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ON A SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING HURWITZ-LERCH ZETA FUNCTION

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Abstract. In this work, we introduce and investigate a new class of analytic functions in the open unit disc U with negative coefficients. The object of the present paper is to determine coefficient estimates, neighborhoods and partial sums for functions f belonging to this class.

Keywords: analytic; starlike; coefficient estimate; partial sums.

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1. Introduction

Let A denote the class of analytic functions f defined on the unit disk $U = \{z : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$. Such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

denoted by S, the subclass of A consisting of functions that are univalent in U.

For $f \in A$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{2}$$

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their convolution (or Hadamard product), denoted by $(f*g)$, is defined as

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n=(g*f)(z),\ (z\in U). \tag{3}$$

Note that $f*g\in A$.

A function $f\in A$ is said to be in $k-US(\gamma)$, the class of $k-$uniformly starlike functions of order $\gamma, 0\leq\gamma<1$, if satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>k\left|\frac{zf'(z)}{f(z)}-1\right|+\gamma,\ (k\geq 0) \tag{4}$$

and a function $f\in A$ is said to be in $k-UC(\gamma)$, the class of $k-$uniformly convex functions of order $\gamma, 0\leq\gamma<1$, if satisfies the condition

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>k\left|\frac{zf''(z)}{f'(z)}\right|+\gamma,\ (k\geq 0). \tag{5}$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [5] and then studied by various authors. In [12], Sakaguchi defined the class S_s of starlike functions with respect to symmetric points as follows: Let $f\in A$. Then f is said to be starlike with respect to symmetric points in U if and only if

$$\Re\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\}>0,\ (z\in U).$$

Recently, Owa et al. [10] defined the class $S_s(\alpha,t)$ as follows:

$$\Re\left\{\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right\}>\alpha,\ (z\in U),$$

where $0\leq\alpha<1, |t|\leq 1, t\neq 1$. Note that $S_s(0,-1)=S_s$ and $S_s(\alpha,-1)=S_s(\alpha)$ is called Sakaguchi function of order α.

In [8] Mustafa and Darus hzve recently introduced a new generalized integral operator $\Im^{\alpha}_{\mu,b}f(z)$ as we show in the following:

Definition 1.1. A general Hurwitz- Lerch Zeta function $\Phi(z,\mu,b)$ defined by

$$\Phi(z,\mu,b)=\sum_{n=0}^{\infty}\frac{z^n}{(n+b)^{\mu}},$$

where $(\mu \in C, b \in C - \mathbb{Z}_{\not{y}}^{-})$ when $|z| < 1$, and $\Re(\mathfrak{b}) > \imath$ when $(|z| = 1)$.

We define the function

$$\Phi^{*}(z,\mu,b) = (b^{\mu} z\Phi(z,\mu,b)) * f(z),$$

then

$$\Phi^{*}(z,\mu,b) = z + \sum_{n=2}^{\infty} \frac{a_n}{(n+b-1)^{\mu}} z^n$$

Definition 1.2. Let the function f be analytic in a simply connected domain of the z-plane containg the orgin. The fractional derivative of f of order α is defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt, \ (0 \leq \alpha < 1),$$

where the multiplycity of $(z-t)^{-\alpha}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Using Definition 1.2 and and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [9] introduced the operator $\Omega^{\alpha} : A \to A$ which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned} \Omega^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z), \ (\alpha \neq 2,3,4,...) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n, \ (z \in U) \end{aligned}$$

For $\alpha \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_{\not{y}}^{-}$, and $0 \leq \alpha < 1$, the generalized integral operator $\mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} f : A \to A$, is defined by

$$\begin{aligned} \mathfrak{J}_{\mu,\mathfrak{b}}^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} \Phi^{*}(z,\alpha,b), \ (\alpha \neq 2,3,4,\ldots) \\ &= z + \sum_{n=2}^{\infty} \Phi_n(\mu,b,\alpha) \ a_n z^n, \ (z \in U). \end{aligned}$$

where $\Phi_n(\mu,b,\alpha) = \frac{\Gamma(n+1) z^{\alpha} D_z^{\alpha} \Phi^{*}(z,\alpha,b)}{\Gamma(n+1-\alpha)} \left(\frac{b}{n-1+b}\right)^{\mu}$

Note that : $\mathfrak{J}_{\mathrm{o},\mathfrak{b}}^{\mathrm{o}} f(z) = f(z)$.

Special cases of this operator include :

(i). $\mathfrak{J}_{\mathrm{o},\mathfrak{b}}^{\alpha} f(z) \equiv \Omega^{\alpha} f(z)$ is Owa and Srivastava operator [9].

(ii). $\mathfrak{J}_{\mu,\mathfrak{b}+\imath}^{\mathrm{o}} f(z) \equiv J_{\mu,b} f(z)$ is the Srivastava and Attiya integral operator[19].

(iii). $\mathfrak{J}^{0}_{1,1}f(z) \equiv A(f)(z)$ is the Alexander integral operator [1].

(iv). $\mathfrak{J}^{0}_{\mu+1,1}f(z) \equiv L(f)(z)$ is the Libera integral operator [7].

(v). $\mathfrak{J}^{0}_{1,\delta}f(z) \equiv L_{\delta}(f)(z)$ is the Bernardi integral operator [3].

(vi). $\mathfrak{J}^{0}_{\sigma,2}f(z) \equiv I^{\sigma}f(z)$ is the Jung-Kim-Kim-Srivastava integral operator [6].

Now, by making use of the Hurwitz - Lerch zeta operator $\mathfrak{J}^{\alpha}_{\mu,b}f$, we define a new subclass of functions belonging to the class A.

Definition 1.3. A function $f \in A$ is said to be in the class $k-US_s(\alpha,b,\mu,\gamma,t)$ if for all $z \in U$

$$\Re\left\{\frac{(1-t)z\left(\mathfrak{J}^{\alpha}_{\mu,b}f(z)\right)'}{\mathfrak{J}^{\alpha}_{\mu,b}f(z)-\mathfrak{J}^{\alpha}_{\mu,b}f(tz)}\right\} \geq k\left|\frac{(1-t)z\left(\mathfrak{J}^{\alpha}_{\mu,b}f(z)\right)'}{\mathfrak{J}^{\alpha}_{\mu,b}f(z)-\mathfrak{J}^{\alpha}_{\mu,b}f(tz)}-1\right|+\gamma,$$

for $k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$.

Furthermore, we say that a function $f \in k-US_s(\alpha,b,\mu,\gamma,t)$ is in the subclass $k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ if $f(z)$ is of the following form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0,\ (n \in \mathbb{N},\ z \in U). \tag{6}$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$.

Firstly, we shall need the following lemmas [2].

Lemma 1.4. *Let w be a complex number. Then*

$$\Re(w) \geq \nu \text{ if and only if } |w-(1+\nu)| \leq |w+(1-\nu)|.$$

Lemma 1.5. *Let w be a complex number and ν, ς be real numbers. Then*

$$\Re\,(w) > \nu|w-1|+\varsigma \text{ if and only if } \Re\{w(1+\nu e^{i\theta})-\nu e^{i\theta}\} > \varsigma,\ \ -\pi < \theta < \pi.$$

2. Coefficient Bounds of the Function Class $k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$

Theorem 2.1. *The function f defined by* (6) *is in the class $k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ if and only if*

$$\sum_{n=2}^{\infty} \phi_n(\alpha,b,\mu)|n(k+1)-u_n(k+\gamma)|a_n \leq 1-\gamma, \tag{7}$$

where $k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$ *and* $u_n = 1 + t + \cdots + t^{n-1}$.

The result is sharp for the function $f(z)$ *given by*

$$f(z) = z - \frac{1-\gamma}{\phi_n(\alpha, b, \mu)|n(k+1) - u_n(k+\gamma)|} z^n.$$

Proof. By Definition 1.3, we get

$$\Re\left\{\frac{(1-t)z\left(\mathfrak{J}^{\alpha}_{\mu,b}f(z)\right)'}{\mathfrak{J}^{\alpha}_{\mu,b}f(z) - \mathfrak{J}^{\alpha}_{\mu,b}f(tz)}\right\} \geq k\left|\frac{(1-t)z\left(\mathfrak{J}^{\alpha}_{\mu,b}f(z)\right)'}{\mathfrak{J}^{\alpha}_{\mu,b}f(z) - \mathfrak{J}^{\alpha}_{\mu,b}f(tz)} - 1\right| + \gamma.$$

Then by Lemma 1.5, we have

$$\Re\left\{\frac{(1-t)z\left(\mathfrak{J}^{\alpha}_{\mu,b}f(z)\right)'}{\mathfrak{J}^{\alpha}_{\mu,b}f(z) - \mathfrak{J}^{\alpha}_{\mu,b}f(tz)}(1+ke^{i\theta}) - ke^{i\theta}\right\} \geq \gamma, \ -\pi < \theta < \pi$$

or equivalently

$$\Re\left\{\frac{(1-t)z\left(\mathfrak{J}^{\alpha}_{\mu,b}f(z)\right)'(1+ke^{i\theta})}{\mathfrak{J}^{\alpha}_{\mu,b}f(z) - \mathfrak{J}^{\alpha}_{\mu,b}f(tz)} - \frac{ke^{i\theta}\left[\mathfrak{J}^{\alpha}_{\mu,b}f(z) - \mathfrak{J}^{\alpha}_{\mu,b}f(tz)\right]}{\mathfrak{J}^{\alpha}_{\mu,b}f(z) - \mathfrak{J}^{\alpha}_{\mu,b}f(tz)}\right\} \geq \gamma. \tag{8}$$

Let $F(z) = (1-t)z\left(\mathfrak{J}^{\alpha}_{\mu,b}f(z)\right)'(1+ke^{i\theta}) - ke^{i\theta}\left[\mathfrak{J}^{\alpha}_{\mu,b}f(z) - \mathfrak{J}^{\alpha}_{\mu,b}f(tz)\right]$
and $E(z) = \mathfrak{J}^{\alpha}_{\mu,b}f(z) - \mathfrak{J}^{\alpha}_{\mu,b}f(tz)$.
By Lemma 1.4, (8) is equivalent to

$$|F(z) + (1-\gamma)E(z)| \geq |F(z) - (1+\gamma)E(z)|, \text{ for } 0 \leq \gamma < 1.$$

But

$$\begin{aligned}|F(z) + (1-\gamma)E(z)| = &\Big|(1-t)\big\{(2-\gamma)z - \sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)(n+u_n(1-\gamma))a_n z^n \\ &- ke^{i\theta}\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)(n-u_n)a_n z^n\big\}\Big| \\ \geq &|1-t|\Big\{(2-\gamma)|z| - \sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)|n+u_n(1-\gamma)|a_n|z^n| \\ &- k\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)|n-u_n|a_n|z^n|\Big\}.\end{aligned}$$

Also

$$\begin{aligned}|F(z)-(1+\gamma)E(z)| = &\Big|(1-t)\big\{-\gamma z-\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)(n-u_n(1+\gamma))a_nz^n\\ &-ke^{i\theta}\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)(n-u_n)a_nz^n\big\}\Big|\\ \le&|1-t|\Big\{\gamma|z|+\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)|n-u_n(1+\gamma)|a_n|z^n|\\ &+k\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)|n-u_n|a_n|z^n|\Big\}.\end{aligned}$$

So

$$\begin{aligned}&|F(z)+(1-\gamma)E(z)|-|F(z)-(1+\gamma)E(z)|\\ \ge&\ |1-t|\Big\{2(1-\gamma)|z|-\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)\big[|n+u_n(1-\gamma)|+|n-u_n(1+\gamma)|\\ &+2k|n-u_n|\ \big]a_n|z^n|\Big\}\\ \ge&\ 2(1-\gamma)|z|-\sum_{n=2}^{\infty}2\phi_n(\alpha,b,\mu)\big|n(k+1)-u_n(k+\gamma)\big|a_n|z^n|\ge 0\end{aligned}$$

or

$$\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)\big|n(k+1)-u_n(k+\gamma)\big|a_n\le 1-\gamma.$$

Conversely, suppose that (7) holds. Then we must show

$$\Re\left\{\frac{(1-t)z\left(\mathfrak{J}_{\mu,b}^{\alpha}f(z)\right)'(1+ke^{i\theta})-ke^{i\theta}\left[\mathfrak{J}_{\mu,b}^{\alpha}f(z)-\mathfrak{J}_{\mu,b}^{\alpha}f(tz)\right]}{\mathfrak{J}_{\mu,b}^{\alpha}f(z)-\mathfrak{J}_{\mu,b}^{\alpha}f(tz)}\right\}\ge\gamma.$$

Upon choosing the values of z on the positive real axis where $0\le|z|=r<1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)[n(1+ke^{i\theta})-u_n(\gamma+ke^{i\theta})]a_nz^{n-1}}{1-\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)u_na_nz^{n-1}}\right\}\ge 0.$$

Since $\Re(-e^{i\theta})\ge-|e^{i\theta}|=-1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)[n(1+k)-u_n(\gamma+k]a_nr^{n-1}}{1-\sum_{n=2}^{\infty}\phi_n(\alpha,b,\mu)u_na_nr^{n-1}}\right\}\ge 0.$$

Letting $r \to 1^-$, we have desired conclusion. $\square$

Corollary 2.2. *If* $f(z) \in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ *then*

$$a_n \le \frac{1-\gamma}{\phi_n(\alpha,b,\mu)|n(k+1)-u_n(k+\gamma)|}$$

where $k \ge 0, |t| \le 1, t \ne 1, 0 \le \gamma < 1$ *and* $u_n = 1+t+\cdots+t^{n-1}$.

3. Neighborhood of the Function Class $k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [4], Ruscheweyh [11] and Santosh et al. [13], we define the neighborhood of a function $f \in T$.

Definition 3.1. Let $k \ge 0, |t| \le 1, t \ne 1, 0 \le \gamma < 1, \alpha \ge 0$ and $u_n = 1+t+\cdots+t^{n-1}$. We define the $\alpha-$neighborhood of a function $f \in T$ and denote by $N_\alpha(f)$ consisting of all functions $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in S(b_n \ge 0, n \in \mathbb{N})$ satisfying

$$\sum_{n=2}^{\infty} \frac{\phi_n(\alpha,b,\mu)|n(k+1)-u_n(k+\gamma)|}{1-\gamma}|a_n - b_n| \le 1-\alpha.$$

Theorem 3.2. *Let* $f(z) \in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ *and* $\Re(\gamma) \ne 1$. *For any complex number* ε *with* $|\varepsilon| < \alpha(\alpha \ge 0)$, *if f satisfies the following condition:*

$$\frac{f(z)+\varepsilon z}{1+\varepsilon} \in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$$

then $N_\alpha(f) \subset k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$.

Proof. It is obvious that $f \in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ if and only if

$$\left| \frac{(1-t)z\left(\mathfrak{J}^\alpha_{\mu,b}f(z)\right)'(1+ke^{i\theta}) - (ke^{i\theta}+1+\gamma)\left(\mathfrak{J}^\alpha_{\mu,b}f(z) - \mathfrak{J}^\alpha_{\mu,b}f(tz)\right)}{(1-t)z\left(\mathfrak{J}^\alpha_{\mu,b}f(z)\right)'(1+ke^{i\theta}) + (1-ke^{i\theta}-\gamma)\left(\mathfrak{J}^\alpha_{\mu,b}f(z) - \mathfrak{J}^\alpha_{\mu,b}f(tz)\right)} \right| < 1,$$

$-\pi < \theta < \pi$.

For any complex number s with $|s| = 1$, we have

$$\frac{(1-t)z\left(\mathfrak{J}^\alpha_{\mu,b}f(z)\right)'(1+ke^{i\theta}) - (ke^{i\theta}+1+\gamma)\left(\mathfrak{J}^\alpha_{\mu,b}f(z) - \mathfrak{J}^\alpha_{\mu,b}f(tz)\right)}{(1-t)z\left(\mathfrak{J}^\alpha_{\mu,b}f(z)\right)'(1+ke^{i\theta}) + (1-ke^{i\theta}-\gamma)\left(\mathfrak{J}^\alpha_{\mu,b}f(z) - \mathfrak{J}^\alpha_{\mu,b}f(tz)\right)} \ne s.$$

In other words, we must have

$$(1-s)(1-t)z\left(\mathfrak{I}^{\alpha}_{\mu,b}f(z)\right)'(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma+s(-1+ke^{i\theta}+\gamma))$$
$$\times\left(\mathfrak{I}^{\alpha}_{\mu,b}f(z)-\mathfrak{I}^{\alpha}_{\mu,b}f(tz)\right)\neq 0$$

which is equivalent to

$$z-\sum_{n=2}^{\infty}\frac{\phi_n(\alpha,b,\mu)\left((n-u_n)(1+ke^{i\theta}-ske^{i\theta})-s(n+u_n)-u_n\gamma(1-s)\right)}{\gamma(s-1)-2s}z^n\neq 0.$$

However, $f\in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ if and only $\frac{(f*h)}{z}\neq 0$, $z\in U-\{0\}$, where $h(z)=z-\sum\limits_{n=2}^{\infty}c_nz^n$ and

$$c_n=\frac{\phi_n(\alpha,b,\mu)\left((n-u_n)(1+ke^{i\theta}-ske^{i\theta})-s(n+u_n)-u_n\gamma(1-s)\right)}{\gamma(s-1)-2s}.$$

We note that

$$|c_n|\leq\frac{\phi_n(\alpha,b,\mu)\,|n(1+k)-u_n(k+\gamma)|}{1-\gamma}$$

since $\frac{f(z)+\varepsilon z}{1+\varepsilon}\in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$, therefore $z^{-1}\left(\frac{f(z)+\varepsilon z}{1+\varepsilon}*h(z)\right)\neq 0$, which is equivalent to

$$\frac{(f*h)(z)}{(1+\varepsilon)z}+\frac{\varepsilon}{1+\varepsilon}\neq 0. \tag{9}$$

Now suppose that $\left|\frac{(f*h)(z)}{z}\right|<\alpha$. Then by (9), we must have

$$\left|\frac{(f*h)(z)}{(1+\varepsilon)z}+\frac{\varepsilon}{1+\varepsilon}\right|\geq\frac{|\varepsilon|}{|1+\varepsilon|}-\frac{1}{|1+\varepsilon|}\left|\frac{(f*h)(z)}{z}\right|$$
$$>\frac{|\varepsilon|-\alpha}{|1+\varepsilon|}\geq 0,$$

this is a contradiction by $|\varepsilon|<\alpha$ and however, we have $\left|\frac{(f*h)(z)}{z}\right|\geq\alpha$.

If $g(z)=z-\sum\limits_{n=2}^{\infty}b_nz^n\in N_\alpha(f)$ then

$$\alpha-\left|\frac{(g*h)(z)}{z}\right|\leq\left|\frac{((f-g)*h)(z)}{z}\right|\leq\sum_{n=2}^{\infty}|a_n-b_n||c_n||z^n|$$
$$<\sum_{n=2}^{\infty}\frac{\phi_n(\alpha,b,\mu)|n(1+k)-u_n(k+\gamma)|}{1-\gamma}|a_n-b_n|$$
$$\leq\alpha.$$

□

4. Partial Sums of the Function Class $k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$

In this section, applying methods used by Silverman [15] and Silvia [16], we investigate the ratio of a function of the form (6) to its sequence of partial sums $f_m(z)=z+\sum_{n=2}^{m}a_nz^n$.

Theorem 4.1. *If $f\in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ then*

$$\Re\left\{\frac{f(z)}{f_m(z)}\right\}\geq\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)-1+\gamma}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}\right) \tag{10}$$

where

$$\delta_n=\delta_n(k,\gamma,u_n)\phi_n(\alpha,b,\mu)\geq\begin{cases}1-\gamma, & \text{if } n=2,3,\cdots,m;\\ \delta_{m+1}\phi_{m+1}(\alpha,b,\mu), & \text{if } n=m+1,m+2,\cdots\end{cases} \tag{11}$$

and

$$\delta_n=\delta_n(k,\gamma,u_n)=n(1+k)-u_n(k+\gamma).$$

The result in (10) *is sharp with the following given by*

$$f(z)=z+\frac{1-\gamma}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}z^{m+1}. \tag{12}$$

Proof. Define the function w, we may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)}&=\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\left\{\frac{f(z)}{f_m(z)}-\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)-1+\gamma}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}\right\}\\ &=\left\{\frac{1+\sum_{n=2}^{m}a_nz^{n-1}+\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}a_nz^{n-1}}{1+\sum_{n=2}^{m}a_nz^{n-1}}\right\}.\end{aligned} \tag{13}$$

It suffices to show that $|w(z)|\leq 1$. Now, from (13), we can obtain

$$w(z)=\frac{\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}a_nz^{n-1}}{2+2\sum_{n=2}^{m}a_nz^{n-1}+\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}a_nz^{n-1}}.$$

Hence we obtain

$$|w(z)|\leq\frac{\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{m}|a_n|-\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}|a_n|}.$$

Now $|w(z)| \leq 1$ if

$$2\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}|a_n| \leq 2-2\sum_{n=2}^{m}|a_n|,$$

or, equivalently

$$\sum_{n=2}^{m}|a_n| + \left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}|a_n| \leq 1.$$

From the condition (7), it is sufficient to show that

$$\sum_{n=2}^{m}|a_n| + \left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\frac{\delta_n\phi_n(\alpha,b,\mu)}{1-\gamma}|a_n|$$

which is equivalent to

(14)
$$\sum_{n=2}^{m}\left(\frac{\delta_n\phi_n(\alpha,b,\mu)-1+\gamma}{1-\gamma}\right)|a_n| + \sum_{n=m+1}^{\infty}\left(\frac{\delta_n\phi_n(\alpha,b,\mu)-\delta_{n+1}\phi_{n+1}(\alpha,b,\mu)}{1-\gamma}\right)|a_n| \geq 0.$$

To see that the function gives by (12) given the sharp result, we observe that for $z = re^{\frac{i\pi}{n}}$

$$\begin{aligned}\frac{f(z)}{f_m(z)} =& 1+\frac{1-\gamma}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}z^n \to 1-\frac{1-\gamma}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}\\ =& \frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)-1+\gamma}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}, \text{ when } r \to 1^-.\end{aligned}$$

□

Theorem 4.2. *If* $f \in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ *then*

$$\Re\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)+1-\gamma}, \quad (z \in U) \tag{15}$$

where $\delta_{m+1}\phi_{m+1}(\alpha,b,\mu) \geq 1-\gamma$ *and*

$$\delta_n\phi_n(\alpha,b,\mu) \geq \begin{cases} 1-\gamma, & \text{if } n = 2,3,\cdots,m; \\ \delta_{m+1}\phi_{m+1}(\alpha,b,\mu), & \text{if } n = m+1,m+2,\cdots. \end{cases} \tag{16}$$

The result (15) *is sharp with the function given by* (12).

Proof. We write

$$\frac{1+w(z)}{1-w(z)}=\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)+1-\gamma}{1-\gamma}\left\{\frac{f_m(z)}{f(z)}-\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)+1-\gamma}\right\}$$
$$=\left\{\frac{1+\sum_{n=2}^{m}a_nz^{n-2}-\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)\sum_{n=m+1}^{\infty}a_nz^{n-1}}{1+\sum_{n=2}^{\infty}a_nz^{n-1}}\right\}$$

$$|w(z)|\leq\frac{\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)+1-\gamma}{1-\gamma}\right)\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{m}|a_n|-\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)+1-\gamma}{1-\gamma}\right)\sum_{n=m+1}^{\infty}|a_n|}\leq 1.$$

This last inequality is equivalent to

$$\sum_{n=2}^{m}|a_n|+\sum_{n=m+1}^{\infty}\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{1-\gamma}\right)|a_n|\leq 1.$$

Making use of (7) to get (14). Finally, equality holds in (15) for the extremal furcation $f(z)$ given by (12). □

Theorem 4.3. *If* $f\in k-\widetilde{US}_s(\alpha,b,\mu,\gamma,t)$ *then*

$$\Re\left\{\frac{f'(z)}{f'_m(z)}\right\}\geq\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)-(1-\gamma)(m+1)}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}\right),\quad (z\in U)\tag{17}$$

$$\text{and }\Re\left\{\frac{f'_m(z)}{f'(z)}\right\}\geq\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)+(1-\gamma)(m-1)}\right),\quad (z\in U)\tag{18}$$

where $\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)\geq(m+1)(1-\gamma)$ *and*

$$\delta_n\phi_n(\alpha,b,\mu)\geq\begin{cases} n(1-\gamma), & \text{if } n=1,2,3,\cdots,m;\\ n\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{m+1}\right), & \text{if } n=m+1,m+2,\cdots.\end{cases}$$

The results are sharp with the function given by (12).

Proof. We write

$$\frac{1+w(z)}{1-w(z)}=\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\left\{\frac{f'(z)}{f'_m(z)}-\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)-(1-\gamma)(m+1)}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}\right)\right\}$$

where

$$w(z)=\frac{\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\sum_{n=m+1}^{\infty}na_nz^{n-1}}{2+2\sum_{n=2}^{m}na_nz^{n-1}+\left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\sum_{n=m+1}^{\infty}na_nz^{n-1}}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{m} n|a_n| + \left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\sum_{n=m+1}^{\infty} n|a_n| \leq 1.$$

From the condition (7), it is sufficient to show that

$$\sum_{n=2}^{m} n|a_n| + \left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\sum_{n=m+1}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty}\frac{\delta_n\phi_n(\alpha,b,\mu)}{1-\gamma}|a_n|$$

which is equivalent to

$$\sum_{n=2}^{m}\frac{\delta_n\phi_n(\alpha,b,\mu)-(1-\gamma)n}{1-\gamma}|a_n| + \sum_{n=m+1}^{\infty}\frac{(m+1)\delta_n\phi_n(\alpha,b,\mu)-n\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}|a_n| \geq 0.$$

To prove the result (18), define the function $w(z)$

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \left(\frac{(m+1)(1-\gamma)+\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\\ &\quad\times\left\{\frac{f_m'(z)}{f'(z)} - \left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)+(m+1)(1-\gamma)}\right)\right\}\end{aligned}$$

where

$$w(z) = \frac{-\left(1+\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{2+2\sum\limits_{n=2}^{m} na_n z^{n-1} + \left(1-\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{m} n|a_n| + \left(\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)\sum_{n=m+1}^{\infty} n|a_n| \leq 1. \tag{19}$$

It suffices to show that the left hand side of (19) is bounded above by the condition

$$\sum_{n=2}^{\infty}\frac{\delta_n\phi_n(\alpha,b,\mu)}{1-\gamma}|a_n|$$

which is equivalent to

$$\sum_{n=2}^{m}\left(\frac{\delta_n\phi_n(\alpha,b,\mu)}{1-\gamma}-n\right)|a_n| + \sum_{n=m+1}^{\infty}\left(\frac{\delta_n\phi_n(\alpha,b,\mu)}{1-\gamma}-\frac{\delta_{m+1}\phi_{m+1}(\alpha,b,\mu)}{(m+1)(1-\gamma)}\right)n|a_n| \geq 0.$$

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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