Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:183
https://doi.org/10.28919/jmcs/7510
ISSN: 1927-5307

GROUP MEAN CORDIAL LABELING OF SOME PATH AND CYCLE RELATED GRAPHS

R.N. RAJALEKSHMI*, R. KALA

Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, India

Abstract. Let G be a (p,q) graph and let A be a group. Let $f : V(G) \longrightarrow A$ be a map. For each edge uv assign the label $\left\lfloor \frac{o(f(u))+o(f(v))}{2} \right\rfloor$. Here $o(f(u))$ denotes the order of $f(u)$ as an element of the group A. Let $\mathbb{I}$ be the set of all integers that are labels of the edges of G. f is called a group mean cordial labeling if the following conditions hold:

(1) For $x,y \in A$, $|v_f(x) - v_f(y)| \leq 1$, where $v_f(x)$ is the number of vertices labeled with x.

(2) For $i, j \in \mathbb{I}$, $|e_f(i) - e_f(j)| \leq 1$, where $e_f(i)$ denote the number of edges labeled with i.

A graph with a group mean cordial labeling is called a group mean cordial graph. In this paper, we take A as the group of fourth roots of unity and prove that, the graphs Ladder, Slanting Ladder, Triangular Ladder, Fan, Flower and Sunflower are group mean cordial graphs.

Keywords: cordial labeling; mean labeling; group mean cordial labeling.

2010 AMS Subject Classification: 05C78.

1. INTRODUCTION

Graphs considered here are finite, undirected and simple. Terms not defined here are used in the sense of Harary [4] and Gallian [3]. Somasundaram and Ponraj [6] introduced the concept

*Corresponding author

E-mail address: rajalekshmimoni@gmail.com

Received May 17, 2022

of mean labeling of graphs.

Definition 1.1. [6] A graph G with p vertices and q edges is a mean graph if there is an injective function f from the vertices of G to $0,1,2,\ldots,q$ such that when each edge uv is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd then the resulting edge labels are distinct.

Cahit [2] introduced the concept of cordial labeling.

Definition 1.2.[2] Let $f : V(G) \to \{0,1\}$ be any function. For each edge xy assign the label $|f(x)-f(y)|$. f is called a cordial labeling if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1. Also the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1.

Ponraj et al. [5] introduced mean cordial labeling of graphs.

Definition 1.3.[5] Let f be a function from the vertex set $V(G)$ to $\{0,1,2\}$. For each edge uv assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a *mean cordial labeling* if $\left|v_f(i)-v_f(j)\right| \leq 1$ and $\left|e_f(i)-e_f(j)\right| \leq 1$, $i,j \in \{0,1,2\}$, where $v_f(x)$ and $e_f(x)$ respectively denote the number of vertices and edges labeled with x $(x = 0,1,2)$. A graph with a mean cordial labeling is called a mean cordial graph.

Athisayanathan et al. [1] introduced the concept of group A cordial labeling.

Definition 1.4.[1] Let A be a group. We denote the order of an element $a \in A$ by $o(a)$. Let $f : V(G) \to A$ be a function. For each edge uv assign the label 1 if $(o(f(u)),o(f(v))) = 1$or 0 otherwise. f is called a group A Cordial labeling if $|v_f(a)-v_f(b)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$, where $v_f(x)$ and $e_f(n)$ respectively denote the number of vertices labelled with an element x and number of edges labelled with $n(n = 0, 1)$. A graph which admits a group A Cordial labeling is called a group A Cordial graph.

Motivated by these , we define group mean cordial labeling of graphs.

For any real number x, we denoted by $\lfloor x \rfloor$, the greatest integer smaller than or equal to x and by $\lceil x \rceil$, we mean the smallest integer greater than or equal to x.

2. PRELIMINARIES

Definition 2.1. The graph $L_n = P_n \times P_2$ is called a Ladder graph.

Definition 2.2. The Trianular ladder, $T(L_n)$ is a graph obtained from the Ladder graph, L_n by adding the edges $u_j v_{j+1}, (1 \leq j \leq n-1)$, where $u_j, v_j (1 \leq j \leq n)$, are the vertices of L_n.

Definition 2.3. A Slanting ladder, $S(L_n)$ is the graph obtained from two paths $u_1 u_2 ... u_n$ and $v_1 v_2 ... v_n$ by joining each v_j with u_{j+1}, $1 \leq j \leq n-1$.

Definition 2.4. The graph $F_n = P_n + K_1$ is called a Fan graph where $P_n : u_1 u_2 ... u_n$ is a Path.

Definition 2.5. The Flower graph Fl_n is a graph obtained from a Helm by joining each pendent vertex to the central vertex of the Helm.

Definition 2.6. The Sunflower graph SF_n is obtained from a Wheel with the central vertex v, the cycle $C_n : u_1 u_2 ... u_n u_1$ and additional vertices $v_1, v_2, ..., v_n$ where v_j is joined by edges to u_j, u_{j+1} where u_{j+1} is taken modulo n.

3. MAIN RESULTS

Definition 3.1. Let G be a (p,q) graph and let A be a group. Let f be a map from $V(G)$ to A. For each edge uv assign the label $\left\lfloor \frac{o(f(u))+o(f(v))}{2} \right\rfloor$. Let $\mathbb{I}$ be the set of all integers that are labels of the edges of G. f is called group mean cordial labeling if the following conditions hold:
(1) For $x, y \in A$, $|v_f(x) - v_f(y)| \leq 1$, where $v_f(x)$ is the number of vertices labeled with x.
(2) For $i, j \in \mathbb{I}$, $|e_f(i) - e_f(j)| \leq 1$, where $e_f(i)$ denote the number of edges labeled with i.
A graph with a group mean cordial labeling is called a group mean cordial graph.

In this paper, we take the group A as the group $\{1, -1, i, -i\}$ which is the group of fourth roots of unity, that is cyclic with generators i and $-i$.

Example 3.2. Figure 1 is a simple example of a group mean cordial graph.

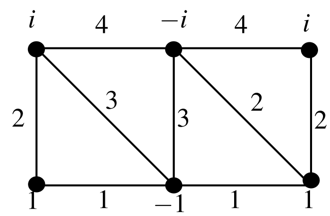

FIGURE 1

Theorem 3.3. The Ladder, L_n is a group mean cordial graph for every n.

Proof. Let $V(L_n) = \{u_j, v_j : 1 \leq j \leq n\}$. Then $E(L_n) = \{u_1u_2, u_2u_3, ..., u_{n-1}u_n\} \cup \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\} \cup \{u_jv_j : 1 \leq j \leq n\}$. This graph has $2n$ vertices and $3n-2$ edges.

Define $f : V(L_n) \longrightarrow \{1, -1, i, -i\}$ by,

$$f(u_j) = \begin{cases} 1 & \text{if} \quad j \equiv 0,2 \pmod 8 \\ -1 & \text{if} \quad j \equiv 1,3 \pmod 8 \\ i & \text{if} \quad j \equiv 4,6 \pmod 8 \\ -i & \text{if} \quad j \equiv 5,7 \pmod 8 \end{cases}$$

and

$$f(v_j) = \begin{cases} 1 & \text{if} \quad j \equiv 4,6 \pmod 8 \\ -1 & \text{if} \quad j \equiv 5,7 \pmod 8 \\ i & \text{if} \quad j \equiv 0,2 \pmod 8 \\ -i & \text{if} \quad j \equiv 1,3 \pmod 8 \end{cases}$$

The following Tables 1 and 2 show that f is a group mean cordial labeling for the graph L_n.

Nature of n	$v_f(1)$	$v_f(-1)$	$v_f(i)$	$v_f(-i)$
n is odd	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$
n is even	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$

TABLE 1

Nature of n	$e_f(1)$	$e_f(2)$	$e_f(3)$	$e_f(4)$
$n \equiv 0\,(mod\,4)$	$\frac{3n}{4}-1$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}-1$
$n \equiv 1\,(mod\,4)$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$	$\frac{3n+1}{4}$	$\frac{3n-3}{4}$
$n \equiv 2\,(mod\,4)$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$
$n \equiv 3\,(mod\,4)$	$\frac{3n-1}{4}$	$\frac{3n-5}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$

TABLE 2

Theorem 3.4. The Slanting Ladder, $S(L_n)$ is a group mean cordial graph for every n.

Proof. Let $V(S(L_n)) = \{u_j, v_j : 1 \leq j \leq n\}$. Then $E(S(L_n)) = \{u_1u_2, u_2u_3, ..., u_{n-1}u_n\} \cup \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\} \cup \{u_{j+1}v_j : 1 \leq j \leq n-1\}$. This graph has $2n$ vertices and $3n-3$ edges. Define $f : V(S(L_n)) \longrightarrow \{1, -1, i, -i\}$ by,

$$f(u_j) = \begin{cases} 1 & \text{if} \quad j \equiv 1,3 \pmod 4 \\ -i & \text{if} \quad j \equiv 2 \pmod 4 \\ -1 & \text{if} \quad j \equiv 0 \pmod 4 \end{cases}$$

and

$$f(v_j) = \begin{cases} i & \text{if} \quad j \equiv 1,3 \pmod 4 \\ -1 & \text{if} \quad j \equiv 2 \pmod 4 \\ -i & \text{if} \quad j \equiv 0 \pmod 4 \end{cases}$$

The following Tables 3 and 4 show that f is a group mean cordial labeling for the graph $S(L_n)$.

Nature of n	$v_f(1)$	$v_f(-1)$	$v_f(i)$	$v_f(-i)$
n is odd	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$
n is even	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$

TABLE 3

Nature of n	$e_f(1)$	$e_f(2)$	$e_f(3)$	$e_f(4)$
$n \equiv 0\,(mod\,4)$	$\frac{3n}{4}-1$	$\frac{3n}{4}-1$	$\frac{3n}{4}$	$\frac{3n}{4}-1$
$n \equiv 1\,(mod\,4)$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$
$n \equiv 2\,(mod\,4)$	$\frac{3n-6}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$
$n \equiv 3\,(mod\,4)$	$\frac{3n-5}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-5}{4}$

TABLE 4

Theorem 3.5. The Triangular Ladder, $T(L_n)$ is a group mean cordial graph for every n.

Proof. Let $V(T(L_n)) = \{u_j, v_j : 1 \leq j \leq n\}$. Then $E(T(L_n)) = \{u_1u_2, u_2u_3, ..., u_{n-1}u_n\} \cup \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\} \cup \{u_jv_j : 1 \leq j \leq n\} \cup \{u_{j+1}v_j : 1 \leq j \leq n-1\}$. This graph has $2n$ vertices and $4n-3$ edges.

Define $f : V(T(L_n)) \longrightarrow \{1, -1, i, -i\}$ by,

$$f(u_j) = \begin{cases} 1 & \text{if} \quad j \equiv 1 \pmod 2 \\ -1 & \text{if} \quad j \equiv 0 \pmod 2 \end{cases}$$

and

$$f(v_j) = \begin{cases} i & \text{if} \quad j \equiv 1 \pmod 2 \\ -i & \text{if} \quad j \equiv 0 \pmod 2 \end{cases}$$

Table 3 proves the vertex group mean cordial condition. Table 5 proves the edge group mean cordial condition.

Nature of n	$e_f(1)$	$e_f(2)$	$e_f(3)$	$e_f(4)$
n is odd	$n-1$	n	$n-1$	$n-1$
n is even	$n-1$	$n-1$	n	$n-1$

TABLE 5

Theorem 3.6. The Fan graph, F_n is a group mean cordial graph for every n.

Proof. Let $P_n : u_1u_2...u_n$ be a path. Let $V(K_1) = \{v\}$. Then $V(F_n) = V(P_n) \cup V(K_1)$. Also $E(F_n) = E(P_n) \cup \{vu_j : 1 \leq j \leq n\}$. The order and size of the Fan graph are $n+1$ and $2n-1$ respectively.

Define $f : V(F_n) \longrightarrow \{1, -1, i, -i\}$ by,

Case 1: $n \equiv 0, 1 \pmod 4$.

First, define $f(v) = -1$.Next define,

$$f(u_j) = \begin{cases} -1 & \text{if } 1 \le j \le \lfloor \frac{n}{4} \rfloor \\ 1 & \text{if } \lfloor \frac{n}{4} \rfloor + 1 \le j \le \lfloor \frac{n}{2} \rfloor \\ i & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \le j \le \lceil \frac{3n}{4} \rceil \\ -i & \text{if } \lceil \frac{3n}{4} \rceil + 1 \le j \le n \end{cases}$$

Case 2: $n \equiv 2,3 \pmod 4$.

Define $f(v) = -1$ and $f(u_n) = 1$.Then define,

$$f(u_j) = \begin{cases} -1 & \text{if } 1 \le j \le \lfloor \frac{n-2}{4} \rfloor \\ 1 & \text{if } \lfloor \frac{n-2}{4} \rfloor + 1 \le j \le \lfloor \frac{n-2}{2} \rfloor \\ i & \text{if } \lfloor \frac{n-2}{2} \rfloor + 1 \le j \le \lfloor \frac{3n-2}{4} \rfloor \\ -i & \text{if } \lfloor \frac{3n-2}{4} \rfloor + 1 \le j \le n-1. \end{cases}$$

The vertex and edge group mean cordial conditions are proved by the following tables 6 and 7.

Nature of n	$v_f(1)$	$v_f(-1)$	$v_f(i)$	$v_f(-i)$
$n \equiv 0\,(mod\,4)$	$\frac{n}{4}$	$\frac{n}{4}+1$	$\frac{n}{4}$	$\frac{n}{4}$
$n \equiv 1\,(mod\,4)$	$\frac{n-1}{4}$	$\frac{n+3}{4}$	$\frac{n+3}{4}$	$\frac{n-1}{4}$
$n \equiv 2\,(mod\,4)$	$\frac{n+2}{4}$	$\frac{n+2}{4}$	$\frac{n+2}{4}$	$\frac{n-2}{4}$
$n \equiv 3\,(mod\,4)$	$\frac{n+1}{4}$	$\frac{n+1}{4}$	$\frac{n+1}{4}$	$\frac{n+1}{4}$

TABLE 6

Nature of n	$e_f(1)$	$e_f(2)$	$e_f(3)$	$e_f(4)$
n is even	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}-1$
n is odd	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$

TABLE 7

Theorem 3.7. The Flower graph, Fl_n is a group mean cordial graph for every n.

Proof. Let $V(Fl_n) = \{u_j, v_j : 1 \le j \le n\} \cup \{v\}$. Then $E(Fl_n) = \{u_j u_{j+1} : 1 \le j \le n-1\} \cup \{u_n u_1\} \cup \{u_j v_j, u_j v, v_j v : 1 \le j \le n\}$. The order and size of this graph are $2n+1$ and $4n$ respectively.

Define $f : V(Fl_n) \longrightarrow \{1,-1,i,-i\}$ as follows:

Case 1: n is even.

$f(v) = i$

$f(u_{2j-1}) = 1 \quad if \quad 1 \leq j \leq \frac{n}{2}$

$f(u_{2j}) = -1 \quad if \quad 1 \leq j \leq \frac{n}{2}$

$f(v_j) = i \quad if \quad 1 \leq j \leq \frac{n}{2}$

$f(v_{\frac{n}{2}+j}) = -i \quad if \quad 1 \leq j \leq \frac{n}{2}$.

By this labeling, we get $v_f(1) = v_f(-1) = v_f(-i) = \frac{n}{2}$ and $v_f(i) = \frac{n}{2}+1$. Also, $e_f(1) = e_f(2) = e_f(3) = e_f(4) = n$.

Case 2: n is odd.

Here define,

$f(u_1) = f(v) = i$; $f(u_{n-1}) = f(u_n) = 1$ and $f(v_{n-1}) = f(v_n) = -1$.

$f(u_{2j+1}) = 1 \quad if \quad 1 \leq j \leq \frac{n-3}{2}$

$f(u_{2j}) = -1 \quad if \quad 1 \leq j \leq \frac{n-3}{2}$

$f(v_j) = i \quad if \quad 1 \leq j \leq \frac{n-3}{2}$

$f(v_{\frac{n-3}{2}+j}) = -i \quad if \quad 1 \leq j \leq \frac{n-1}{2}$.

In this case, we get $v_f(1) = v_f(-1) = v_f(i) = \frac{n+1}{2}$ and $v_f(-i) = \frac{n-1}{2}$. Also, $e_f(1) = e_f(2) = e_f(3) = e_f(4) = n$.

Hence the Flower graph Fl_n is a group mean cordial graph for every n.

Theorem 3.8. The Sunflower graph, SF_n is a group mean cordial graph for every n.

Proof. Let $C_n : u_1u_2...u_nu_1$ be a cycle. Let v be the central vertex.Let $v_1, v_2, ..., v_n$ be the newly added vertices. Then $E(SF_n) = E(C_n) \cup \{u_jv, u_jv_j : 1 \leq j \leq n\} \cup \{u_{j+1}v_j : 1 \leq j \leq n-1\} \cup \{u_1v_n\}$. The order and size of this graph are $2n+1$ and $4n$ respectively.

Define $f : V(SF_n) \longrightarrow \{1,-1,i,-i\}$ as follows:

Case 1: $n \equiv 0 \pmod 4$.

$f(v) = -1$ and

$$f(u_j) = f(v_j) = \begin{cases} 1 & \text{if } j \equiv 1 \pmod 4 \\ -1 & \text{if } j \equiv 2 \pmod 4 \\ i & \text{if } j \equiv 3 \pmod 4 \\ -i & \text{if } j \equiv 0 \pmod 4. \end{cases}$$

In this case, we get $v_f(-1) = \frac{n}{2} + 1$. Also $v_f(1) = v_f(i) = v_f(-i) = \frac{n}{2}$.

Case 2: $n \equiv 1 \pmod 4$.

The group mean cordial labeling of SF_5 is given in Figure 2

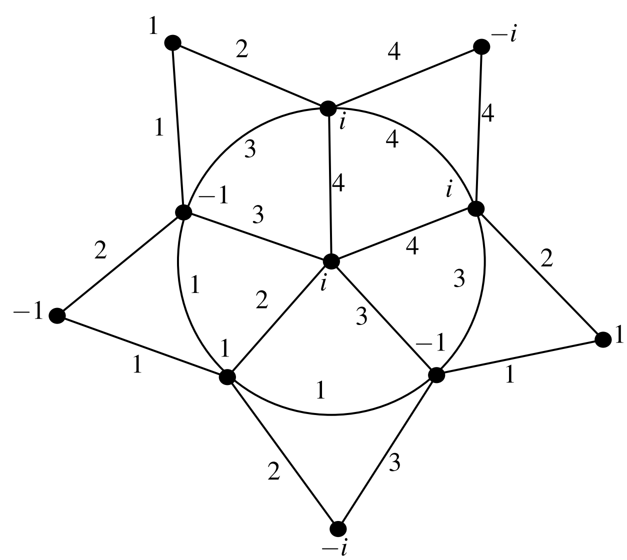

FIGURE 2

Let $n \geq 9$. Define $f(v) = -1$

Label $u_j, v_j (1 \leq j \leq n-9)$ as follows:

$$f(u_j) = f(v_j) = \begin{cases} -1 & \text{if } j \equiv 1 \pmod 4 \\ 1 & \text{if } j \equiv 2 \pmod 4 \\ i & \text{if } j \equiv 3 \pmod 4 \\ -i & \text{if } j \equiv 0 \pmod 4. \end{cases}$$

Next define,

$f(u_{n-8}) = -1; f(u_{n-7}) = f(u_{n-5}) = f(u_{n-2}) = 1;$

$f(u_{n-6}) = f(u_{n-4}) = f(u_{n-3}) = f(u_{n-1}) = f(u_n) = i;$

$f(v_{n-8}) = f(v_{n-5}) = 1; f(v_{n-2}) = f(v_{n-3}) = -1;$

$f(v_{n-7}) = f(v_{n-6}) = f(v_{n-4}) = f(v_{n-1}) = f(v_n) = -i$

In this case, we get $v_f(-1) = \frac{n-1}{2}$. Also $v_f(1) = v_f(i) = v_f(-i) = \frac{n+1}{2}$.

Case 3: $n \equiv 2 \pmod 4$.

Label the vertices v and $u_j, v_j (1 \leq j \leq n-6)$ as in case 1. Next assign 1 to the vertices $u_{n-5}, u_{n-2}, v_{n-5}$ and v_{n-3}. Assign -1 to the vertices v_{n-2}, v_n. Then, assign i to the vertices u_{n-4}, u_{n-3} and u_{n-1}. Finally assign $-i$ to the vertices u_n, v_{n-4} and v_{n-1}.

In this case, we get $v_f(1) = \frac{n}{2} + 1$. Also $v_f(-1) = v_f(i) = v_f(-i) = \frac{n}{2}$.

Case 4: $n \equiv 3 \pmod 4$.

Label the vertices v and $u_j, v_j (1 \leq j \leq n-3)$ as in case 1. Assign 1 to the vertices u_{n-2}, v_n and -1 to the vertex v_{n-2}. Next assign i to the vertices u_{n-1}, u_n and $-i$ to the vertex v_{n-1}.

In this case, we get $v_f(-i) = \frac{n-1}{2}$. Also $v_f(1) = v_f(-1) = v_f(i) = \frac{n+1}{2}$.

In all the cases,we get $e_f(x) = n$, for all $x \in \{1, 2, 3, 4\}$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] S. Athisayanathan, R. Ponraj, M.K. Karthik Chidambaram, Group A cordial labeling of graphs, Int. J. Appl. Math. Sci. 10 (2017), 1-11.

[2] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, Ars Comb. 23 (1987), 201-207.

[3] J.A. Gallian, A Dynamic survey of graph labeling, Electron. J. Comb. 2015 (2015), #DS6.

[4] F. Harary, Graph Theory, Addison Wesley, Reading Mass, 1972.

[5] R. Ponraj, M. Sivakumar, M. Sundaram, Mean cordial labeling of graphs, Open J. Discr. Math. 2(2012), 145-148.

[6] S. Somasundaram, R. Ponraj, Mean labeling of graphs, Natl. Acad. Sci. Lett. 26 (2003), 210-213.