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APPLICATION OF SELBERG TYPE INEQUALITIES IN HILBERT C*-MODULES

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Abstract. In this paper we prove some applications (inequality in [3], inequality in [9] and inequality in [12]) of Selberg and refinement type inequalities in Hilbert C*-modules.

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1. Introduction

The Selberg inequality in [8]. Let $y_1, \ldots, y_n$, be non-zero vectors in a Hilbert space X with inner product $\langle , \rangle$. Then, for all $x \in X$,

$$\sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{n} |\langle y_j, y_k\rangle|} \leq ||x||^2 \tag{1.1}$$

In [9]the Selberg inequality is refined as follows: If $\langle y, y_i\rangle = 0$ for given $\{y_i\}$, then

$$|\langle y, x\rangle|^2 + \sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{n} |\langle y_j, y_k\rangle|} \|y\|^2 \leq ||x||^2 \|y\|^2 \tag{1.2}$$

holds for all x.

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In [3] Dragomir obtained the following reverse of the triangle inequality by using an argument based on the Selberg inequality in Hilbert space.

Theorem 1.1. [Dragomir] *Let $(X, \langle , \rangle)$ be a complex inner product Hilbert space and $x_1, \ldots, x_n$, $y_1, \ldots, y_m$ be a non zero vectors in X such that there exist the nonnegative real numbers $\rho_j, \mu_j, j \in \{1, \ldots, m\}$ with*

$$Re \langle x_i, y_j \rangle \geq \rho_j \|x_i\| \|y_j\| \ , \ Im \langle x_i, y_j \rangle \geq \mu_j \|x_i\| \|y_j\| \tag{1.3}$$

for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then

$$(\sum_{j=1}^{m} \frac{(\rho_j^2 + \mu_j^2) \|y_j\|^2}{\sum_{k=1}^{m} |\langle y_j, y_k \rangle|})^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|. \tag{1.4}$$

In [9] M.Fujii and R.Nakamoto by refinement of seleberg inequality, give a simple Proof to an extension of Diaz-Metcalf inequality due to Fujii-Yamada.

Theorem 1.2. [Fujii,Nakamoto] *Let $(X, \langle , \rangle)$ be a complexe Hilbert space and $z_1, \cdots$, z_n be non zero vectors in X , $x_1, \cdots , x_m \in X$ such that there exist the nonnegative real numbers a_k , $k \in \{1, \ldots, n\}$ with*

$$0 \leq a_k \|x_i\| \leq Re \langle z_k, x_i \rangle \tag{1.5}$$

for all $i \in \{1, \cdots m\}$ and $k \in \{1, \cdots , n\}$. If $\langle y, z_i \rangle = 0$ for all i , then

$$| \langle x_1 + \cdots + x_m, y \rangle |^2 + (\sum_{k=1}^{n} \frac{a_k^2}{c_k})(\|x_1\| + \cdots + \|x_m\|)^2 ||y||^2 \leq \|x_1 + \cdots + x_m\|^2 ||y||^2 \tag{1.6}$$

where $c_k = \sum_{j=1}^{n} |\langle z_j, z_k \rangle|$.

In [12] C.-S. Lin obtain by refinement of seleberg inequality the following inequality.

Theorem 1.3. [Lin] *Let $x, y \in X$ and $\langle y, z_i \rangle = 0$ for given non-zero vectors $z_i \in X$ and $i = 1, 2, ..., n$. Then, for any $k \in \{1, 2, \ldots, n\}$,*

$$|\langle y, x \rangle|^2 + \|y\|^2 [\|z_k\|^{-2} \left| \langle z_k, x \rangle - \left\langle \sum_{i=1}^{n} \frac{\langle z_i, x \rangle}{\sum_{j=1}^{n} \|\langle z_i, z_j \rangle\|} z_i, z_k \right\rangle \right|^2 + \sum_{i=1}^{n} \frac{|\langle z_i, x \rangle|^2}{\sum_{j=1}^{n} |\langle z_i, z_j \rangle|}]$$
$$\leq \|x\|^2 \|y\|^2 .$$

And he obtain generalized and sharpened Cauchy-Schwarz inequality and Bessels inequality

Theorem 1.4. [Lin] *Let* $x, y \in X$ *, and* z_i *be unit vectors with* $\langle y, z_i \rangle = 0$*,* $z_i \in X$ *,* $i = 1, 2, \ldots, n$*. Then*

$$|\langle y, x \rangle|^2 + \|y\|^2 \left[|\langle z_k, u_n \rangle|^2 + \sum_{i=1}^{n} |\langle z_i, u_{i-1} \rangle|^2\right] \leq \|x\|^2 \|y\|^2$$

or

$$|\langle y, x \rangle| \leq |u_n - \langle z_k, u_n \rangle z_k| \|y\|$$

for $k \in \{1, 2, \ldots, n\}$ *, where* $u_0 = x$ *and* $u_i = u_{i-1} - \langle z_i, u_{i-1} \rangle z_i$ *,* $i = 1, 2, \ldots, n$*.*
In particular , if z_i *is unit orthogonal vectors , then*

$$|\langle y, x \rangle|^2 + \|y\|^2 \sum_{i=1}^{n} |\langle z_i, x \rangle|^2 \leq |x|^2 \|y\|^2$$

$$\text{or} \quad |\langle y, x \rangle| \leq |x - \langle z_k, x \rangle z_k| \|y\|$$

In [1] we give an extension of Selberg and refinement inequality in Hilbert C^*- module
.
The goal of this paper is to show some applications of Selberg and refinement inequality in Hilbert C^*-module via ([3] , [9], [12]) .

2. Preliminaries in Hilbert C^*-modules

In this section we briefly recall the definitions and examples of Hilbert C^*-modules. For information about Hilbert C^*-module, we refer to ([6,7,11]). Our reference for C^*-algebras is([2]).
Let A be a C^*-algebra (not necessarily unitary) and X be a complex linear space.

Definition 2.1. *A pre-Hilbert A-module is a right A-module X equipped with a sesquilinear map* $\langle ., . \rangle : X \times X \to A$ *satisfying*

(1) $\langle x, x \rangle \geq 0; \langle x, x \rangle = 0$ *if and only if* $x = 0$ *for all* x *in* X*,*
(2) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta(\langle x, z \rangle)$ *for all* x, y, z *in* X, α, β *in* $\mathbb{C}$*,*

(3) $\langle x, y\rangle = \langle y, x\rangle^*$ *for all* x, y *in* X,

(4) $\langle x, y.a\rangle = \langle x, y\rangle\, a$ *for all* x, y *in* X, a *in* A.

The map $\langle ., .\rangle$ is called an A-valued inner product of X, and for $x \in X$, we define $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$is a norm on X ,where the latter norm denotes that in the C^*-algebra A . This norm makes X into a right normed module over A . A pre-Hilbert module X is called a Hilbert A-module if it is complete with respect to its norm.

Two typical examples of Hilbert C^*-modules are as follows:

(I) Every Hilbert space is a Hilbert C^*-module.

(II) Every C^*algebra A is a Hilbert A -module via $\langle a, b\rangle = a^*b(a, b \in A)$.

Notice that the inner product structure of a C^*-algebra is essentially more complicated than complex numbers. One may define an A -valued norm $|.|$ by $|x| = \langle x, x\rangle^{\frac{1}{2}}$. Clearly, $\|x\| = \||x|\|$ for each $x \in X$. It is known that $|.|$ does not satisfy the triangle inequality in general.

3.MAIN RESULT

We start our work by presenting some applications of the Selbergs inequality for Hilbert C^*-modules.

Lemma 3.1. *Let* A *be a* C^**- alegebra,* $a \in A$ *and* $\lambda \in \mathbb{R}^+$. *If* $0 \leq a \leq \lambda$,*then*

$$a^2 \leq \lambda^2. \tag{3.1}$$

Theorem 3.2. *Let* X *be a Hilbert* A *module,* $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_m$ *be a non zero vectors in* X *such that there exist the nonnegative real numbers* $\rho_j, \mu_j, j \in \{1, \ldots, m\}$ *with*

$$Re\,\langle x_i, y_j\rangle \geq \rho_j\, \|x_i\|\, \|y_j\| \ \ , \ Im\,\langle x_i, y_j\rangle \geq \mu_j\, \|x_i\|\, \|y_j\| \tag{3.2}$$

and

$$\left|\left\langle y_j, \sum_{i=1}^{n} x_i \right\rangle\right|^2 \geq \left|\left\langle \sum_{i=1}^{n} x_i, y_j \right\rangle\right|^2 \tag{3.3}$$

for each $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, m\}$. *Then*

$$(\sum_{j=1}^{m} \frac{(\rho_j^2 + \mu_j^2) \left\|y_j\right\|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k \rangle\|})^{\frac{1}{2}} \sum_{i=1}^{n} |x_i| \leq \left| \sum_{i=1}^{n} x_i \right|. \tag{3.4}$$

Proof. Using Selberg inequality , we have

$$\sum_{j=1}^{m} \frac{|\langle y_j, \sum_{i=1}^{n} x_i \rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k \rangle\|} \leq \left| \sum_{i=1}^{n} x_i \right|^2. \tag{3.5}$$

Since

$$\frac{1}{2} \left| \left\langle \sum_{i=1}^{n} x_i, y_j \right\rangle \right|^2 + \frac{1}{2} \left| \left\langle y_j, \sum_{i=1}^{n} x_i \right\rangle \right|^2 = (\sum_{i=1}^{n} Re \langle y_j, x_i \rangle)^2 + (\sum_{i=1}^{n} Im \langle y_j, x_i \rangle)^2.$$

Then by (3.2),(3.3) and Lemma (3.1) we obtain

$$\begin{aligned} \left| \left\langle y_j, \sum_{i=1}^{n} x_i \right\rangle \right|^2 &\geq \rho_j^2 \left\|y_j\right\|^2 (\sum_{i=1}^{n} \|x_i\|)^2 + \mu_j^2 \left\|y_j\right\|^2 (\sum_{i=1}^{n} \|x_i\|)^2 \\ &= (\rho_j^2 + \mu_j^2) \left\|y_j\right\|^2 (\sum_{i=1}^{n} \|x_i\|)^2. \end{aligned} \tag{3.6}$$

For any $j \in \{1, \ldots, m\}$. Therefore by 3.5 we get

$$(\sum_{j=1}^{m} \frac{(\rho_j^2 + \mu_j^2) \left\|y_j\right\|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k \rangle\|})(\sum_{i=1}^{n} \|x_i\|)^2 \leq \left| \sum_{i=1}^{n} x_i \right|^2$$

and

$$(\sum_{j=1}^{m} \frac{(\rho_j^2 + \mu_j^2) \left\|y_j\right\|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k \rangle\|})(\sum_{i=1}^{n} |x_i|)^2 \leq \left| \sum_{i=1}^{n} x_i \right|^2.$$

We pass to root square and the result follows.

Remark 3.3. If only the first condition of 3.2 and the condition 3.3 is available , then

$$(\sum_{j=1}^{m} \frac{\rho_j^2 \left\|y_j\right\|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k \rangle\|})^{\frac{1}{2}} \sum_{i=1}^{n} |x_i| \leq \left| \sum_{i=1}^{n} x_i \right|. \tag{3.7}$$

Remark 3.4. If in Theorem (3.2) $y_1, \ldots, y_m$ be a sequence of unit vectors , then

$$(\sum_{j=1}^{m} (\rho_j^2 + \mu_j^2))^{\frac{1}{2}} \sum_{i=1}^{n} |x_i| \leq \left| \sum_{i=1}^{n} x_i \right|. \tag{3.8}$$

This inequalitie is a type of Diaz-Metcalf inequality in C^*-module.

Remark 3.5. If we have $\langle y_j, \sum_{i=1}^{n} x_i\rangle$ in commutator $Z(A)$ for all $j = 1, \ldots, m$ the condition (3.3) is verified .

Theorem 3.6. *Let X be a Hilbert A - module , let $z_1, \cdots, z_n$ be non zero vectors in X and $x_1, \cdots, x_m \in X$ such that*

$$0 \leq a_k \|x_i\| \leq Re\langle z_k, x_i\rangle\,,\ 0 \leq b_k \|x_i\| \leq Im\langle z_k, x_i\rangle \tag{3.9}$$

and

$$\left|\left\langle z_k, \sum_{i=1}^{m} x_i\right\rangle\right|^2 \geq \left|\left\langle \sum_{i=1}^{m} x_i, z_k\right\rangle\right|^2 \tag{3.10}$$

for all $i \in 1, \cdots m, k \in 1, \cdots, n$. If $\langle y, z_i\rangle = 0$ for all i , then

$$|\langle x_1 + \cdots + x_m, y\rangle|^2 + (\sum_{k=1}^{n} \frac{a_k^2 + b_k^2}{c_k})(|x_1| + \cdots + |x_m|)^2 ||y||^2 \leq |x_1 + \cdots + x_m|^2 ||y||^2. \tag{3.11}$$

where $c_k = \sum_{j=1}^{n} \|\langle z_j, z_k\rangle\|$.

Proof. Put $x = x_1 + \cdots + x_n$. Then by using (3.9) , (3.10) and the refinement of Selberg inequality we have

$$\begin{aligned}
&\|y\|^2\{|x|^2 - \sum_{k=1}^{n} \frac{a_k^2 + b_k^2}{c_k}(\|x_1\| + \cdots + \|x_m\|)^2\} \\
&\geq \|y\|^2\{|x|^2 - \sum_{k=1}^{n} \frac{\mathrm{Re}\langle x, z_k\rangle^2 + \mathrm{Im}\langle x, z_k\rangle^2}{c_k} \\
&= \|y\|^2\{|x|^2 - \frac{1}{2}\sum_{k=1}^{n} \frac{|\langle x, z_k\rangle|^2}{c_k} - \frac{1}{2}\sum_{k=1}^{n} \frac{\langle z_k, x\rangle^2}{c_k}\}, \\
&\geq \|y\|^2\{|x|^2 - \sum_{k=1}^{n} \frac{|\langle x, z_k\rangle|^2}{c_k}\}, \\
&\geq |\langle y, x\rangle|^2
\end{aligned}$$

as desired.

Remark 3.7. If in Theorem 3.2 $z_1, \ldots, z_n$ be a sequence of unit vectors , then

$$|(x_1 + \cdots + x_m, y)|^2 + (\sum_{j=1}^{m}(a_j^2 + b_j^2))(\sum_{i=1}^{n}|x_i|)^2||y||^2 \leq \left|\sum_{i=1}^{n} x_i\right|^2 ||y||^2. \tag{3.12}$$

This inequalitie is a type of extension of Diaz-Metcalf inequality in C^*-module.

Remark 3.8 If we have $\langle z_k, \sum_{i=1}^{m} x_i \rangle$ in commutator $Z(A)$ for all $k = 1, \ldots, n$ the condition (3.10) is verified .

Theorem 3.9. *Let $x, y \in X$ and $\langle y, z_i \rangle = 0$ for given non-zero vectors $z_i \in X, i = 1, 2, ..., n$. Then, for any $k \in \{1, 2, \ldots, n\}$*

$$|\langle y, x\rangle|^2 + \|y\|^2 [\|z_k\|^{-2} \left| \langle z_k, x\rangle - \left\langle \sum_{i=1}^{n} \frac{\langle z_i, x\rangle}{\sum_{j=1}^{n} \|\langle z_i, z_j\rangle\|} z_i, z_k \right\rangle \right|^2 + \sum_{i=1}^{n} \frac{|\langle z_i, x\rangle|^2}{\sum_{j=1}^{n} \|\langle z_i, z_j\rangle\|}]$$
$$\leq |x|^2 \|y\|^2 .$$

Proof. Let $u = x - \sum_{i=1}^{n} \frac{\langle z_i, x\rangle}{\sum_{j=1}^{n} \|\langle z_i, z_j\rangle\|} z_i$. Then by Proof of Theorem (3.2) in [1] we get

$$|u|^2 \leq |x|^2 - \sum_{i=1}^{n} \frac{|\langle z_i, x\rangle|^2}{\sum_{j=1}^{n} \|\langle z_i, z_j\rangle\|}.$$

Then $\langle y, u\rangle = \langle y, x\rangle$ as $\langle z_i, y\rangle = 0, i = 1, 2, \ldots, n$, and so by refinement of Seleberg inequality in [1] we get

$$\begin{aligned} |\langle y, x\rangle|^2 + \|z_k\|^{-2} \|y\|^2 |\langle z_k, u\rangle|^2 \leq & \quad |u|^2 \|y\|^2 \\ \leq & \quad \|y\|^2 [|x|^2 - \sum_{i=1}^{n} \frac{|\langle z_i, x\rangle|^2}{\sum_{j=1}^{n} \|\langle z_i, z_j\rangle\|}] \end{aligned}$$

thus the desired inequality thus follows.

We should obtain generalized and sharpened Cauchy-Schwarz inequality and Bessels inequality in C^*- module as follows.

Theorem 3.10. *Let $x, y \in X$, and z_i be unit vectors with $\langle y, z_i\rangle = 0$, $z_i \in X, i = 1, 2, \ldots, n$. Then*

$$|\langle y, x\rangle|^2 + \|y\|^2 [|\langle z_k, u_n\rangle|^2 + \sum_{i=1}^{n} |\langle z_i, u_{i-1}\rangle|^2]$$
$$\leq |x|^2 \|y\|^2$$
$$\text{or} \quad |\langle y, x\rangle| \leq |u_n - z_k \langle z_k, u_n\rangle| \, \|y\|$$

for $k \in \{1, 2, \ldots, n\}$ *, where* $u_0 = x$ *and* $u_i = u_{i-1} - z_i \langle z_i, u_{i-1} \rangle$ *,* $i = 1, 2, \ldots, n$ *.*

In particular , if z_i *are unit orthogonal vectors, then*

$$|\langle y, x \rangle|^2 + \|y\|^2 \sum_{i=1}^{n} |\langle z_i, x \rangle|^2 \le |x|^2 \|y\|^2 \tag{3.13}$$

or

$$|\langle y, x \rangle| \le |x - z_k \langle z_k, x \rangle| \|y\| . \tag{3.14}$$

Proof. By a simple calcul we get

$$u_n = x - \sum_{i=1}^{n} z_i \langle z_i, u_{i-1} \rangle .$$

Due to the definition of u_i and $\langle z_i, z_i \rangle = 1$, we have

$$|u_i|^2 = |u_{i-1}|^2 - |\langle z_i, u_{i-1} \rangle|^2 .$$

Setting $i = 1, 2, \ldots, n$ in above yields

$$|u_n|^2 = |x|^2 - \sum_{i=1}^{n} |\langle z_i, u_{i-1} \rangle|^2 .$$

Now we have $\langle y, z_k \rangle = 0$ for $k \in \{1, 2, \ldots, n\}$, by applying the refinement of Seleberg inequality in [1] , we obtain

$$|\langle y, u_n \rangle|^2 + \|y\|^2 |\langle z_k, u_n \rangle|^2 \le |u_n|^2 \|y\|^2$$

and

$$|u_n|^2 \|y\|^2 = \|y\|^2 [|x|^2 - \sum_{i=1}^{n} |\langle z_i, u_{i-1} \rangle|^2]$$

but $|\langle y, u_n \rangle|^2 = |\langle y, x \rangle|^2$ because $\langle y, z_i \rangle = 0$ then the first inequality holds.

For the second , we have

$$|u_n - z_k \langle z_k, u_n \rangle|^2 = |x|^2 - [|\langle z_k, u_n \rangle|^2 + \sum_{i=1}^{n} |\langle z_i, u_{i-1} \rangle|^2]$$

and we pass to square root we get the disered result .

Now if $\{z_i\}_i$, is a set of unit orthogonal vectors then

$$\begin{aligned}\langle z_k, u_n\rangle &= \left\langle z_k, x - \sum_{i=1}^{n} z_i \langle z_i, u_{i-1}\rangle \right\rangle \\ &= \langle z_k, x\rangle - \langle z_k, u_{k-1}\rangle \\ &= \langle z_k, x\rangle - \left\langle z_k, x - \sum_{i=1}^{k-1} z_i \langle z_i, u_{i-1}\rangle \right\rangle \\ &= 0.\end{aligned}$$

Also , $\langle z_i, u_{i-1}\rangle = \langle z_i, x\rangle$, $i = 1, 2, \ldots, n$ by a similar comptation, we have the particular case, this complet the Proof of Theorem .

Remark 3.11. If y is a non-zero vector orthogonal to vector x in (3.13) we obtain the Bessel inequality in Hilbert C^*-module.

If x is orthogonal to z_k in (3.14) we get the Cauchy-Shwartz inequality in Hilbert C^*-module.

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