpha>t>0.\end{aligned}$$

Definition 3.4. Let S be a continuous random variable such that $S\sim\Gamma_v(m,\alpha)$. Then the k^{th} moment of S is defined by

$$E_v(S^k)=\frac{v^k}{\Gamma_v(m)}\int_0^\infty \alpha^{\frac{m}{v}}\left(\frac{s}{v}\right)^{\frac{m}{v}+k-1}e^{-\alpha s}ds,\ k=1,2,\cdots. \tag{16}$$

Proposition 3.5. *Equation* (16) *has the following equivalent form.*

$$E_v(S^k)=\frac{1}{\alpha^k v^k}m(m+v)\cdots(m+(k-1)v),\ k=1,2,\cdots. \tag{17}$$

Proof. Let $u=\alpha s$ in (16). Then by change of variable, we have

$$\begin{aligned}E_v(S^k) &= \frac{v^k}{\alpha^k\Gamma_v(m)}\int_0^\infty\left(\frac{u}{v}\right)^{\frac{m}{v}+k-1}e^{-u}du \\ &= \frac{v^k}{\alpha^k\Gamma_v(m)}\Gamma_v(m+kv) \\ &= \frac{1}{\alpha^k v^k}m(m+v)\cdots(m+(k-1)v)\end{aligned}$$

as a result of equation (10) and that completes the proof.

Theorem 3.6. *Let X and Y be continuous random variables such that $X\sim\Gamma_v(ax+by,\alpha)$ and $Y\sim\Gamma_v(x,\alpha)$ where $a,b\geq 0$ with $a+b=1$ and $x,y>0$. Then for $v>0$, the inequality*

$$E_v(X)^{ak}\Gamma_v(ax+by)\leq E_v(Y)^k[\Gamma_v(x)]^a[\Gamma_v(y)]^b,\ k=1,2,\cdots \tag{18}$$

holds.

Proof. Let us define the mappings $f,g,h:[0,\infty)\longrightarrow[0,\infty)$ by

$$f(t)=\alpha^{\frac{ax}{v}}\left(\frac{t}{v}\right)^{a\left(\frac{x}{v}-1\right)+ak},\quad g(t)=\alpha^{\frac{by}{v}}\left(\frac{t}{v}\right)^{b\left(\frac{y}{v}-1\right)},\quad h(t)=e^{-\alpha t}$$

for $v>0$ and $t\in[0,\infty)$. Applying weighted Hölder's inequality to the functions f, g and h, we have

$$\int_0^\infty\left(\frac{t}{v}\right)^{a\left(\frac{x}{v}-1\right)+ak}\alpha^{\frac{ax}{v}}\left(\frac{t}{v}\right)^{b\left(\frac{y}{v}-1\right)}\alpha^{\frac{by}{v}}e^{-\alpha t}dt\le\left[\int_0^\infty\left(\alpha^{\frac{ax}{v}}\left(\frac{t}{v}\right)^{a\left(\frac{x}{v}+k-1\right)}\right)^{\frac{1}{a}}e^{-\alpha t}dt\right]^a$$
$$\times\left[\int_0^\infty\left(\alpha^{\frac{by}{v}}\left(\frac{t}{v}\right)^{b\left(\frac{y}{v}-1\right)}\right)^{\frac{1}{b}}e^{-\alpha t}dt\right]^b.$$

This implies that

$$\int_0^\infty\alpha^{\frac{ax+by}{v}}\left(\frac{t}{v}\right)^{\left(\frac{ax+by}{v}+ak-a-b\right)}e^{-\alpha t}dt\le\left[\int_0^\infty\alpha^{\frac{x}{v}}\left(\frac{t}{v}\right)^{\frac{x}{v}-1+k}e^{-\alpha t}dt\right]^a \tag{19}$$
$$\times\left[\int_0^\infty\alpha^{\frac{y}{v}}\left(\frac{t}{v}\right)^{\frac{y}{v}-1}e^{-\alpha t}dt\right]^b.$$

From (16), we deduce the following relations for the random variables X and Y.

$$\frac{1}{v^{ak}}E_v(X)^{ak}\Gamma_v(ax+by)=\int_0^\infty\alpha^{\frac{ax+by}{v}}\left(\frac{t}{v}\right)^{\frac{ax+by}{v}+ak-1}e^{-\alpha t}dt \tag{20}$$

and

$$\frac{1}{v^{k}}E_v(Y)^{k}\Gamma_v(x)=\int_0^\infty\alpha^{\frac{x}{v}}\left(\frac{t}{v}\right)^{\frac{x}{v}+k-1}e^{-\alpha t}dt. \tag{21}$$

By substituting the relations (20) and (21) into (19), we obtain inequality (18) and that completes the proof.

Theorem 3.7. *Let X and Y be continuous random variables such that $X\sim\Gamma_v(ax+by,\alpha)$ and $Y\sim\Gamma_v(x,\alpha)$ where $a,b\ge0$ with $a+b=1$ and $x,y>0$. Then the inequality*

$$\Gamma_v(ax+by)E_v\left(\frac{1}{X}\right)^{ak}\le\left[\Gamma_v(x)E_v\left(\frac{1}{Y}\right)^k\right]^a[\Gamma_v(y)]^b,\ k=1,2,\cdots. \tag{22}$$

holds.

Proof. Let us consider the mappings $f,g,h:[0,\infty)\longrightarrow[0,\infty)$ given by

$$f(t)=\alpha^{\frac{ax}{v}}\left(\frac{t}{v}\right)^{a\left(\frac{x}{v}-1\right)-ak},\quad g(t)=\alpha^{\frac{by}{v}}\left(\frac{t}{v}\right)^{b\left(\frac{y}{v}-1\right)},\quad h(t)=e^{-\alpha t}$$

for $v>0$ and $t\in[0,\infty)$. Applying weighted Hölder's inequality to the functions f, g and h, we have

$$\begin{aligned}\int_0^\infty\left(\frac{t}{v}\right)^{a\left(\frac{x}{v}-1\right)-ak}\alpha^{\frac{ax}{v}}\left(\frac{t}{v}\right)^{b\left(\frac{y}{v}-1\right)}\alpha^{\frac{by}{v}}e^{-\alpha t}dt\leq&\left[\int_0^\infty\left(\alpha^{\frac{ax}{v}}\left(\frac{t}{v}\right)^{a\left(\frac{x}{v}-k-1\right)}\right)^{\frac{1}{a}}e^{-\alpha t}dt\right]^a\\&\times\left[\int_0^\infty\left(\alpha^{\frac{by}{v}}\left(\frac{t}{v}\right)^{b\left(\frac{y}{v}-1\right)}\right)^{\frac{1}{b}}e^{-\alpha t}dt\right]^b.\end{aligned}$$

This implies that

$$\begin{aligned}\int_0^\infty\alpha^{\frac{ax+by}{v}}\left(\frac{t}{v}\right)^{\left(\frac{ax+by}{v}-ak-a-b\right)}e^{-\alpha t}dt\leq&\left[\int_0^\infty\alpha^{\frac{x}{v}}\left(\frac{t}{v}\right)^{\frac{x}{v}-1-k}e^{-\alpha t}dt\right]^a\\&\times\left[\int_0^\infty\alpha^{\frac{y}{v}}\left(\frac{t}{v}\right)^{\frac{y}{v}-1}e^{-\alpha t}dt\right]^b.\end{aligned}\tag{23}$$

From (16), we deduce the following relations for the reciprocal of X and Y.

$$\frac{1}{v^{ak}}E_v\left(\frac{1}{X}\right)^{ak}\Gamma_v(ax+by)=\int_0^\infty\alpha^{\frac{ax+by}{v}}\left(\frac{t}{v}\right)^{\frac{ax+by}{v}-ak-1}e^{-\alpha t}dt\tag{24}$$

and

$$\frac{1}{v^k}E_v\left(\frac{1}{Y}\right)^k\Gamma_v(x)=\int_0^\infty\alpha^{\frac{x}{v}}\left(\frac{t}{v}\right)^{\frac{x}{v}-k-1}e^{-\alpha t}dt.\tag{25}$$

By substituting (24) and (25) into (23), we obtain inequality (22) and that completes the proof.

Theorem 3.8. *Let X and Y be continuous random variables such that $X\sim\Gamma_v(x+y+m,\alpha)$ and $Y\sim\Gamma_v(y+m,\alpha)$ where $x,y,m>0$. Then for $v>0$, the inequality*

$$\Gamma_v(m)\Gamma_v(x+y+m)E_v(X)^k\geq\Gamma_v(x+m)\Gamma_v(y+m)E_v(Y)^k,\ k=1,2,\cdots\tag{26}$$

holds.

Proof. Let $v > 0$ and consider the mappings $f, g, h : [0, \infty) \longrightarrow [0, \infty)$ given by

$$f(t) = \alpha^{\frac{x}{v}} \left(\frac{t}{v}\right)^{\frac{x}{v}}, \quad g(t) = \alpha^{\frac{y}{v}} \left(\frac{t}{v}\right)^{\frac{y}{v}+k}, \quad h(t) = \alpha^{\frac{m}{v}} \left(\frac{t}{v}\right)^{\frac{m}{v}-1} e^{-\alpha t}$$

for $t \in [0, \infty)$. By substituting these mappings into the Chebyshev's integral inequality, we have

$$\int_0^\infty \alpha^{\frac{m}{v}} \left(\frac{t}{v}\right)^{\left(\frac{m}{v}-1\right)} e^{-\alpha t} dt \int_0^\infty \alpha^{\frac{x+y+m}{v}} \left(\frac{t}{v}\right)^{\left(\frac{x+y+m}{v}+k-1\right)} e^{-\alpha t} dt$$

$$\geq \int_0^\infty \alpha^{\frac{x+m}{v}} \left(\frac{t}{v}\right)^{\left(\frac{x+m}{v}-1\right)} e^{-\alpha t} dt \int_0^\infty \alpha^{\frac{m+y}{v}} \left(\frac{t}{v}\right)^{\left(\frac{m+y}{v}+k-1\right)} e^{-\alpha t} dt. \tag{27}$$

From (16), we derive the following:

$$\frac{1}{v^k} E_v(X)^k \Gamma_v(x+y+m) = \int_0^\infty \alpha^{\frac{x+y+m}{v}} \left(\frac{t}{v}\right)^{\frac{x+y+m}{v}+k-1} e^{-\alpha t} dt \tag{28}$$

$$\frac{1}{v^k} E_v(X)^k \Gamma_v(x+m) = \int_0^\infty \alpha^{\frac{x+m}{v}} \left(\frac{t}{v}\right)^{\frac{x+m}{v}+k-1} e^{-\alpha t} dt. \tag{29}$$

$$\frac{1}{v^k} E_v(Y)^k \Gamma_v(y+m) = \int_0^\infty \alpha^{\frac{y+m}{v}} \left(\frac{t}{v}\right)^{\frac{y+m}{v}+k-1} e^{-\alpha t} dt. \tag{30}$$

On substituting (28) through (30) into (27), we obtain inequality (26) and that completes the proof.

Theorem 3.9. *Let X and Y be continuous random variables such that $X \sim \Gamma_v(ax+by, \alpha)$ and $Y \sim \Gamma_v(x, \alpha)$ where $a, b \geq 0$ with $a + b = 1$ and $x, y > 0$. Let $(Var)_v(X)$ and $(Var)_v(Y)$ be variances of X and Y respectively. Then for $v > 0$, the inequality*

$$\frac{[(Var)_v(X^a) + E_v(X^a)]^2}{\left[(Var)_v(Y) + \left(\frac{x}{\alpha v}\right)^2\right]^a} \leq \frac{[\Gamma_v(x)]^a [\Gamma_v(y)]^b}{\Gamma_v(ax+by)}, \; k = 1, 2, \cdots \tag{31}$$

holds.

Proof. Using Theorem 3.6 and taking $k = 2$, we have

$$E_v(X)^{2a} \Gamma_v(ax+by) \leq E_v(Y)^2 [\Gamma_v(x)]^a [\Gamma_v(y)]^b. \tag{32}$$

Expressing $E_v(.)$ in terms of $(Var)_v(.)$ in (32), we have

$$[(Var)_v(X^a)+E_v(X^a)]^2\Gamma_v(ax+by) \leq [(Var)_v(Y)+(E_v(Y)^2]^a[\Gamma_v(x)]^a[\Gamma_v(y)]^b \tag{33}$$

Since $Y \sim \Gamma_v(x,\alpha)$, we recall from Proposition 3.3 that $E_v(Y)=\frac{x}{\alpha v}$. A direct substitution into (33) and rearranging the terms gives inequality (31) and that completes the proof.

Theorem 3.10. *Let X, Y, W and Z be continuous random variables such that $X \sim \Gamma_v(m,\alpha)$, $Y \sim \Gamma_v(x+y+m,\alpha)$, $W \sim \Gamma_v(x+m,\alpha)$ and $Z \sim \Gamma_v(y+m,\alpha)$, where m,x,y and $\alpha>0$. Then the inequality*

$$E_v(X)^kE_v(Y)^k\Gamma_v(y+m)\Gamma_v(x+y+m) \geq E_v(W)^kE_v(Z)^k\Gamma_v(x+m),\ k=1,2,\cdots \tag{34}$$

holds.

Proof. Let $v>0$ and consider the mappings $f,g,h:[0,\infty)\longrightarrow[0,\infty)$ given by

$$f(t)=\alpha^{\frac{x}{v}}\left(\frac{t}{v}\right)^{\frac{x}{v}},\quad g(t)=\alpha^{\frac{y}{v}}\left(\frac{t}{v}\right)^{\frac{y}{v}},\quad h(t)=\alpha^{\frac{m}{v}}\left(\frac{t}{v}\right)^{\frac{m}{v}+k-1}e^{-\alpha t}$$

for $t\in\ [0,\infty)$. By substituting these mappings into the Chebyshev's integral inequality, we have

$$\begin{aligned}&\int_0^\infty \alpha^{\frac{m}{v}}\left(\frac{t}{v}\right)^{\left(\frac{m}{v}+k-1\right)}e^{-\alpha t}dt\int_0^\infty \alpha^{\frac{m+x+y}{v}}\left(\frac{t}{v}\right)^{\left(\frac{m+x+y}{v}+k-1\right)}e^{-\alpha t}dt\\ &\qquad \geq \int_0^\infty \alpha^{\frac{m+x}{v}}\left(\frac{t}{v}\right)^{\left(\frac{m+x}{v}+k-1\right)}e^{-\alpha t}dt\int_0^\infty \alpha^{\frac{m+y}{v}}\left(\frac{t}{v}\right)^{\left(\frac{m+y}{v}+k-1\right)}e^{-\alpha t}dt.\end{aligned} \tag{35}$$

From (16), we derive the following:

$$\frac{1}{v^k}E_v(X)^k\Gamma_v(m)=\int_0^\infty\left(\frac{t}{v}\right)^{\frac{m}{v}+k-1}e^{-\alpha t}dt, \tag{36}$$

$$\frac{1}{v^k}E_v(Y)^k\Gamma_v(x+y+m)=\int_0^\infty\alpha^{\frac{x+y+m}{v}}\left(\frac{t}{v}\right)^{\frac{x+y+m}{v}+k-1}e^{-\alpha t}dt, \tag{37}$$

$$\frac{1}{v^k}E_v(W)^k\Gamma_v(x+m)=\int_0^\infty\alpha^{\frac{x+m}{v}}\left(\frac{t}{v}\right)^{\frac{x+m}{v}+k-1}e^{-\alpha t}dt, \tag{38}$$

$$\frac{1}{v^k}E_v(Z)^k\Gamma_v(y+m)=\int_0^\infty\alpha^{\frac{y+m}{v}}\left(\frac{t}{v}\right)^{\frac{y+m}{v}+k-1}e^{-\alpha t}dt. \tag{39}$$

On substituting (36) through (39) into (35), we obtain inequality (34) and that completes the proof.

Theorem 3.11. *Let X and Y be continuous random variables such that $X \sim \Gamma_v(x+y+m,\alpha)$ and $Y \sim \Gamma_v(y+m,\alpha)$. Let $(Var)_v(X)$ and $(Var)_v(X)$ be variances of X and Y respectively. Then for $v>0$, the inequality*

$$\frac{\alpha^2 v^2 (Var)_v(X)+(x+y+m)^2}{\alpha^2 v^2 (Var)_v(Y)+(x+m)^2} \geq \frac{\Gamma_v(x+m)\Gamma_v(y+m)}{\Gamma_v(x+y+m)},\ k=1,2,\cdots \tag{40}$$

holds.

Proof. Using Theorem 3.10 with $k=2$, we have

$$\Gamma_v(m)\Gamma_v(x+y+m)E_v(X)^2 \geq \Gamma_v(x+m)\Gamma_v(y+m)E_v(Y)^2. \tag{41}$$

Expressing $E_v(.)$ in terms of $(Var)_v(.)$, and rearranging the terms, we have

$$\frac{(Var)_v(X)+[E_v(X)]^2}{(Var)_v(Y)+[E_v(Y)]^2} \geq \frac{\Gamma_v(x+m)\Gamma_v(y+m)}{\Gamma_v(x+y+m)}. \tag{42}$$

From Proposition 3.3, we have $E_v(X)=\frac{x+y+m}{\alpha v}$ and $E_v(Y)=\frac{y+m}{\alpha v}$. By substituting these expressions into (42), we obtain inequality (40) and that completes the proof.

Remark 3.12. By setting $v=1$ in (18) and (34), we obtain Theorem 3.7 of [4] and Theorem 3.9 of [4] respectively.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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