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CIRIĆ AND ALMOST CONTRACTIONS IN CONVEX GENERALIZED b-METRIC SPACES

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Abstract. This manuscript intends to extend the work for Cirić contraction and almost contraction, condition (B), in the context of convex generalized b-metric spaces. We demonstrate the existence of a fixed point using Mann's iteration and prove its uniqueness.

Keywords: convex structure; generalized b-metric space; mann's iteration; cirić contraction, almost contraction.

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1. Introduction

The origins of the fixed point theory can be traced hundred years back to Banach's work. In 1922, he proved the famous fixed point theorem, stating that every contraction mapping on a complete metric space has only one fixed point. Since then, his work has been extended in various ways, including changing the framework of the metric space, bringing very powerful nonlinear analysis results, expanding fixed-point theory's field in multiple directions, and implementing new contraction kinds. In 1974, Cirić[4] proposed the concept of quasi-contraction

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as a general statement of the Banach contraction principle. Also, the weak contraction was outlined by Berinde[8]. It was renamed almost contraction by Berinde [9] in 2008. Furthermore, Babu et al. [1] worked on the open problem stated by Berinde [8], and as a result, the maps satisfying the condition (B) was introduced.

Bakhtin[2] pioneered the idea of b-metric spaces, which Czerwik[6] elaborated to broaden the Banach contraction's domain. Takahashi[11], in 1970, defined convexity and invented "convex metric space" to characterize a metric space with convexity. Singh et al.[5] introduced generalized b-metric spaces, which Singh and Singh[10] extended with convexity. The idea of convexity in b-metric spaces was delineated by Chen et al.[3] with the demonstration of Banach and Kannan's type fixed point theorems in these areas. Ćirić and almost contractions in convex b-metric spaces were proved by Rathee et al.[7]. Present paper reveals that fixed point exists for Ćirić contraction and almost contraction when the complete generalized b-metric space possesses a convex structure. The following is the structure of this article: First, some basic definitions related to the main theorems are defined, followed by an existence and uniqueness fixed point theorem for Ćirić contraction and almost contraction in generalized b-metric spaces and some deductions with examples.

2. PRELIMINARIES

Definition 2.1. [5] Assuming $H_s(\neq \phi)$ be a set and $s_1, s_2 \geq 1$ be two real numbers such that $b_{s_{12}}$ holds the following conditions true for every $\vartheta, \xi, \dagger \in H_s$,

(1) $b_{s_{12}}(\vartheta, \xi) = 0$ if and only if $\vartheta = \xi$,
(2) $b_{s_{12}}(\vartheta, \xi) = b_{s_{12}}(\xi, \vartheta)$,
(3) $b_{s_{12}}(\vartheta, \xi) \leq s_1 b_{s_{12}}(\vartheta, \dagger) + s_2 b_{s_{12}}(\dagger, \xi)$.

Such a function is called s_1, s_2 b-metric or generalized b-metric and the space $(H_s, b_{s_{12}})$ so formed is called s_1, s_2 b-metric space or generalized b-metric space.

Definition 2.2. [6]In a s_1, s_2 b-metric space, consider a sequence $\{\vartheta_n\}$. Then,

(1) The sequence $\{\vartheta_n\}$ is Cauchy in $(H_s, b_{s_{12}})$ if for each $\varepsilon > 0$ and
$\forall\, n, m > l$, $\exists\, l \in \mathbb{N}$ with $b_{s_{12}}(\vartheta_n, \vartheta_m) < \varepsilon$.

(2) The sequence $\{\vartheta_n\}$ converges to $\vartheta^* \in H_s$ in $(H_s, b_{s_{12}})$ if $\lim_{n\to\infty} b_{s_{12}}(\vartheta_n, \vartheta^*) = 0$.

(3) If all the Cauchy sequences in H_s converge, s_1, s_2 b-metric space is complete.

Definition 2.3. [12]A continuous function $\varpi : H_s \times H_s \times [0,1] \to H_s$ is said to be a convex structure on H_s for each $\dagger, \vartheta, \xi \in H_s$ and $\varsigma \in [0,1]$, if

$$b_{s_{12}}(\dagger, \varpi(\vartheta, \xi; \varsigma)) \leq \varsigma b_{s_{12}}(\dagger, \vartheta) + (1-\varsigma) b_{s_{12}}(\dagger, \xi).$$

Example 2.4. Let $H_s = [1,5]$ and define $b_{s_{12}}$ by:

$$b_{s_{12}}(\vartheta, \xi) = \begin{cases} 5^{|\vartheta-\xi|}, & \vartheta \neq \xi \\ 0, & \vartheta = \xi \end{cases}$$

and

$$\begin{aligned} b_{s_{12}}(\vartheta, \xi) &\leq 5^{|\vartheta-\dagger|+|\dagger-\xi|} \\ &= 5^{\frac{1}{5}|\vartheta-\dagger|+\frac{4}{5}|\dagger-\xi|} 5^{\frac{4}{5}|\vartheta-\dagger|+\frac{1}{5}|\dagger-\xi|} \\ &\leq \left(\frac{1}{5}5^{|\vartheta-\dagger|} + \frac{4}{5}5^{|\dagger-\xi|}\right) \sup_{\vartheta,\xi,\dagger\in H} 5^{\frac{4}{5}|\vartheta-\dagger|+\frac{1}{5}|\dagger-\xi|} \\ &= 5b_{s_{12}}(\vartheta, \dagger) + 20b_{s_{12}}(\dagger, \xi). \end{aligned}$$

So, $(H_s, b_{s_{12}})$ is a s_1, s_2 b-generalized metric space with $s_1 = 5$ and $s_2 = 20$. However, it is not a metric space as

$$b_{s_{12}}(1,5) > b_{s_{12}}(1,3) + b_{s_{12}}(3,5)$$

For convexity, define $\varpi(\vartheta, \xi; \varsigma) = \varsigma u + (1-\varsigma)v$ with $\varsigma \in [0,1]$, then

$$\begin{aligned} b_{s_{12}}(\dagger, \varpi(\vartheta, \xi; \varsigma)) &= b_{s_{12}}(\dagger, \varsigma u + (1-\varsigma)\xi) \\ &= 5^{|\dagger-\varsigma u-(1-\varsigma)\xi)|} \\ &= 5^{|\varsigma(\dagger-\vartheta)+(1-\varsigma)(\dagger-\xi)|} \\ &\leq 5^{|\varsigma(\dagger-\vartheta)|+|(1-\varsigma)(\dagger-\xi)|} \\ &\leq \varsigma 5^{|(\dagger-\vartheta)|} + (1-\varsigma)5^{|(\dagger-\vartheta)|} \\ &= \varsigma b_{s_{12}}(\vartheta, \dagger) + (1-\varsigma) b_{s_{12}}(\dagger, \xi), \end{aligned}$$

and hence, $(H_s, b_{s_{12}}, \varpi)$ is a convex generalized s_1, s_2 b-metric space.

3. Main Result for Ćiric Contraction

Theorem 3.1. *Assume $(H_s, b_{s_{12}}, \varpi)$ is a complete s_1, s_2 $b-$metric space with constants $s_1, s_2 > 1$ and $T_s : H_s \to H_s$ be defined as*

$$\begin{aligned} b_{s_{12}}(T_s\vartheta, T_s\xi) \leq\ & \kappa_s \max\{b_{s_{12}}(\vartheta,\xi), b_{s_{12}}(\vartheta, T_s\vartheta), b_{s_{12}}(\xi, T_s\xi), b_{s_{12}}(\vartheta, T_s\xi), \\ & b_{s_{12}}(\xi, T_s\vartheta)\} \end{aligned} \tag{1}$$

$\forall\ \vartheta,\xi \in H_s$ and $\kappa_s \in [0,1)$. Then T_s possesses a fixed point in H_s that is unique if $\kappa_s < \min\left\{\frac{1}{s_2(s_2+s_1^2)}, \frac{1}{s_1^3 s_2}, \frac{1}{s_1^4}\right\}$, $s_2^2 < \min\left\{s_1^3, s_2+s_1^2, \frac{s_1^4}{s_2}\right\}$ and $0 \leq \varsigma_{n-1} < \min\left\{\frac{1-s_2^2\kappa_s}{s_1^2} - s_2\kappa_s, \frac{\frac{1}{s_1^3 s_2}-\kappa_s}{\frac{1}{s_1 s_2}-\kappa_s}, \frac{\frac{1}{s_1^4}-\kappa_s}{\frac{1}{s_1^2}-\kappa_s}\right\}$ for each $n \in \mathbb{N}$, where $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$, $0 \leq \varsigma_{n-1} < 1$.

Proof. For any $n \in \mathbb{N}$

$$b_{s_{12}}(\vartheta_n, \vartheta_{n+1}) = b_{s_{12}}(\vartheta_n, \varpi(\vartheta_n, T_s\vartheta_n; \varsigma_n)) \leq (1-\varsigma_n) b_{s_{12}}(\vartheta_n, T_s\vartheta_n)$$

and

$$\begin{aligned} b_{s_{12}}(\vartheta_n, T_s\vartheta_n) \leq\ & s_1 b_{s_{12}}(\vartheta_n, T_s\vartheta_{n-1}) + s_2 b_{s_{12}}(T_s\vartheta_{n-1}, T_s\vartheta_n) \\ \leq\ & s_1 b_{s_{12}}(\varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1}), T_s\vartheta_{n-1}) + s_2\kappa_s \\ & \max\{b_{s_{12}}(\vartheta_{n-1}, \vartheta_n), b_{s_{12}}(\vartheta_{n-1}, T_s\vartheta_{n-1}), b_{s_{12}}(\vartheta_n, T_s\vartheta_n), \\ & b_{s_{12}}(\vartheta_{n-1}, T_s\vartheta_n), b_{s_{12}}(\vartheta_n, T_s\vartheta_{n-1})\} \\ \leq\ & s_1\varsigma_{n-1} b_{s_{12}}(\vartheta_{n-1}, T_s\vartheta_{n-1}) + s_2\kappa_s \max\{(1-\varsigma_{n-1}) b_{s_{12}} \\ & (\vartheta_{n-1}, T_s\vartheta_{n-1}), b_{s_{12}}(\vartheta_{n-1}, T_s\vartheta_{n-1}), b_{s_{12}}(\vartheta_n, T_s\vartheta_n), \\ & b_{s_{12}}(\vartheta_{n-1}, T_s\vartheta_n), b_{s_{12}}(\vartheta_n, T_s\vartheta_{n-1})\} \\ \leq\ & s_1\varsigma_{n-1} b_{s_{12}}(\vartheta_{n-1}, T_s\vartheta_{n-1}) + s_2\kappa_s \max\{b_{s_{12}}(\vartheta_{n-1}, T_s\vartheta_{n-1}), \\ & b_{s_{12}}(\vartheta_n, T_s\vartheta_n), s_1 b_{s_{12}}(\vartheta_{n-1}, \vartheta_n) + s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n), \\ & b_{s_{12}}(\varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1}), T_s\vartheta_{n-1})\} \end{aligned}$$

$$\begin{aligned}
&\le s_1\varsigma_{n-1}b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})+s_2\kappa_s\max\{b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}),\\
&\quad b_{s_{12}}(\vartheta_n,T_s\vartheta_n),s_1(1-\varsigma_{n-1})b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&\quad +s_2b_{s_{12}}(\vartheta_n,T_s\vartheta_n),\varsigma_{n-1}b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\}\\
&\le s_1\varsigma_{n-1}b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})+s_2\kappa_s\max\{b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}),\\
&\quad s_1(1-\varsigma_{n-1})b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})+s_2b_{s_{12}}(\vartheta_n,T_s\vartheta_n))\}\\
&\le s_1\varsigma_{n-1}b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})+s_2\kappa_s\max\{b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}),\\
&\quad s_1b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})+s_2b_{s_{12}}(\vartheta_n,T_s\vartheta_n)\}\\
&= s_1\varsigma_{n-1}b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})+s_1s_2\kappa_sb_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&\quad +s_2^2\kappa_sb_{s_{12}}(\vartheta_n,T_s\vartheta_n))\\
&= [s_1\varsigma_{n-1}+s_1s_2\kappa_s]b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})+s_2^2\kappa_sb_{s_{12}}(\vartheta_n,T_s\vartheta_n)
\end{aligned}$$

$$\implies (1-s_2^2\kappa_s)b_{s_{12}}(\vartheta_n,T_s\vartheta_n)\le[s_1\varsigma_{n-1}+s_1s_2\kappa_s]b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})$$

$$\begin{aligned}
b_{s_{12}}(\vartheta_n,T_s\vartheta_n) &\le \frac{s_1\varsigma_{n-1}+s_1s_2\kappa_s}{1-s_2^2\kappa_s}b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&< \frac{1}{s_1}b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})
\end{aligned}$$

with inequalities $\kappa_s<\min\left\{\frac{1}{s_2(s_2+s_1^2)},\frac{1}{s_1^3s_2},\frac{1}{s_1^4}\right\}$, $s_2^2<\min\left\{s_1^3,s_2+s_1^2,\frac{s_1^4}{s_2}\right\}$ and
$0\le\varsigma_{n-1}<\min\left\{\frac{1-s_2^2\kappa_s}{s_1^2}-s_2\kappa_s,\frac{\frac{1}{s_1^3s_2}-\kappa_s}{\frac{1}{s_1s_2}-\kappa_s},\frac{\frac{1}{s_1^4}-\kappa_s}{\frac{1}{s_1^2}-\kappa_s}\right\},\qquad n\in\mathbb{N}$

Thus,

$$b_{s_{12}}(\vartheta_n,T_s\vartheta_n)<\frac{1}{s_1}b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}) \tag{2}$$

$\implies\{b_{s_{12}}(\vartheta_n,T_s\vartheta_n)\}$ is a decreasing non-negative real numbers sequence. Therefore, $\exists\,\widehat{\zeta}\ge 0$ with

$$\lim_{n\to\infty}b_{s_{12}}(\vartheta_n,T_s\vartheta_n)=\widehat{\zeta}$$

We claim that $\widehat{\zeta}=0$. Assume that $\widehat{\zeta}>0$. Taking $n\to\infty$ in (2),

$$\widehat{\zeta}<\frac{1}{s_1}\widehat{\zeta}$$

which is a contradiction. Hence $\widehat{\zeta}=0$, that is,

$$\lim_{n\to\infty} b_{s_{12}}(\vartheta_n, T_s\vartheta_n) = 0$$

Here, we claim that $\{\vartheta_n\}$ is a Cauchy sequence.

Suppose that $\{\vartheta_n\}$ cannot be a Cauchy sequence,implying $\exists\ \varepsilon > 0$ and $\{\vartheta_{m_\iota}\}$ and $\{\vartheta_{n_\iota}\}$, subsequences of $\{\vartheta_n\}$, m_ι being the least natural cardinal with $m_\iota > n_\iota > \iota$ satisfying

$$b_{s_{12}}(\vartheta_{m_\iota}, \vartheta_{n_\iota}) \geq \varepsilon$$

and

$$b_{s_{12}}(\vartheta_{m_\iota - 1}, \vartheta_{n_\iota}) < \varepsilon$$

Then, we conclude that

$$\varepsilon \leq b_{s_{12}}(\vartheta_{m_\iota}, \vartheta_{n_\iota}) \leq s_1 b_{s_{12}}(\vartheta_{m_\iota}, \vartheta_{n_\iota+1}) + s_2 b_{s_{12}}(\vartheta_{n_\iota+1}, \vartheta_{n_\iota}),$$

which implies that

$$\frac{\varepsilon}{s_1} \leq \limsup_{\kappa_s\to\infty} b_{s_{12}}(\vartheta_{m_\iota}, \vartheta_{n_\iota+1})$$

Noticing that

$$\begin{aligned}
b_{s_{12}}(\vartheta_{m_\iota}, \vartheta_{n_\iota+1}) &= b_{s_{12}}(\varpi(\vartheta_{m_\iota-1}, T_s\vartheta_{m_\iota-1}; \varsigma_{m_\iota-1}), \vartheta_{n_\iota+1}) \\
&\leq \varsigma_{m_\iota-1} b_{s_{12}}(\vartheta_{m_\iota-1}, \vartheta_{n_\iota+1}) + (1-\varsigma_{m_\iota-1}) b_{s_{12}} \\
&\quad (T_s\vartheta_{m_\iota-1}, \vartheta_{n_\iota+1}) \\
&\leq \varsigma_{m_\iota-1} b_{s_{12}}(\vartheta_{m_\iota-1}, \vartheta_{n_\iota+1}) + (1-\varsigma_{m_\iota-1}) s_1 \\
&\quad b_{s_{12}}(T_s\vartheta_{m_\iota-1}, T_s\vartheta_{n_\iota+1}) + s_2 b_{s_{12}}(T_s\vartheta_{n_\iota+1}, \vartheta_{n_\iota+1})] \\
&\leq \varsigma_{m_\iota-1} b_{s_{12}}(\vartheta_{m_\iota-1}, \vartheta_{n_\iota+1}) + (1-\varsigma_{m_\iota-1}) s_2 b_{s_{12}} \\
&\quad (T_s\vartheta_{n_\iota+1}, \vartheta_{n_\iota+1}) + (1-\varsigma_{m_\iota-1}) s_1 \kappa_s \max\{b_{s_{12}} \\
&\quad (\vartheta_{m_\iota-1}, \vartheta_{n_\iota+1}), b_{s_{12}}(\vartheta_{m_\iota-1}, T_s\vartheta_{m_\iota-1}), b_{s_{12}} \\
&\quad (\vartheta_{n_\iota+1}, T_s\vartheta_{n_\iota+1}), b_{s_{12}}(\vartheta_{m_\iota-1}, T_s\vartheta_{n_\iota+1}), b_{s_{12}} \\
&\quad (\vartheta_{n_\iota+1}, T_s\vartheta_{m_\iota-1})\} \\
&\leq s_1 \varsigma_{m_\iota-1} b_{s_{12}}(\vartheta_{m_\iota-1}, \vartheta_{n_\iota}) + s_2 \varsigma_{m_\iota-1} b_{s_{12}}(\vartheta_{n_\iota}, \vartheta_{n_\iota+1}) +
\end{aligned}$$

$$
\begin{aligned}
& (1-\varsigma_{m_\iota -1}) s_2 b_{s_{12}}(T_s\vartheta_{n_\iota+1}, \vartheta_{n_\iota+1}) + (1-\varsigma_{m_\iota -1}) s_1 \kappa_s \\
& \max\{s_1 b_{s_{12}}(\vartheta_{m_\iota-1}, \vartheta_{n_\iota}) + s_2 b_{s_{12}}(\vartheta_{n_\iota}, \vartheta_{n_\iota+1}), b_{s_{12}} \\
& (\vartheta_{m_\iota-1}, T_s\vartheta_{m_\iota-1}), b_{s_{12}}(\vartheta_{n_\iota+1}, T_s\vartheta_{n_\iota+1}), s_1 \\
& b_{s_{12}}(\vartheta_{m_\iota-1}, \vartheta_{n_\iota+1}) + s_2 b_{s_{12}}(\vartheta_{n_\iota+1}, T_s\vartheta_{n_\iota+1}), s_1 \\
& b_{s_{12}}(\vartheta_{n_\iota+1}, \vartheta_{m_\iota-1}) + s_2 b_{s_{12}}(\vartheta_{m_\iota-1}, T_s\vartheta_{m_\iota-1})\} \\
\leq\; & s_1 \varsigma_{m_\iota-1} b_{s_{12}}(\vartheta_{m_\iota-1}, \vartheta_{n_\iota}) + s_2 \varsigma_{m_\iota-1} b_{s_{12}}(\vartheta_{n_\iota}, \vartheta_{n_\iota+1}) \\
& +(1-\varsigma_{m_\iota -1}) s_2 b_{s_{12}}(T_s\vartheta_{n_\iota+1}, \vartheta_{n_\iota+1}) + (1-\varsigma_{m_\iota -1}) s_1 \kappa_s \\
& \max\{s_1 b_{s_{12}}(\vartheta_{m_\iota-1}, \vartheta_{n_\iota}) + s_2 b_{s_{12}}(\vartheta_{n_\iota}, \vartheta_{n_\iota+1}), \\
& b_{s_{12}}(\vartheta_{m_\iota-1}, T_s\vartheta_{m_\iota-1}), b_{s_{12}}(\vartheta_{n_\iota+1}, T_s\vartheta_{n_\iota+1}), s_1^2 b_{s_{12}} \\
& (\vartheta_{m_\iota-1}, \vartheta_{n_\iota}) + s_1 s_2 b_{s_{12}}(\vartheta_{n_\iota}, \vartheta_{n_\iota+1}) + s_2 b_{s_{12}} \\
& (\vartheta_{n_\iota+1}, T_s\vartheta_{n_\iota+1}), s_1^2 b_{s_{12}}(\vartheta_{n_\iota+1}, \vartheta_{n_\iota}) \\
& + s_1 s_2 b_{s_{12}}(\vartheta_{n_\iota}, \vartheta_{m_\iota-1}) + s_2 b_{s_{12}}(\vartheta_{m_\iota-1}, T_s\vartheta_{m_\iota-1})\} \\
<\; & s_1 \varsigma_{m_\iota-1}\varepsilon + s_1 \kappa_s (1-\varsigma_{m_\iota-1}) \max\{s_1\varepsilon, s_1^2\varepsilon, s_1 s_2 \varepsilon\} \\
<\; & s_1 \varsigma_{m_\iota-1}\varepsilon + s_1^2 \kappa_s (1-\varsigma_{m_\iota-1}) \varepsilon \max\{s_1, s_2\}
\end{aligned}
$$

If $s_1 > s_2$, then

$$
b_{s_{12}}(\vartheta_{m_\iota}, \vartheta_{n_\iota+1}) < s_1 \varepsilon \left(\varsigma_{m_\iota-1}(1-s_1^2\kappa_s) + s_1^2\kappa_s\right) < \frac{\varepsilon}{s_1}
$$

if $s_2 > s_1$, then

$$
\begin{aligned}
b_{s_{12}}(\vartheta_{m_\iota}, \vartheta_{n_\iota+1}) \;&<\; s_1 \varsigma_{m_\iota-1}\varepsilon + s_1^2 s_2 \kappa_s (1-\varsigma_{m_\iota-1})\varepsilon \\
&=\; s_1 \varepsilon \left(\varsigma_{m_\iota-1}(1-s_1 s_2\kappa_s) + s_1 s_2 \kappa_s\right) < \frac{\varepsilon}{s_1} \qquad (3)
\end{aligned}
$$

Thus, we obtain

$$
\frac{\varepsilon}{s_1} \leq \limsup_{\kappa_s\to\infty} b_{s_{12}}(\vartheta_{m_\iota}, \vartheta_{n_\iota+1}) < \frac{\varepsilon}{s_1}
$$

which is a contradiction.

Thus, $\{\vartheta_n\}$ being a Cauchy sequence in H_s. and owing to completeness of H_s, $\exists\, \vartheta^* \in H$ such that $\lim_{n\to\infty} b_{s_{12}}(\vartheta_n, \vartheta^*) = 0$.

Now we verify that ϑ^* is a fixed point of T_s. For this,

$$\begin{aligned}
b_{s_{12}}(\vartheta^*, T_s\vartheta^*) &\leq s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta^*) \\
&\leq s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_1 s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n) + s_2^2 \\
&\quad b_{s_{12}}(T_s\vartheta_n, T_s\vartheta^*) \\
&\leq s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_1 s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n) + s_2^2 \kappa_s \\
&\quad \max\{b_{s_{12}}(\vartheta_n, \vartheta^*), b_{s_{12}}(\vartheta_n, T_s\vartheta_n), b_{s_{12}} \\
&\quad (\vartheta^*, T_s\vartheta^*), b_{s_{12}}(\vartheta_n, T_s\vartheta^*), b_{s_{12}}(\vartheta^*, T_s\vartheta_n)\} \\
&\leq s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_1 s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n) + s_2^2 \kappa_s \\
&\quad \max\{b_{s_{12}}(\vartheta_n, \vartheta^*), b_{s_{12}}(\vartheta_n, T_s\vartheta_n) b_{s_{12}} \\
&\quad (\vartheta^*, T_s\vartheta^*), s_1 b_{s_{12}}(\vartheta_n, \vartheta^*), + s_2 b_{s_{12}}(\vartheta^*, T_s\vartheta^*), \\
&\quad s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n)\}
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
b_{s_{12}}(\vartheta^*, T_s\vartheta^*) &\leq s_2^2 \kappa_s \max\{b_{s_{12}}(\vartheta^*, T_s\vartheta^*), s_2 b_{s_{12}}(\vartheta^*, T_s\vartheta^*)\} \\
&= s_2^3 \kappa_s b_{s_{12}}(\vartheta^*, T_s\vartheta^*) \\
&< b_{s_{12}}(\vartheta^*, T_s\vartheta^*)
\end{aligned}$$

So, $b_{s_{12}}(\vartheta^*, T_s\vartheta^*) = 0 \implies T_s\vartheta^* = \vartheta^*$.

Hence, ϑ^* is a fixed point of T_s.

To prove that this fixed point so obtained is unique, consider $q \in H_s$ such that $T_s q = q$, then

$$\begin{aligned}
0 < b_{s_{12}}(\vartheta^*, q) &= b_{s_{12}}(T_s\vartheta^*, Tq) \\
&\leq \kappa_s \max\{b_{s_{12}}(\vartheta^*, q), b_{s_{12}}(\vartheta^*, T_s\vartheta^*), b_{s_{12}}(q, Tq), b_{s_{12}}(\vartheta^*, Tq), \\
&\quad b_{s_{12}}(q, T_s\vartheta^*)\} \\
&\leq \kappa_s \max\{b_{s_{12}}(\vartheta^*, q), s_1 b_{s_{12}}(\vartheta^*, q) + s_2 b_{s_{12}}(q, Tq), s_1 b_{s_{12}}(q, \vartheta^*) \\
&\quad + s_2 b_{s_{12}}(\vartheta^*, T_s\vartheta^*)\} \\
&= s_1 \kappa_s b_{s_{12}}(\vartheta^*, q)
\end{aligned}$$

$$< \frac{1}{s_1^3} b_{s_{12}}(\vartheta^*, q) < b_{s_{12}}(\vartheta^*, q)$$

that is a contradictory statement. Hence, $\vartheta^* = q$.

□

Following is the corresponding result for Chatterjae type contraction in s_1, s_2 b-metric space which is a implication of Theorem 3.1:

Corollary 3.2. *Assume $(H_s, b_{s_{12}}, \varpi)$ is a complete s_1, s_2 $b-$metric space with constants $s_1, s_2 > 1$ and $T_s : H_s \to H_s$ be defined as*

$$b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \kappa_s [b_{s_{12}}(\vartheta, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)] \tag{4}$$

$\forall\ \vartheta, \xi \in H_s$ and $\kappa_s \in \left[0, \frac{1}{2}\right)$. Then T_s possesses a fixed point in H_s that is unique if $\kappa_s < \min\left\{\frac{1}{2s_2(s_2+s_1^2)}, \frac{1}{2s_1^3 s_2}, \frac{1}{2s_1^4}\right\}$, $s_2^2 < \min\left\{s_1^3, s_2 + s_1^2, \frac{s_1^4}{s_2}\right\}$ and $0 \leq \varsigma_{n-1} < \min\left\{\frac{1-2s_2^2\kappa_s}{s_1^2} - s_2\kappa_s, \frac{\frac{1}{s_1^3 s_2} - 2\kappa_s}{\frac{1}{s_1 s_2} - 2\kappa_s}, \frac{\frac{1}{s_1^4} - 2\kappa_s}{\frac{1}{s_1^2} - 2\kappa_s}\right\}$ for each $n \in \mathbb{N}$, where $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$, $0 \leq \varsigma_{n-1} < 1$.

If $s_1 = s_2 = s$ in Theorem 3.1, then we have Theorem 1 of [7] in convex b-metric spaces:

Corollary 3.3. [7] *Assume $(H_s, b_{s_{12}}, \varpi)$ is a complete $b-$metric space with $s > 1$ and $T_s : H_s \to H_s$ be defined as*

$$\begin{aligned} b_{s_{12}}(T_s\vartheta, T_s\xi) \quad &\leq \quad \kappa_s \max\{b_{s_{12}}(\vartheta, \xi), b_{s_{12}}(\vartheta, T_s\vartheta), b_{s_{12}}(\xi, T_s\xi), b_{s_{12}}(\vartheta, T_s\xi), \\ &\qquad b_{s_{12}}(\xi, T_s\vartheta)\} \end{aligned} \tag{5}$$

$\forall\ \vartheta, \xi \in H_s$ and $\kappa_s \in [0, 1)$. Then T_s possesses a fixed point in H_s that is unique if $\kappa_s < \min\left\{\frac{1}{s^2(s+1)}, \frac{1}{s^4}\right\}$ and $0 \leq \varsigma_{n-1} < \min\left\{\frac{1}{s^2} - (s+1)\kappa_s, \frac{\frac{1}{s^4} - \kappa_s}{\frac{1}{s^2} - \kappa_s}\right\}$ for each $n \in \mathbb{N}$, where $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$, $0 \leq \varsigma_{n-1} < 1$.

By using Lemma 1 of [7], we have Theorem 2 of [7] in convex b-metric spaces:

Corollary 3.4. [7] *Assume* $(H_s, b_{s_{12}}, \varpi)$ *is a complete* $b-$*metric space with* $s > 1$ *and* $T_s : H_s \to H_s$ *be defined as*

$$
\begin{aligned}
b_{s_{12}}(T_s\vartheta, T_s\xi) \quad \leq \quad & \kappa_s \max\{b_{s_{12}}(\vartheta,\xi), b_{s_{12}}(\vartheta, T_s\vartheta), b_{s_{12}}(\xi, T_s\xi), b_{s_{12}}(\vartheta, T_s\xi), \\
& b_{s_{12}}(\xi, T_s\vartheta)\}, \qquad (6)
\end{aligned}
$$

$\forall\ \vartheta, \xi \in H_s$ *and* $\kappa_s \in [0,1)$*. Then* T_s *possesses a fixed point in* H_s *that is unique If* $\kappa_s < \frac{1}{s^4}$ *and* $0 \leq \varsigma_{n-1} < \frac{\frac{1}{s^4}-\kappa_s}{\frac{1}{s^2}-\kappa_s}$*, where* $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$*,* $0 \leq \varsigma_{n-1} < 1$.

Direct implication of Corollary 3.4, which is also Corollary 1 of [7], is as under:

Corollary 3.5. [7] *Assume* $(H_s, b_{s_{12}}, \varpi)$ *is a complete* $b-$*metric space with* $s > 1$ *and* $T_s : H_s \to H_s$ *be defined as*

$$
b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \kappa_s \left[b_{s_{12}}(\vartheta, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)\right], \qquad (7)
$$

$\forall\ \vartheta, \xi \in H_s$ *and* $\kappa_s \in \left[0, \frac{1}{2}\right)$*. Then* T_s *possesses a fixed point in* H_s *that is unique if* $\kappa_s < \frac{1}{2s^4}$ *and* $0 \leq \varsigma_{n-1} < \frac{\frac{1}{s^4}-2\kappa_s}{\frac{1}{s^2}-2\kappa_s}$ *for each* $n \in \mathbb{N}$*, where* $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$*,* $0 \leq \varsigma_{n-1} < 1$.

4. Main Result for Almost Contraction

Theorem 4.1. *Assume* $(H_s, b_{s_{12}}, \varpi)$ *is a complete* s_1, s_2 $b-$*metric spaces with constants* $s_1, s_2 > 1$ *and* $T_s : H_s \to H_s$ *be condition(B) defined as*

$$
\begin{aligned}
b_{s_{12}}(T_s\vartheta, T_s\xi) \quad \leq \quad & \kappa_s b_{s_{12}}(\vartheta,\xi) + L\min\{b_{s_{12}}(\vartheta, T_s\vartheta), b_{s_{12}}(\xi, T_s\xi), b_{s_{12}}(\vartheta, T_s\xi), \\
& b_{s_{12}}(\xi, T_s\vartheta)\}, \qquad (8)
\end{aligned}
$$

$\forall\ \vartheta, \xi \in H_s$ *and* $\kappa_s \in [0,1)$*. Then* T_s *possesses a fixed point in* H_s *that is unique if* $\kappa_s < \min\left\{\frac{1}{s_1^3}, \frac{1}{s_2^3}\right\}$ *and* $0 \leq \varsigma_{n-1} < \min\left\{\frac{\frac{1}{s_2^3}-\kappa_s}{\frac{s_1}{s_2}-\kappa_s+L}, \frac{\frac{1}{s_1^3}-\kappa_s}{1-\kappa_s+L}\right\}$ *for each* $n \in \mathbb{N}$*, where* $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$*,* $0 \leq \varsigma_{n-1} < 1$.

Proof. For any $n \in \mathbb{N}$

$$
b_{s_{12}}(\vartheta_n, \vartheta_{n+1}) = b_{s_{12}}(\vartheta_n, \varpi(\vartheta_n, T_s\vartheta_n; \varsigma_n)) \leq (1-\varsigma_n) b_{s_{12}}(\vartheta_n, T_s\vartheta_n)
$$

and

$$
b_{s_{12}}(\vartheta_n, T_s\vartheta_n) \quad \leq \quad s_1 b_{s_{12}}(\vartheta_n, T_s\vartheta_{n-1}) + s_2 b_{s_{12}}(T_s\vartheta_{n-1}, T_s\vartheta_n)
$$

$$
\begin{aligned}
&\leq s_1 b_{s_{12}}(\varpi(\vartheta_{n-1},T_s\vartheta_{n-1};\varsigma_{n-1}),T_s\vartheta_{n-1}) + s_2\kappa_s b_{s_{12}}(\vartheta_{n-1},\vartheta_n)\\
&\quad +s_2 L\min\{ b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}), b_{s_{12}}(\vartheta_n,T_s\vartheta_n), b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_n),\\
&\quad b_{s_{12}}(\vartheta_n,T_s\vartheta_{n-1})\}\\
&\leq s_1\varsigma_{n-1} b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}) + s_2\kappa_s(1-\varsigma_{n-1}) b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&\quad +s_2 L\min\{ b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}), s_1 b_{s_{12}}(\vartheta_n,\vartheta_{n-1}) + s_2 b_{s_{12}}\\
&\quad (\vartheta_{n-1},T_s\vartheta_n), b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_n), b_{s_{12}}(\varpi(\vartheta_{n-1},T_s\vartheta_{n-1};\varsigma_{n-1}),\\
&\quad T_s\vartheta_{n-1})\}\\
&\leq s_1\varsigma_{n-1} b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}) + s_2\kappa_s(1-\varsigma_{n-1}) b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&\quad +s_2 L\min\{ b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}), b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_n), \varsigma_{n-1}\\
&\quad b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\}\\
&\leq s_1\varsigma_{n-1} b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}) + s_2\kappa_s(1-\varsigma_{n-1}) b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&\quad +s_2 L\min\{ s_1 b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}) + s_2 b_{s_{12}}(T_s\vartheta_{n-1},T_s\vartheta_n), \varsigma_{n-1}\\
&\quad b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\}\\
&= s_1\varsigma_{n-1} b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}) + s_2\kappa_s(1-\varsigma_{n-1}) b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&\quad +s_2 L\varsigma_{n-1} b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&= [\varsigma_{n-1}(s_1 - s_2\kappa_s + s_2 L) + s_2\kappa_s] b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1})\\
&< \frac{1}{s_2^2} b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}),
\end{aligned}
$$

with inequalities $\kappa_s < \min\left\{\frac{1}{s_1^3},\frac{1}{s_2^3}\right\}$ and $0 \leq \varsigma_{n-1} < \min\left\{\frac{\frac{1}{s_2^3}-\kappa_s}{\frac{s_1}{s_2}-\kappa_s+L},\frac{\frac{1}{s_1^3}-\kappa_s}{\frac{1}{s_1}-\kappa_s+L}\right\}, \quad n\in\mathbb{N}$

Thus,

$$
b_{s_{12}}(\vartheta_n,T_s\vartheta_n) < \frac{1}{s_2^2} b_{s_{12}}(\vartheta_{n-1},T_s\vartheta_{n-1}) \tag{9}
$$

$\implies \{b_{s_{12}}(\vartheta_n,T_s\vartheta_n)\}$ is a decreasing non-negative real numbers sequence. Therefore, $\exists\ \widehat{\zeta} \geq 0$ with

$$
\lim_{n\to\infty} b_{s_{12}}(\vartheta_n,T_s\vartheta_n) = \widehat{\zeta}
$$

We claim that $\widehat{\zeta}=0$. Assume that $\widehat{\zeta}>0$. Taking $n\to\infty$ in (9),

$$\widehat{\zeta}<\frac{1}{s_2^2}\widehat{\zeta}$$

which is a contradiction. Hence $\widehat{\zeta}=0$, that is,

$$\lim_{n\to\infty} b_{s_{12}}(\vartheta_n, T_s\vartheta_n)=0$$

Here, we claim that $\{\vartheta_n\}$ is a Cauchy sequence.

Suppose that $\{\vartheta_n\}$ cannot be a Cauchy sequence,implying $\exists\ \varepsilon>0$ and $\{\vartheta_{m_\iota}\}$ and $\{\vartheta_{n_\iota}\}$, subsequences of $\{\vartheta_n\}$, m_ι being the least natural cardinal with $m_\iota>n_\iota>\iota$ satisfying

$$b_{s_{12}}(\vartheta_{m_\iota},\vartheta_{n_\iota})\geq\varepsilon$$

and

$$b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota})<\varepsilon$$

Then, we conclude that

$$\varepsilon\leq b_{s_{12}}(\vartheta_{m_\iota},\vartheta_{n_\iota})\leq s_1 b_{s_{12}}(\vartheta_{m_\iota},\vartheta_{n_\iota+1})+s_2 b_{s_{12}}(\vartheta_{n_\iota+1},\vartheta_{n_\iota}),$$

which implies that

$$\frac{\varepsilon}{s_1}\leq\limsup_{\kappa_s\to\infty} b_{s_{12}}(\vartheta_{m_\iota},\vartheta_{n_\iota+1})$$

Noticing that

$$\begin{aligned}
b_{s_{12}}(\vartheta_{m_\iota},\vartheta_{n_\iota+1}) &= b_{s_{12}}(\varpi(\vartheta_{m_\iota-1},T_s\vartheta_{m_\iota-1};\varsigma_{m_\iota-1}),\vartheta_{n_\iota+1})\\
&\leq \varsigma_{m_\iota-1}b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota+1})+(1-\varsigma_{m_\iota-1})b_{s_{12}}\\
&\quad (T_s\vartheta_{m_\iota-1},\vartheta_{n_\iota+1})\\
&\leq \varsigma_{m_\iota-1}b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota+1})+(1-\varsigma_{m_\iota-1})[s_1 b_{s_{12}}\\
&\quad (T_s\vartheta_{m_\iota-1},T_s\vartheta_{n_\iota+1})+s_2 b_{s_{12}}(T_s\vartheta_{n_\iota+1},\vartheta_{n_\iota+1})]\\
&\leq \varsigma_{m_\iota-1}b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota+1})+(1-\varsigma_{m_\iota-1})s_2 b_{s_{12}}\\
&\quad (T_s\vartheta_{n_\iota+1},\vartheta_{n_\iota+1})+(1-\varsigma_{m_\iota-1})s_1[\kappa_s b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota+1})\\
&\quad +L\min\{b_{s_{12}}(\vartheta_{m_\iota-1},T_s\vartheta_{m_\iota-1}),b_{s_{12}}(\vartheta_{n_\iota+1},T_s\vartheta_{n_\iota+1}),b_{s_{12}}
\end{aligned}$$

$$
\begin{aligned}
&(\vartheta_{m_\iota-1},T_s\vartheta_{n_\iota+1}),b_{s_{12}}(\vartheta_{n_\iota+1},T_s\vartheta_{m_\iota-1})\}] \\
\leq\ & s_1\varsigma_{m_\iota-1}b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota})+s_2\varsigma_{m_\iota-1}b_{s_{12}}(\vartheta_{n_\iota},\vartheta_{n_\iota+1}) \\
&+(1-\varsigma_{m_\iota-1})s_2b_{s_{12}}(T_s\vartheta_{n_\iota+1},\vartheta_{n_\iota+1})+(1-\varsigma_{m_\iota-1})s_1 \\
&[\kappa_s\{s_1b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota})+s_2b_{s_{12}}(\vartheta_{n_\iota},\vartheta_{n_\iota+1})\} \\
&+L\min\{b_{s_{12}}(\vartheta_{m_\iota-1},T_s\vartheta_{m_\iota-1}),b_{s_{12}}(\vartheta_{n_\iota+1},T_s\vartheta_{n_\iota+1}), \\
&s_1b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota+1})+s_2b_{s_{12}}(\vartheta_{n_\iota+1}, \\
&T_s\vartheta_{n_\iota+1}),s_1b_{s_{12}}(\vartheta_{n_\iota+1},\vartheta_{m_\iota-1})+s_2b_{s_{12}}(\vartheta_{m_\iota-1},T_s\vartheta_{m_\iota-1})\}] \\
\leq\ & s_1\varsigma_{m_\iota-1}b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota})+s_2\varsigma_{m_\iota-1}b_{s_{12}}(\vartheta_{n_\iota},\vartheta_{n_\iota+1}) \\
&+(1-\varsigma_{m_\iota-1})s_2b_{s_{12}}(T_s\vartheta_{n_\iota+1},\vartheta_{n_\iota+1})+(1-\varsigma_{m_\iota-1})s_1 \\
&[\kappa_s\{s_1b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota})+s_2b_{s_{12}}(\vartheta_{n_\iota},\vartheta_{n_\iota+1})\} \\
&+L\min\{b_{s_{12}}(\vartheta_{m_\iota-1},T_s\vartheta_{m_\iota-1}),b_{s_{12}}(\vartheta_{n_\iota+1},T_s\vartheta_{n_\iota+1}), \\
&s_1^2b_{s_{12}}(\vartheta_{m_\iota-1},\vartheta_{n_\iota})s_1s_2b_{s_{12}}(\vartheta_{n_\iota},\vartheta_{n_\iota+1}) \\
&+s_2b_{s_{12}}(\vartheta_{n_\iota+1},T_s\vartheta_{n_\iota+1}),s_1^2b_{s_{12}}(\vartheta_{n_\iota+1},\vartheta_{n_\iota}) \\
&+s_1s_2b_{s_{12}}(\vartheta_{n_\iota},\vartheta_{m_\iota-1})+s_2b_{s_{12}}(\vartheta_{m_\iota-1},T_s\vartheta_{m_\iota-1})\} \\
<\ & s_1\varsigma_{m_\iota-1}\varepsilon+s_1(1-\varsigma_{m_\iota-1})[\kappa_ss_1\varepsilon+L\min\{0,0,s_1^2\varepsilon,s_1s_2\varepsilon\}] \\
=\ & s_1\varepsilon\left(\varsigma_{m_\iota-1}(1-s_1\kappa_s)+s_1\kappa_s\right) \\
<\ & s_1\varepsilon\left(\frac{\frac{1}{s_1^3}-\kappa_s}{1-\kappa_s+L}(1-s_1\kappa_s)+s_1\kappa_s\right) \\
=\ & s_1\varepsilon\left(\frac{\frac{1}{s_1^2}-s_1\kappa_s}{s_1-s_1\kappa_s+s_1L}(1-s_1\kappa_s)+s_1\kappa_s\right) \\
\leq\ & s_1\varepsilon\left(\frac{\frac{1}{s_1^2}-s_1\kappa_s}{1-s_1\kappa_s}(1-s_1\kappa_s)+s_1\kappa_s\right)=\frac{\varepsilon}{s_1}.
\end{aligned}
$$

Thus, we obtain

$$\frac{\varepsilon}{s_1}\leq\limsup_{\kappa_s\to\infty}b_{s_{12}}(\vartheta_{m_\iota},\vartheta_{n_\iota+1})<\frac{\varepsilon}{s_1}$$

which is a contradiction.

Thus, $\{\vartheta_n\}$ being a Cauchy sequence in H_s. and owing to completeness of H_s, $\exists\ \vartheta^* \in H$ such that $\lim_{n\to\infty} b_{s_{12}}(\vartheta_n, \vartheta^*) = 0$.

Now we verify that ϑ^* is a fixed point of T_s. For this,

$$\begin{aligned}
b_{s_{12}}(\vartheta^*, T_s\vartheta^*) &\leq s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta^*) \\
&\leq s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_1 s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n) + s_2^2 b_{s_{12}}(T_s\vartheta_n, T_s\vartheta^*) \\
&\leq s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_1 s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n) + s_2^2[\kappa_s b_{s_{12}}(\vartheta_n, \vartheta^*) \\
&\quad + L\min\{b_{s_{12}}(\vartheta_n, T_s\vartheta_n), b_{s_{12}}(\vartheta^*, T_s\vartheta^*), b_{s_{12}}(\vartheta_n, T_s\vartheta^*), \\
&\quad b_{s_{12}}(\vartheta^*, T_s\vartheta_n)\}] \\
&\leq s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_1 s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n) + s_2^2[\kappa_s b_{s_{12}}(\vartheta_n, \vartheta^*) \\
&\quad + L\min\{b_{s_{12}}(\vartheta_n, T_s\vartheta_n), b_{s_{12}}(\vartheta^*, T_s\vartheta^*), s_1 b_{s_{12}}(\vartheta_n, \vartheta^*) \\
&\quad + s_2 b_{s_{12}}(\vartheta^*, T_s\vartheta^*), s_1 b_{s_{12}}(\vartheta^*, \vartheta_n) + s_2 b_{s_{12}}(\vartheta_n, T_s\vartheta_n)\}].
\end{aligned}$$

Letting $n \to \infty$, we get

$$b_{s_{12}}(\vartheta^*, T_s\vartheta^*) \leq s_2^2 \min\{0, b_{s_{12}}(\vartheta^*, T_s\vartheta^*), s_2 b_{s_{12}}(\vartheta^*, T_s\vartheta^*)\} = 0.$$

So, $b_{s_{12}}(\vartheta^*, T_s\vartheta^*) = 0 \implies T_s\vartheta^* = \vartheta^*$.

Hence, ϑ^* is a fixed point of T_s.

To prove that this fixed point so obtained is unique, consider $q \in H_s$ such that $T_s q = q$, then

$$\begin{aligned}
0 < b_{s_{12}}(\vartheta^*, q) &= b_{s_{12}}(T_s\vartheta^*, Tq) \\
&\leq \kappa_s b_{s_{12}}(\vartheta^*, q) + L\min\{b_{s_{12}}(\vartheta^*, T_s\vartheta^*), b_{s_{12}}(q, Tq), b_{s_{12}}(\vartheta^*, Tq), \\
&\quad b_{s_{12}}(q, T_s\vartheta^*)\} \\
&\leq \kappa_s b_{s_{12}}(\vartheta^*, q) + L\min\{0, 0, s_1 b_{s_{12}}(\vartheta^*, q) + s_2 b_{s_{12}}(q, Tq), \\
&\quad s_1 b_{s_{12}}(q, \vartheta^*) + s_2 b_{s_{12}}(\vartheta^*, T_s\vartheta^*)\} \\
&= \kappa_s b_{s_{12}}(\vartheta^*, q) \\
&< \frac{1}{s_1^3} b_{s_{12}}(\vartheta^*, q) < b_{s_{12}}(\vartheta^*, q),
\end{aligned}$$

that is a contradictory statement. Hence, $\vartheta^* = q$.

□

If we take $L=0$ in Theorem 4.1, then we obtain the following result.

Corollary 4.2. *Assume $(H_s, b_{s_{12}}, \varpi)$ is a complete s_1, s_2 $b-$metric spaces with constants $s_1, s_2 > 1$ and $T_s : H_s \to H_s$ be defined as*

$$b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \kappa_s b_{s_{12}}(\vartheta, \xi), \tag{10}$$

$\forall\ \vartheta, \xi \in H_s$ and $\kappa_s \in \left[0, \frac{1}{2}\right)$. Then T_s possesses a fixed point in H_s that is unique if $\kappa_s < \min\left\{\frac{1}{s_1^3}, \frac{1}{s_2^3}\right\}$ and $0 \leq \varsigma_{n-1} < \min\left\{\frac{\frac{1}{s_2^3}-\kappa_s}{\frac{s_1}{s_2}-\kappa_s}, \frac{\frac{1}{s_1^3}-\kappa_s}{1-\kappa_s}\right\}$ for each $n \in \mathbb{N}$, where $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$, $0 \leq \varsigma_{n-1} < 1$.

If $s_1 < s_2$, then we have a version of Theorem 1 of [10].

Corollary 4.3. [10] *Assume $(H_s, b_{s_{12}}, \varpi)$ is a complete s_1, s_2 $b-$metric spaces with constants $s_1, s_2 > 1$ and $T_s : H_s \to H_s$ be defined as*

$$b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \kappa_s b_{s_{12}}(\vartheta, \xi), \tag{11}$$

$\forall\ \vartheta, \xi \in H_s$ and $\kappa_s \in \left[0, \frac{1}{2}\right)$. Then T_s possesses a fixed point in H_s that is unique if $\kappa_s < \frac{1}{s_2^3}$ and $0 \leq \varsigma_{n-1} < \min\left\{\frac{\frac{1}{s_2^3}-\kappa_s}{\frac{s_1}{s_2}-\kappa_s}, \frac{\frac{1}{s_1^3}-\kappa_s}{1-\kappa_s}\right\}$ for each $n \in \mathbb{N}$, where $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$, $0 \leq \varsigma_{n-1} < 1$.

For $s_1 = s_2 = s$, we arrive at Theorem 3 of [7].

Corollary 4.4. [7] *Assume $(H_s, b_{s_{12}}, \varpi)$ is a complete $b-$metric spaces with $s > 1$ and $T_s : H_s \to H_s$ be defined as*

$$\begin{aligned} b_{s_{12}}(T_s\vartheta, T_s\xi) \leq\ & \kappa_s b_{s_{12}}(\vartheta, \xi) + L\min\{b_{s_{12}}(\vartheta, T_s\vartheta), b_{s_{12}}(\xi, T_s\xi), \\ & b_{s_{12}}(\vartheta, T_s\xi), b_{s_{12}}(\xi, T_s\vartheta)\}, \end{aligned} \tag{12}$$

$\forall\ \vartheta, \xi \in H_s$ and $\kappa_s \in \left[0, \frac{1}{2}\right)$. Then T_s possesses a fixed point in H_s that is unique if $\kappa_s < \frac{1}{s^3}$ and $0 \leq \varsigma_{n-1} < \frac{\frac{1}{s^3}-\kappa_s}{1-\kappa_s+L}$ for each $n \in \mathbb{N}$, where $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$, $0 \leq \varsigma_{n-1} < 1$.

Proposition 4.5. *Let $(H_s, b_{s_{12}})$ be an s_1, s_2 b-metric spaces. Then any map $T_s : H_s \to H_s$ that satisfies Chatterjea contraction also satisfies condition(B) if $\kappa_s < \frac{1}{s_1+s_2^2}$.*

Proof. Chatterjea contractive condition and property of $s_1 s_2$ b-metric implies that

$$\begin{aligned}
b_{s_{12}}(T_s\vartheta, T_s\xi) &\leq \kappa_s\left[b_{s_{12}}(\vartheta, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)\right] \\
&\leq \kappa_s\left[s_1 b_{s_{12}}(\vartheta, \xi) + s_2 b_{s_{12}}(\xi, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)\right] \\
&\leq \kappa_s[s_1 b_{s_{12}}(\vartheta, \xi) + s_1 s_2 b_{s_{12}}(\xi, T_s\vartheta) + s_2^2 b_{s_{12}}(T_s\vartheta, T_s\xi) \\
&\quad + b_{s_{12}}(\xi, T_s\vartheta)],
\end{aligned}$$

which follows

$$b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \frac{\kappa_s s_1}{1-\kappa_s s_2^2} b_{s_{12}}(\vartheta, \xi) + \frac{\kappa_s(s_1 s_2 + 1)}{1-\kappa_s s_2^2} b_{s_{12}}(\xi, T_s\vartheta). \tag{13}$$

In the similar fashion,

$$\begin{aligned}
b_{s_{12}}(T_s\vartheta, T_s\xi) &\leq \kappa_s\left[b_{s_{12}}(\vartheta, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)\right] \\
&\leq \kappa_s\left[b_{s_{12}}(\vartheta, T_s\xi) + s_1 b_{s_{12}}(\xi, \vartheta) + s_2 b_{s_{12}}(\vartheta, T_s\vartheta)\right] \\
&\leq \kappa_s[b_{s_{12}}(\vartheta, T_s\xi) + s_1 b_{s_{12}}(\xi, \vartheta) + s_1 s_2 b_{s_{12}}(\vartheta, T_s\xi) \\
&\quad + s_2^2 b_{s_{12}}(T_s\xi, T_s\vartheta)],
\end{aligned}$$

which provides

$$b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \frac{\kappa_s s_1}{1-\kappa_s s_2^2} b_{s_{12}}(\vartheta, \xi) + \frac{\kappa_s(s_1 s_2 + 1)}{1-\kappa_s s_2^2} b_{s_{12}}(\vartheta, T_s\xi). \tag{14}$$

Similarly, the inequality follows

$$\begin{aligned}
b_{s_{12}}(T_s\vartheta, T_s\xi) &\leq \kappa_s\left[b_{s_{12}}(\vartheta, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)\right] \\
&\leq \kappa_s[s_1 b_{s_{12}}(\vartheta, \xi) + s_2 b_{s_{12}}(\xi, T_s\xi) + s_1 b_{s_{12}}(\xi, T_s\xi) \\
&\quad + s_2 b_{s_{12}}(T_s\xi, T_s\vartheta)],
\end{aligned}$$

that yields

$$b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \frac{\kappa_s s_1}{1-\kappa_s s_2} b_{s_{12}}(\vartheta, \xi) + \frac{\kappa_s(s_1 + s_2)}{1-\kappa_s s_2} b_{s_{12}}(\xi, T_s\xi)$$

$$\leq \frac{\kappa_s s_1}{1-\kappa_s s_2^2} b_{s_{12}}(\vartheta,\xi) + \frac{\kappa_s(s_1 s_2+1)}{1-\kappa_s s_2^2} b_{s_{12}}(\xi, T_s\xi). \tag{15}$$

Similar argument reveals

$$\begin{aligned} b_{s_{12}}(T_s\vartheta, T_s\xi) &\leq \kappa_s\left[b_{s_{12}}(\vartheta, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)\right] \\ &\leq \kappa_s[s_1 b_{s_{12}}(\vartheta, T_s\vartheta) + s_2 b_{s_{12}}(T_s\vartheta, T_s\xi) + s_1 b_{s_{12}}(\xi,\vartheta) \\ &\quad + s_2 b_{s_{12}}(\vartheta, T_s\vartheta)], \end{aligned}$$

which results into

$$\begin{aligned} b_{s_{12}}(T_s\vartheta, T_s\xi) &\leq \frac{\kappa_s s_1}{1-\kappa_s s_2} b_{s_{12}}(\vartheta,\xi) + \frac{\kappa_s(s_1+s_2)}{1-\kappa_s s_2} b_{s_{12}}(\vartheta, T_s\vartheta) \\ &\leq \frac{\kappa_s s_1}{1-\kappa_s s_2^2} b_{s_{12}}(\vartheta,\xi) + \frac{\kappa_s(s_1 s_2+1)}{1-\kappa_s s_2^2} b_{s_{12}}(\vartheta, T_s\vartheta). \qquad (16) \end{aligned}$$

Now, by using equations (13)-(16), we have

$$\begin{aligned} b_{s_{12}}(T_s\vartheta, T_s\xi) &\leq \frac{\kappa_s s_1}{1-\kappa_s s_2^2} b_{s_{12}}(\vartheta,\xi) + \frac{\kappa_s(s_1 s_2+1)}{1-\kappa_s s_2^2} \min\{b_{s_{12}}(\vartheta, T_s\vartheta), \\ &\quad b_{s_{12}}(\xi, T_s\xi), b_{s_{12}}(\vartheta, T_s\xi), b_{s_{12}}(\xi, T_s\vartheta)\} \\ &\leq p b_{s_{12}}(\vartheta,\xi) + L\min\{b_{s_{12}}(\vartheta, T_s\vartheta), \\ &\quad b_{s_{12}}(\xi, T_s\xi), b_{s_{12}}(\vartheta, T_s\xi), b_{s_{12}}(\xi, T_s\vartheta)\}, \end{aligned}$$

where $p = \frac{\kappa_s s_1}{1-\kappa_s s_2^2} < 1$ (as $\kappa_s < \frac{1}{s_1+s_2^2}$) and $L = \frac{\kappa_s(s_1 s_2+1)}{1-\kappa_s s_2^2} \geq 0$.Therefore, T_s satisfies condition (B) □

Proposition 4.6. *If* $s_1, s_2 > 1$ *and* $\kappa_s \in [0, 1/2)$ *such that* $\varsigma < \min\left\{\frac{\frac{1}{s_2^3 s_1} - \frac{\kappa_s}{s_1 s_2} - \kappa_s}{\frac{s_1+\kappa_s s_2}{s_1 s_2} - \kappa_s}, \frac{\frac{1}{s_1^4} - \frac{\kappa_s s_2^2}{s_1^4} - \kappa_s}{\frac{1+\kappa_s}{s_1} - \kappa_s + \kappa_s \frac{s_2}{s_1}(s_1 - s_2)}\right\}$, *then* $\varsigma < \min\left\{\frac{\frac{1}{s_2^3} - p}{\frac{s_1}{s_2} - p + L}, \frac{\frac{1}{s_1^3} - p}{1-p+L}\right\}$, *where* $p = \frac{\kappa_s s_1}{1-\kappa_s s_2^2}$ *and* $L = \frac{\kappa_s(s_1 s_2+1)}{1-\kappa_s s_2^2}$.

Proof. Observe that

$$\begin{aligned} \varsigma &< \min\left\{\frac{\frac{1}{s_2^3 s_1} - \frac{\kappa_s}{s_1 s_2} - \kappa_s}{\frac{s_1+\kappa_s s_2}{s_1 s_2} - \kappa_s}, \frac{\frac{1}{s_1^4} - \frac{\kappa_s s_2^2}{s_1^4} - \kappa_s}{\frac{1+\kappa_s}{s_1} - \kappa_s + \kappa_s \frac{s_2}{s_1}(s_1 - s_2)}\right\} \\ &= \min\{\frac{\frac{1}{s_2^3} - \frac{\kappa_s}{s_2} - \kappa_s s_1}{\frac{1}{s_2}(s_1 + \kappa_s s_2 - \kappa_s s_1 s_2 + \kappa_s s_1 s_2^2 - \kappa_s s_1 s_2^2)}, \end{aligned}$$

$$\left.\frac{\frac{1}{s_1^3} - \frac{\kappa_s s_2^2}{s_1^3} - \kappa_s s_1}{\frac{1}{s_1}(1 + \kappa_s - \kappa_s s_1 + \kappa_s s_2(s_1 - s_2))}\right\}$$

$$= \min\left\{ \frac{\frac{(1-\kappa_s s_2^2)-\kappa_s s_1 s_2^3}{s_2^3(1-\kappa_s s_2^2)}}{\frac{s_1+\kappa_s s_2-\kappa_s s_1 s_2+\kappa_s s_1 s_2^2-\kappa_s s_1 s_2^2}{s_2(1-\kappa_s s_2^2)}}, \frac{\frac{(1-\kappa_s s_2^2)-\kappa_s s_1^4}{s_1^3(1-\kappa_s s_2^2)}}{\frac{1-\kappa_s s_2^2-\kappa_s s_1+\kappa_s(s_1 s_2+1)}{s_1(1-\kappa_s s_2^2)}} \right\},$$

that yields

$$\varsigma < \min\left\{ \frac{\frac{1}{s_2^3} - \frac{\kappa_s s_1}{1-\kappa_s s_2^2}}{\frac{s_1}{s_2} - \frac{\kappa_s s_1}{1-\kappa_s s_2^2} + \frac{\kappa_s(s_1 s_2+1)}{1-\kappa_s s_2^2}}, \frac{\frac{1}{s_1^3} - \frac{\kappa_s s_1}{1-\kappa_s s_2^2}}{1 - \frac{\kappa_s s_1}{1-\kappa_s s_2^2} + \frac{\kappa_s(s_1 s_2+1)}{1-\kappa_s s_2^2}} \right\}$$

$$\Rightarrow \varsigma < \min\left\{ \frac{\frac{1}{s_2^3} - p}{\frac{s_1}{s_2} - p + L}, \frac{\frac{1}{s_1^3} - p}{1 - p + L} \right\}.$$

□

Thus, following result is implied for Chatterjea contraction.

Corollary 4.7. *Assume $(H_s, b_{s_{12}}, \varpi)$ is a complete s_1, s_2 $b-$metric spaces with constants $s_1, s_2 >$ 1 and $T_s : H_s \to H_s$ be defined as*

$$b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \kappa_s \left[b_{s_{12}}(\vartheta, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)\right], \tag{17}$$

$\forall$ $\vartheta, \xi \in H_s$ and $\kappa_s \in \left[0, \frac{1}{2}\right)$. Then T_s possesses a fixed point in H_s that is unique if $\kappa_s <$ $\min\left\{\frac{1}{s_2^2(s_1 s_2+1)}, \frac{1}{s_1^4+s_2^2}\right\}$ and
$$0 \leq \varsigma_{n-1} < \min\left\{ \frac{\frac{1}{s_2^3 s_1} - \frac{\kappa_s}{s_1 s_2} - \kappa_s}{\frac{s_1+\kappa_s s_2}{s_1 s_2} - \kappa_s}, \frac{\frac{1}{s_1^4} - \frac{\kappa_s s_2^2}{s_1^4} - \kappa_s}{\frac{1+\kappa_s}{s_1} - \kappa_s + \kappa_s \frac{s_2}{s_1}(s_1 - s_2)} \right\},$$
where $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$, $0 \leq \varsigma_{n-1} < 1$.

If $s_1 = s_2 = s$ in Corollary 4.7, we obtain Corollary 3 of [7]

Corollary 4.8. [7] *Assume $(H_s, b_{s_{12}}, \varpi)$ is a complete $b-$metric spaces with constant $s > 1$ and $T_s : H_s \to H_s$ be defined as*

$$b_{s_{12}}(T_s\vartheta, T_s\xi) \leq \kappa_s \left[b_{s_{12}}(\vartheta, T_s\xi) + b_{s_{12}}(\xi, T_s\vartheta)\right], \tag{18}$$

$\forall\ \vartheta, \xi \in H_s$ *and* $\kappa_s \in \left[0, \frac{1}{2}\right)$. *Then* T_s *possesses a fixed point in* H_s *that is unique if* $\kappa_s < \frac{1}{s^2(s^2+1)}$ *and* $0 \leq \varsigma_{n-1} < \frac{\frac{1}{s^4} - \frac{\kappa_s}{s^2} - \kappa_s}{\frac{1+\kappa_s}{s} - \kappa_s}$,

where $\vartheta_n = \varpi(\vartheta_{n-1}, T_s\vartheta_{n-1}; \varsigma_{n-1})$, $0 \leq \varsigma_{n-1} < 1$.

5. CONCLUSION

The work on Cirić contraction and almost contraction in convex generalised b-metric spaces was extended in this present paper. We demonstrated the existence of a fixed point and its uniqueness using Mann's iteration. As a specific case of our main result, we demonstrated the various developments in the existing literature.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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