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FRAMELET SCALING SET WITH MATRIX DILATION IN $L^2(\mathbb{R}^2)$

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Abstract. In this paper, we have constructed the non-overlapping frame scaling set with 2×2 expansive matrix dilation for the time frequency analysis in $\mathbb{R}^2$. The FMRA (Frame Multiresolution Analysis) always contains a frame scaling set. We have investigated that frequency domain of any frame scaling function contains a non-overlapping scaling set.

Keywords: wavelet; band-limited wavelet; frame; framelet; FMRA.

2010 AMS Subject Classification: 42C15, 42C40.

1. Introduction

Wavelets has various applications in real life problem. One of the important consequence of wavelet is frames. The time frequency analysis plays a vital role in the field of image processing and signal processing that enables us to the study of signal in terms of time and frequency domain simultaneously. The interest for the study of signal processing and their transform are tightly connected in terms of mathematical aspects. An n-square matrix A is called an *expansive matrix* if the absolute value of each of eigen values are greater than 1. An *orthonormal wavelet*

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is a function ψ on $\mathbb{R}^n$ such that

$$\{\psi_{j,k}(\cdot) = |det(A)|^{j/2}\psi(A^j \cdot -k), j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^n)$, where A is an expansive matrix. For any function $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, its Fourier transform is defined as

$$\hat{f}(s) = \frac{1}{2\pi}\int_{\mathbb{R}^n} f(t)e^{-i<t,s>}dt$$

where $< t, s >$ denotes standard inner product of $\mathbb{R}^n$. Let $V = \{f_\nu\}_{\nu\in\mathbb{Z}} \subset L^2(\mathbb{R}^n)$ be a Bessel sequence then there is $\beta > 0$ such that

$$\sum_{\nu\in\mathbb{Z}} | < f, f_\nu > |^2 \leqslant \beta||f||^2 \text{ for any } f \in L^2(\mathbb{R}^n).$$

Also, if there exist positive constants $0 < \alpha \leqslant \beta < \infty$ such that

$$\alpha||f||^2 \leqslant \sum_{\nu\in\mathbb{Z}} | < f, f_\nu > |^2 \leqslant \beta||f||^2 \text{ for any } f \in L^2(\mathbb{R}^n),$$

then $\{f_\nu\}_{\nu\in\mathbb{Z}} \subset L^2(\mathbb{R}^{\mathrm{n}})$ is called *frame* and if α and β are chosen such way $\alpha = \beta$ then it is known as *tight frame*. Frame is called a *normalized tight frame* or *Parseval frame* if $\alpha = \beta = 1$. If the affine system $\{f^l_{j,k} = |det(A)|^{j/2}f^l(A^j \cdot -k) : l = 1,2,3,...,N\}$ is a frame for $L^2(\mathbb{R}^n)$ then $\{f^{(1)}, f^{(2)}, ..., f^{(N)}\}$ is called a *framelet* [12]. The framelet are naturally an extension of wavelet theory. If the Fourier transform of the framelet $\{f^{(1)}, f^{(2)}, ..., f^{(N)}\}$ is the characteristic function for the sets $W_1, W_2, ..., W_N$ that is $|\hat{f^l}| = \chi_{W_l}, l = 1,2,...,N$, then the set $W = \cup_{j=1}^N W_j$ is known as *framelet set* of order N. A function $f \in L^2(\mathbb{R}^n)$ is said to be a *Band-Limited* if the support of $\hat{f}$ is contained in finite interval. If ϕ is scaling function associated with a band-limited frame $|\hat{\phi}| = \chi_S$ then the measurable set S is called scaling set.

In [2], [4], and [7] author's have studied various results and properties of frame multiresolution analysis and in [5], Z. Zhang have given the idea of framelet scaling set and framelet set for 2 dilation in $L^2(\mathbb{R})$. From the motivation of his work in [3], we are able to find non-overlapping scaling set associated to frequency domain with the A-dilation in $L^2(\mathbb{R}^2)$. There are various interesting theory related to the time-frequency analysis. There are many different tools to split the time frequency domain into non-overlapping time-frequency domain.

2. PRELIMINARIES

Definition 2.1. [6] *Let E be a measurable subset of $\mathbb{R}$. If the set $E^* \subset E$ satisfies the conditions: (i) $E^* + 2\pi\mathbb{Z} = E + 2\pi\mathbb{Z}$ and (ii) $E^* \cap (E^* + 2\pi k) = \emptyset,\ k \in \mathbb{Z}\backslash\{0\}$, then the set E^* is called 2π-translation kernel of E.*

Definition 2.2. [3] *Let A be an 2-square expansive matrix and $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence of subspace of $L^2(\mathbb{R}^2)$ satisfying the following conditions:*

(i) $\overline{\cup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R}^2)$ and $\cap_{n\in\mathbb{Z}} V_n = \{(0,0)\}$.

(ii) $V_n \subset V_{n+1}, \forall n \in \mathbb{Z}$.

(iii) $f(w) \in V_n$ if and only if $f(A^\tau w) \in V_{n+1}$.

(iv) There exists a function $\phi(w) \in V_0$ such that $\{\phi(w-k) : k \in \mathbb{Z}^2\}$ is a frame for V_0,

then the sequence $\{V_n\}_{n\in\mathbb{Z}}$ is known as Frame Multiresolution Analysis (FMRA) and the function ϕ is called a frame scaling function.

For any $g(w) \in V_1, g(A^{-1}w) \in V_0$, so we can write

$$g(w) = |det(A)| \sum_{\nu\in\mathbb{Z}^2} c_\nu \phi(Aw-\nu). \tag{1}$$

If we define low pass filter $H_g(w) = \sum_{\nu\in\mathbb{Z}^2} c_\nu e^{-i<\nu,w>}$, then by taking Fourier transform of both side of equation (1) we get $\hat{g}(A^\tau w) = H_g(w)\hat{\phi}(w)$, here A^τ is transpose of matrix A. In particular

$$\hat{\phi}(A^\tau w) = H(w)\hat{\phi}(w). \tag{2}$$

Equation (2) is known as A-dilation scale equation of the respective FMRA [7].

Theorem 2.3. [5] *Let G be a bounded closed set in $\mathbb{R}^2$. If G is the support of frequency domain of a band- limited FMRA then*

(1) $G \subset AG$.

(2) $\cup_{n\in\mathbb{Z}} A^n G = \mathbb{R}^2$.

(3) $(G\backslash A^{-1}G) \cap (A^{-1}G + 2\pi k) \cong \emptyset,\ k \in \mathbb{Z}^2$

Lemma 2.4. [10] *The function ϕ is an A-dilation scaling function for an FMRA if and only if the following conditions holds*

(1) $\sum_{\nu\in\mathbb{Z}^2} |\hat{\phi}(w+2\pi\nu)|^2 = 1/(2\pi)^2$.

(2) $\lim_{j\to\infty} |\hat{\phi}((A^\tau)^{-j}w)| = 1/(2\pi)$ *a.e.*

(3) There exists a $2\pi\mathbb{Z}^2$*-translation periodic function* $H(w) \in L^2((-\pi,\pi)^2)$ *such that* $\hat{\phi}(A^\tau w) = H(w)\hat{\phi}(w)$.

Lemma 2.5. [11] *Let A be one of the following six matrices*

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}.$$

Then there exists vector l_A *and* q_A *in* $\mathbb{Z}^2$ *with the following properties*

(1) $\mathbb{Z}^2 = A^\tau\mathbb{Z}^2 \cup (l_A + A^\tau\mathbb{Z}^2)$,

(2) $< q_A,\, A^\tau\mathbb{Z}^2 > \subseteq\ 2\mathbb{Z}$ *and* $< q_A,\, (l_A + A^\tau\mathbb{Z}^2) > \subseteq\ 2\mathbb{Z}+1$,

(3) $A^\tau q_A \in (2\mathbb{Z})^2$,

(4) $A\mathbb{Z}^2 = A^\tau\mathbb{Z}^2$.

3. Main Results

In this section we prove some results in the form of lemmas and main result in theorem 3.6 that insures for construction of non-overlapping scaling sets in frequency domain. Following is the definition of $2\pi-$ translation kernel in $\mathbb{R}^2$ which is analogous to Definition (2.1).

Definition 3.1. *Let S be a non-empty subset of* $\mathbb{R}^2$*. Then* $S^* \subset S$ *is called as* 2π*-translation kernel of S if the following two conditions are satisfied:*

(1) $S + 2\pi\mathbb{Z}^2 = S^* + 2\pi\mathbb{Z}^2$.

(2) $S^* \cap (S^* + 2\pi k) = \emptyset,\ k \in \mathbb{Z}^2\backslash\{(0,0)\}$

By considering the theorem 2.3, for any frequency domain $\Omega = supp(\hat{\phi})$, if Ω is bounded then $\exists\ k \in \mathbb{Z}_+$ such that $\Omega \subset A^k \cdot (-\pi,\pi)^2$. Let us define $\Omega_j = A^{-j}(\Omega\backslash A^{-1}\Omega)$. By $\Omega \subset A\Omega$ we have

$$\Omega = (\Omega \backslash A^{-1}\Omega) \cup (A^{-1}\Omega)$$

$$A^{-1}\Omega = (A^{-1}\Omega \backslash A^{-2}\Omega) \cup (A^{-2}\Omega)$$

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$$A^{-k}\Omega = (A^{-k}\Omega \backslash A^{-k-1}\Omega) \cup (A^{-k-1}\Omega)$$

$$\Omega = (\cup_{j=0}^{k-1}\Omega_j) \cup (A^{-k}\Omega)$$

are all disjoint. Thus by Theorem 2.3, we have $\Omega_j \cap A^{-j}(A^{-1}\Omega + 2\pi k) = \emptyset,\ (j = 0,1,2,...)$ and by A-scale equation of FMRA we have $\hat{\phi}(A^\tau w) = H(w)\hat{\phi}(w)$ for $H(w) \in L^2((-\pi,\pi)^2)$. Hence for $\Omega = supp(\hat{\phi})$ and $\hat{\phi}(w) = 0$ if $w \in \Omega + 2\pi\mathbb{Z}^2$.

Lemma 3.2. *If $w \in \Omega^\perp = \Omega + 2\pi\mathbb{Z}^2$ then $|H(w)|^2 + |H(w+(\pi,\pi))|^2 = 1_{\Omega^\perp(Aw)}$, where $1_{\Omega^\perp}$ is the characteristic function of $\Omega^\perp$.*

Proof. Since we have

$$\sum_{\nu \in \mathbb{Z}^2} |\hat{\phi}(w + 2\pi\nu)|^2 = 0 \ or \ \frac{1}{(2\pi)^2}.$$

Thus for $w \in \Omega$, we have

$$\sum_{\nu \in \mathbb{Z}^2} |\hat{\phi}(w + 2\pi\nu)|^2 = \frac{1}{(2\pi)^2} 1_{\Omega^\perp}(w)$$

$$\frac{1}{(2\pi)^2} 1_{\Omega^\perp}(w) = \sum_{\nu \in \mathbb{Z}^2} |\hat{\phi}(A^\tau(A^{-\tau}w + 2\pi(A^{-\tau}\nu))|^2.$$

By using A-dilation scale equation,

$$\frac{1}{(2\pi)^2} 1_{\Omega^\perp}(w) = \sum_{\nu \in \mathbb{Z}^2} |\hat{\phi}(A^{-\tau}w + 2\pi(A^{-\tau}\nu)) \cdot H(A^{-\tau}w + 2\pi(A^{-\tau}\nu))|^2$$

$$\frac{1}{(2\pi)^2} 1_{\Omega^\perp}(w) = \sum_{\nu \in A^\tau\mathbb{Z}^2 \cup (l_A + A^\tau\mathbb{Z}^2)\mathbb{Z}^2} |\hat{\phi}(A^{-\tau}w + 2\pi(A^{-\tau}\nu)) \cdot H(A^{-\tau}w + 2\pi(A^{-\tau}\nu))|^2$$

$$
\begin{aligned}
\frac{1}{(2\pi)^2}1_{\Omega^\perp}(w) = &\sum_{\nu\in A^\tau\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(A^{-\tau}\nu))\cdot H(A^{-\tau}w+2\pi(A^{-\tau}\nu))|^2\\
&+\sum_{\nu\in(l_A+A^\tau\mathbb{Z}^2)}|\hat{\phi}(A^{-\tau}w+2\pi(A^{-\tau}\nu))\cdot H(A^{-\tau}w+2\pi(A^{-\tau}\nu))|^2\\
\frac{1}{(2\pi)^2}1_{\Omega^\perp}(w) = &\sum_{m\in\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(m))\cdot H(A^{-\tau}w+2\pi(m))|^2\\
&+\sum_{m\in\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(A^{-\tau}(l_A+A^\tau m))\cdot H(A^{-\tau}w+2\pi(A^{-\tau}(l_A+A^\tau m)))|^2.
\end{aligned}
$$

Since H is $2\pi\mathbb{Z}^2$ periodic and $2\pi l_A=(\pi,\pi)$, hence

$$
\begin{aligned}
\frac{1}{(2\pi)^2}1_{\Omega^\perp}(w) = &\sum_{m\in\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(m))\cdot H(A^{-\tau}w)|^2\\
&+\sum_{l_A\in\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(A^{-\tau}l_A)\cdot H(A^{-\tau}w+(\pi,\pi)|^2\\
\frac{1}{(2\pi)^2}1_{\Omega^\perp}(w) = &|H(A^{-\tau}w)|^2\sum_{m\in\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(m))|^2\\
&+|H(A^{-\tau}w+(\pi,\pi)|^2\sum_{l_A\in\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(A^{-\tau}l_A)|^2\\
\frac{1}{(2\pi)^2}1_{\Omega^\perp}(w) = &\sum_{m\in\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(m))\cdot H(A^{-\tau}w)|^2\\
&+\sum_{l_A\in\mathbb{Z}^2}|\hat{\phi}(A^{-\tau}w+2\pi(A^{-\tau}l_A)\cdot H(A^{-\tau}w+(\pi,\pi)|^2\\
\frac{1}{(2\pi)^2}1_{\Omega^\perp}(w) = &\frac{1}{(2\pi)^2}|H(A^{-\tau}w)|^2+\frac{1}{(2\pi)^2}|H(A^{-\tau}w+(\pi,\pi)|^2\\
1_{\Omega^\perp}(w) = &|H(A^{-\tau}w)|^2+|H(A^{-\tau}w+(\pi,\pi)|^2\ (w\in\mathbb{R}^2).
\end{aligned}
$$

From A-dilation scale equation we have $supp(\hat{\phi}(A\,\cdot))=supp(H)\cap\Omega$ and
$1_{\Omega^\perp}(Aw)=|H(w)|^2+|H(w+(\pi,\pi))|^2.$ □

Now let us consider a partition for the band limited scaling function ϕ and their support Ω, $E_k=\Omega_k=A^{-k}(\Omega\backslash A^{-1}\Omega)$, $F=\{w\in\mathbb{R}^2 : H(w)=1\}$, $D_{k-j}=E_{k-j}\cap F$, $C_{k-j}=E_{k-j}\backslash F$ and $E_{k-j-1}=(AD_{k-j})\cup(AC_{k-j})^*,\ j=0,1,2,3,...,k-1$ (where X^* represents 2π-translation kernel of X).

Lemma 3.3. *Let us consider $\{E_{k-j}\}_{j=1,2,...,k}$ as defined above, then the following holds:*

(1) $E_{k-j} \subset \Omega_{k-j}$ *for* $j = 1,2,3,...,k,$

(2) $(E_{k-j} + 2\pi\nu) \cap E_{k-l} = \emptyset$ *for* $l \neq j$ *and* $\nu \in \mathbb{Z}^2$.

Proof. Since we have assumed above $E_{k-j-1} = (AD_{k-j}) \cup (AC_{k-j})^*,\ j = 0,1,2,3,...,k-1$ therefore for $j = 0$ we have

$$\begin{aligned} E_{k-1} =& (AD_k) \cup (AC_k)^* \\ \subset& A(D_k \cup C_k) = AE_k = \Omega_{k-1} \\ E_{k-2} =& (AD_{k-1}) \cup (AC_{k-1})^* \\ \subset& A(D_{k-1} \cup C_{k-1}) = AE_{k-1} = \Omega_{k-2} \end{aligned}$$

similarly for j

$$\begin{aligned} E_{k-j} =& (AD_{k-j+1}) \cup (AC_{k-j+1})^* \\ \subset& A(D_{k-j+1} \cup C_{k-j+1}) = AE_{k-j+1} = \Omega_{k-j} \end{aligned}$$

hence proved (1).

Next for the (2), by Theorem 2.3 we get

$$((E_{k-j} + 2\pi\nu) \cap E_{k-l}) \subset ((\Omega_{k-j} + 2\pi\nu) \cap \Omega_{k-l}) = \emptyset \text{ for } j \neq l \text{ and } \nu \in \mathbb{Z}^2,$$

which proves our required result. □

Lemma 3.4. *For any expansive matrix A and band-limited scaling function* ϕ*, the collection of sets* $\{E_{k-j} + 2\pi\nu\}_{\nu \in \mathbb{Z}^2}$ *are pairwise disjoint for* $j = 1,2,3,...,k.$

Proof. For $j = 1$, we have

$$\begin{aligned} E_{k-1} + 2\pi\nu =& ((AD_k) \cup (AC_k)^*) + 2\pi\nu \\ E_{k-1} + 2\pi\nu =& ((AD_k + 2\pi\nu) \cup (AC_k)^* + 2\pi\nu). \end{aligned}$$

Now we prove $(AD_k) \cap (AD_k + 2\pi\nu) = \emptyset,\ (\nu \in \mathbb{Z}^2 \backslash \{(0,0)\})$. As we have assumed $H(w) = 1$ for $w \in D_k \subset F$ and from lemma 3.2 we get $H(w + (\pi,\pi)) = 0$ and $H(w)$ is $2\pi\mathbb{Z}^2$-periodic.

Thus we have for $w \in D_k + (2\nu + (1,1))\pi,\ \nu \in \mathbb{Z}^2$

$$H(w) = 0$$

by A-dilation scale equation we get $\hat{\phi}(A^{\tau}w) = 0$. Since $supp(\hat{\phi}) = \Omega$, we have

$$\Omega \cap (AD_k + 2\pi(A\nu) + 2\pi(1,1)) = \emptyset,\ \nu \neq 0$$

$$\text{As } AD_k \subset A\Omega_k = \Omega_{k-1} \subset \Omega.$$

So we get $AD_k \cap (AD_k + 2\pi(A\nu) + 2\pi(1,1)) = \emptyset$, as $\Omega \subset A^k(-\pi,\pi)^2$ and

$$\Omega = (\cup_{j=0}^{k-1}\Omega_j) \cup (A^{-k}\Omega). \tag{3}$$

It follows that

$$D_k \subset \Omega_k \subset [-\pi,\pi]^2$$

$$\Rightarrow D_k \cap (D_k + 2\pi\nu) = \emptyset,\ \nu \in \mathbb{Z}^2 \setminus \{(0,0)\}.$$

Hence we proved that $AD_k \cap (AD_k + 2\pi\nu) = \emptyset,\ \nu \neq (0,0)$.

Next, let us prove $(AC_k)^* \cap (AD_k + 2\pi\nu) = \emptyset$. From equation (3) it follows that $D_k \subset G_k \subset (-\pi,\pi)^2$ and $D_k \cap C_k = \emptyset$, and hence

$$(D_k + 2\pi\nu) \cap C_k = \emptyset,\ \nu \in \mathbb{Z}^2$$

$$\Rightarrow (AC_k) \cap (AD_k + 2\pi(A\nu)) = \emptyset$$

by $C_k \subset \Omega_k$ we have $AC_k \subset \Omega_{k-1} \subset \Omega$ & by $\Omega \cap (AD_k + 2\pi(A\nu)) = \emptyset$, so we get

$$(AC_k) \cap (AB_k + (2\pi(AD) + 2\pi(1,1))) = \emptyset,\ \nu \in \mathbb{Z}^2.$$

Hence

$$(AC_k)^* \cap (AD_k + (2\pi\nu)) = \emptyset,\ \nu \in \mathbb{Z}^2$$

Thus we have proved that for $j = 1$, $\{E_{k-1} + 2\pi\nu\}_{\nu\in\mathbb{Z}^2}$ is a set of pairwise disjoint sets. Now by principle of mathematical induction, we can arrived at $\{E_{k-j} + 2\pi\nu\}_{\nu\in\mathbb{Z}^2}$ is a disjoint for $j = 1,2,3,...,k$.

By using lemma (3.3), we have

(1) $\{AB_{k-j} + 2\pi\nu\}_{\nu\in\mathbb{Z}^2}$ are pairwise disjoint.

(2) $(AB_{k-j}) \cap (2C_{k-j})^* = \emptyset$

Thus by above argument and relations between E_k, Ω_k and C_{k-j} we conclude that $\{E_{k-j} + 2\pi\nu\}_{\nu\in\mathbb{Z}^2}$ are pairwise disjoint, for $j = 1,2,3,...,k$. □

Lemma 3.5. *Let ϕ be a scaling function and the associated framelet is $\Phi = \{\psi^{(1)}, \psi^{(2)}, ..., \psi^{(l)}\}$ with supp$(\hat{\phi}) = \Omega$, $F = \{t : H(t) = 1\}$. Then $|\hat{\phi}(At)| = |\hat{\phi}(t)|$ if and only if $t \in M = (\mathbb{R}^2 \backslash \Omega) \cup F$.*

Proof. Let us define sets

$$\begin{aligned} M =&\{w : |\hat{\phi}(w)| = |\hat{\phi}(Aw)|\}, \\ T_1 =&\{w : |\hat{\phi}(Aw)| = |\hat{\phi}(w)| = 0\}, \\ \text{and } T_2 =&\{w : |\hat{\phi}(Aw)| = |\hat{\phi}(w)| \neq 0\}. \end{aligned}$$

We have $M = T_1 \cup T_2$. Now if $w \in T_2$ then by A-dilation scale equation $\hat{\phi}(Aw) = H(w)\hat{\phi}(w) \Rightarrow |H(w)| = 1$ and $w \in F \Rightarrow T_2 \subset F$. Hence we have $|\hat{\phi}(Aw)| = |H(w)||\hat{\phi}(w)| \Rightarrow |\hat{\phi}(Aw)| = |\hat{\phi}(w)|$. Therefore $T_2 \subset F \subset M$ and $M = (T_1 \cup T_2) \subset (T_1 \cup F) \subset M$. Also from $supp(\hat{\phi}) = \Omega$, it follows that $\hat{\phi}(Aw) = 0$ for $w \in \mathbb{R}^2 \backslash A^{-1}\Omega$. Thus from this and $A^{-1}\Omega \subset \Omega$ we have

$$\begin{aligned} T_1 =&(\mathbb{R}^2 \backslash \Omega) \cap \left(\mathbb{R}^2 \backslash A^{-1}\Omega\right) = \mathbb{R}^2 \backslash \Omega. \\ M =&(\mathbb{R}^2 \backslash \Omega) \cup F. \end{aligned}$$

□

Theorem 3.6. *If $S = \left(\cup_{j=1}^{k} E_{k-j}\right) \cup A^{-k}\Omega$ is a measurable set associated with the band limited scaling function ϕ along with supp$(\hat{\phi}) \subset A^k(-\pi, \pi)^2$ then S is scaling set and $S \subset \Omega$.*

Proof. Consider the measurable set $S = \left(\cup_{j=1}^{k} E_{k-j}\right) \cup A^{-k}\Omega$
and $S + 2\pi\nu = \left(\cup_{j=1}^{k} E_{k-j} + 2\pi\nu\right) \cup A^{-k}\Omega + 2\pi\nu \; for \; \nu \in \mathbb{Z}^2$.
By lemma 3.2, lemma 3.3, and lemma 3.4, we have

$$E_{k-\nu} \subset \Omega_{k-\nu} \text{ and } \Omega_j \cap (A^l\Omega + 2\pi\nu) = \emptyset, \; l > j, \; \nu \in \mathbb{Z}^2$$

$$((E_{k-j} + 2\pi\nu) \cap (A^{-k} + 2\pi\mu)) \subset ((\Omega_{k-j} + 2\pi\nu) \cap (A^{-k}\Omega + 2\pi\mu)) = \emptyset$$

for $j = 1,,2,3,....,k$ & $\nu \neq \mu$. Again from $\Omega \subset A^{-k}(-\pi, \pi)^2$, we have

$$(A^{-k}\Omega + 2\pi\nu) \cap (A^{-k}\Omega + 2\pi\mu) = \emptyset, \; \nu \neq \mu.$$

Hence $\{S+2\pi\nu\}_{\nu\in\mathbb{Z}^2}$ are pairwise disjoint sets. By Theorem 2.3, we have $\cup_{m\in\mathbb{Z}}A^m\Omega=\mathbb{R}^2$ and $A^{-k}\Omega\subset S$. Thus

$$\mathbb{R}^2 \supset \cup_m A^m S \supset \cup_m A^{(m-k)}\Omega = \cup_{m\in\mathbb{Z}}A^m\Omega = \mathbb{R}^2$$

$$\text{i.e.} \cup_{m\in\mathbb{Z}} A^m\Omega = \mathbb{R}^2.$$

Finally by the construction

$$\begin{aligned} E_k &= \Omega_k = A^{-k}(\Omega\backslash A^{-1}\Omega),\ F=\{w\in\mathbb{R}^2 : |H(w)|=1\} \\ D_{k-j} &= E_{k-j}\cap F,\ C_{k-j}=E_{k-j}\backslash F \\ E_{k-j-1} &= (AD_{k-j}\cup(AC_{k-j+1})^*),\ j=0,1,2,...,k-1 \\ E_{k-j} &= (AD_{k-j+1})\cup(C_{k-j+1})^* \\ &\subset A(D_{k-j+1}\cup C_{k-j+1}) = AE_{k-j+1},\ j=1,2,3,...,k. \end{aligned}$$

By definition of S and $\Omega_j=A^{-j}(\Omega\backslash A^{-1}\Omega)$, we get

$$\begin{aligned} S\subset(\cup_{j=1}^{k-1}(AE_{k-j}))\cup\Omega_{k-1}\cup A^{-k}\Omega &= A(\cup_{j=1}^{k-1}E_{k-j})\cup((A^{k-1})\Omega) \\ &\subset A((\cup_{j=1}^{k-1}\Omega_{k-j})\cup A^{-k}\Omega)\subset AS. \end{aligned}$$

Thus by defining a function ϕ such that $\hat{\phi}=\chi_S$, we get that ϕ is a scaling function and scaling set S is given by

$S=((\cup_{j=1}^{k}E_{k-j})\cup A^{-k}\Omega)\subset((\cup_{j=1}^{k}\Omega_{k-j})\cup A^{-k}\Omega)=\Omega.$ □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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