Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:196
https://doi.org/10.28919/jmcs/7584
ISSN: 1927-5307

ON (m,n)QUASI-IDEALS IN SEMIRINGS

G. MOHANRAJ*, K. KALAIYARASI

Department of Mathematics, Annamalai University, India

Abstract. The notions of quasi-ideal is generalized into (m,n) quasi-ideals which is a generalization of existed (m,n) quasi-ideals. Regular semiring is characterized by the product of generalized quasi-ideals.

Keywords: quasi-ideals; (m,n)quasi-ideals; generalized (m,n)bi-ideals; (m,n)bi-ideals.

2010 AMS Subject Classification: 16Y60.

1. Introduction

Steinfeld prefaced the overview of quasi-ideals for rings and semigroups severally in [8]. Mohanraj et al characterized bi-ideals [1] and quasi-ideals [2] of ternary semigroup. Mohanraj et al classified various type of quasi-ideals in b-semirings [4]. Chinram [9] generalized quasi-ideals in semiring as one way. In this paper, we generalize further into (m,n) quasi-ideals which is a generalization of (m,n)quasi-ideals by Chinram [9]. It is validated by suitable giving example. We characterize regular semiring by generalized (m,n)quasi-ideals.

2. Preliminaries

A algebraic structure (S,+,.) is a semiring in which (S,+) is a commutative semigroup, (S,.) is a semigroup and it satisfies two distributive laws. We say that a semiring S has an absorbing

*Corresponding author

E-mail address: gmohanraaj@gmail.com

Received July 03, 2022

zero, if $a+0=0+a=a$ and $0\cdot a=a\cdot 0=0$ for all $a\in S$. A subset A of S is called subsemiring if A is itself a subsemiring. A subemiring R of S is called right(left) ideal if $RS\subseteq S(SR\subseteq S)$. A subsemiring Q of S is called quasi-ideal if $QS\cap SQ\subseteq Q$. A subsemiring B of S is called bi-ideal if $BSB\subseteq B$. An element a of a semiring A is called regular if $axa=a$ for some $x\in A$ [1]. A subsemiring Q of S is called (m,n)quasi-ideal [9] if $S^mQ\cap QS^n\subseteq Q$ by Chinram.

3. (m,n)QUASI-IDEALS

Hereafter S denotes semiring. Quasi-ideal is generalized as follows:

Definition 3.1. A subsemiring Q of S is called (m,n) quasi-ideal if $Q^mS\cap SQ^n\subseteq Q$ for the positive integers m and n.

Remark 3.2. (i) Every quasi-ideal in S is a $(1,1)$ quasi-ideal.
(ii) $Q^mS\cap SQ^n\subseteq QS^m\cap QS^n\subseteq Q$ implies that every (m,n) quasi-ideal by Chinram [9] is a (m,n) quasi-ideal by us.
(iii) Example 3.3 contrasts (m,n) quasi-ideals from quasi-ideals.
(iv) Example 3.3 gives a (m,n) quasi-ideal which is not a (m,n) quasi-ideal by Chinram [9] for all m and n.

Example 3.3. S is the semiring of 4x4 matrices over non negative integers $\mathbb{Z}^*$.

$$Q=\left\{\begin{pmatrix}0 & a_1 & a_2 & a_3\\ 0 & 0 & b_1 & b_2\\ 0 & 0 & 0 & b_3\\ 0 & 0 & 0 & 0\end{pmatrix}\middle| a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{Z}^*\right\}$$

Clearly Q is a (2,2) quasi-ideal. Now,

$$QS=\left\{\begin{pmatrix}r'_{11} & r''_{12} & r'''_{13} & r''''_{14}\\ r'_{21} & r''_{22} & r'''_{23} & r''''_{24}\\ r'_{31} & r''_{32} & r'''_{33} & r''''_{34}\\ 0 & 0 & 0 & 0\end{pmatrix}\middle| r'_i\in\mathbb{Z}^*\right\}$$

and

$$SQ = \left\{ \begin{pmatrix} 0 & r''_{11} & r''_{12} & r''_{13} \\ 0 & r''_{21} & r''_{22} & r''_{23} \\ 0 & r''_{31} & r''_{32} & r''_{33} \\ 0 & r''_{41} & r''_{42} & r''_{43} \end{pmatrix} \middle| r'_i \in \mathbb{Z}^* \right\}$$

Now,

$$QS \cap SQ = \left\{ \begin{pmatrix} 0 & t'_1 & t''_1 & t'''_1 \\ 0 & t'_2 & t''_2 & t'''_2 \\ 0 & t'_3 & t''_3 & t'''_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| t'_i \in \mathbb{Z}^* \right\}$$

implies that Q is not quasi-ideal. Since $S^n = S$, $QS^m \cap S^n Q = QS \cap SQ \nsubseteq Q$ implies Q is not (m,n)quasi-ideal by Chinram[9] for any m and n.

Definition 3.4. A subset G of semiring in S is called generalized (m,n) bi-ideal if $i)G+G \subseteq G$ $ii)G^m S G^n \subseteq G$ for the positive integers m and n [1]. A generalized (m,n)bi-ideal is (m,n)bi-ideal if $G \cdot G \subseteq G$

Theorem 3.5. Every (m,n) quasi-ideal is a (m,n) bi-ideal.

Proof: Let Q be a (m,n) quasi-ideal.

Then, $Q^m S Q^n \subseteq Q^m SS \subseteq Q^m S$, and

$Q^m S Q^n \subseteq SSQ^n \subseteq SQ^n$ imply

$Q^m S Q^n \subseteq (Q^m S) \cap (SQ^n) \subseteq Q$

Therefore, Q is a (m,n) bi-ideal.

Theorem 3.6. The (m,n)quasi ideal generated by ‘a’, is $\{r_1 a + r_2 a^2 + ... + r_m a^m + (a^m S \cap Sa^n) | r_i \in \mathbb{Z}^*, i = 1 \ to \ m\}$, $m > n$ and is denoted by $\langle a \rangle_{(m,n)q}$.

Proof: Let us take $m > n$. Now,

$$\begin{aligned}
A &= \{r_1a + r_2a^2 + ... + r_ma^m + (a^mS \cap Sa^n) | r \in \mathbb{Z}^*\} \ \ implies \\
A + A &= \{r_1a + r_2a^2 + ... + r_ma^m + a^mS \cap Sa^n | r_i \in \mathbb{Z}^*\} + \\
&\quad \{t_1a + t_2a^2 + ... + t_ma^m + a^mS \cap Sa^n | t_i \in \mathbb{Z}^*\} \\
&= \{(r_1 + t_1)a + (r_2 + t_2)a^2 + ... + (r_m + t_m)a^m + a^mS \cap Sa^n\} \\
&= \{s_1a + s_2a^2 + ... + s_ma^m + a^mS \cap Sa^n | s_i \in \mathbb{Z}^*\} \subseteq A
\end{aligned}$$

$$\begin{aligned}
AA &= \{r_1a + r_2a^2 + ... + r_ma^m + (a^mS \cap Sa^n)\} \cdot \\
&\quad \{t_1a + t_2a^2 + ... + t_ma^m + (a^mS \cap Sa^n) | t \in \mathbb{Z}^*\} \\
&= \{r'a^2 + ... + r'_{m-1}a^m + (a^mS \cap Sa^n) + r''_i a^i (a^mS \cap Sa^n) \\
&\quad + (a^mS \cap Sa^n) t' a^k + a^mS \cap Sa^n | r', r''_i, t' \in \mathbb{Z}^*, i = 1 \text{ to } n\}
\end{aligned}$$

Now, $a^{m+i} \in a^mS \cap Sa^n$, for $i = 1$ to m, $m > n$

For any k, $k = 1$ to m and $r, r_1 \in a^mS \cap Sa^n$

$$\begin{aligned}
a^kr &= a^k(a^ms) \\
&= a^m(a^ks) \in a^mS \\
a^kr &= a^k \cdot (s_1a^n) \\
&= (a^ks_1)a^n \in Sa^n \\
Thus,\ a^k(a^mS \cap Sa^n) &\subseteq a^mS \cap Sa^n \\
Now,\ rr_1 &= (a^ms_1)a^ms_2 \\
&= a^m(s_1a^ms_2) \in a^mS \\
rr_1 &= (a^ms_1)(s'_2a^n) \\
&= (a^ms_1s'_2)a^n \in Sa^n,
\end{aligned}$$

Then, $(a^mS \cap Sa^n)(a^mS \cap Sa^n) \subseteq a^mS \cap Sa^n$.

Therefore, $A \cdot A \subseteq A$.

$$\begin{aligned} A^m &= \{r_1a+r_2a^2+...+r_ma^m+a^mS\cap Sa^n\}...\{r_1a+r_2a^2+ \\ &\quad ...+r_ma^m+a^mS\cap Sa^n\} \\ &= \{r_1'a^m+r_2'a^{m+1}+...+r_m'a^{m^2}+a^i(a^mS\cap Sa^n)^{m-i}+(a^mS\cap Sa^n)^{m-j}a^j+ \\ &\quad (a^mS\cap Sa^n)^m|i,j=1 \text{ to } m-1\} \end{aligned}$$

If $1\le i\le m$, $a^i(a^mS\cap Sa^n)^{m-i}\subseteq a^mS$,

$x\in a^i(a^mS\cap Sa^n)^{m-i}$ implies $x=(a^ksa^n...s)a^n\in Sa^n$

$$\begin{aligned} Thus,\ a^i(a^mS\cap Sa^n)^{m-i} &\subseteq a^mS\cap Sa^n, i=1 \text{ to } m-1 \\ Similarly,(a^mS\cap Sa^n)^{m-j}a^j &\subseteq a^mS\cap Sa^n, j=1 \text{ to } m-1 \\ y\in(a^mS\cap Sa^n)\ implies\ y^m &= a^m(s_1...a^ms_1)\in a^mS \text{ and} \\ y^m &= (s_2a^n...s_2)a^n\in Sa^n \end{aligned}$$

$$Then,\ (a^mS\cap Sa^n)^m\subseteq a^mS\cap Sa^n$$

$$\begin{aligned} Therefore,A^mS &\subseteq a^mS\cap Sa^n \\ Similarly,SA^n &\subseteq a^mS\cap Sa^n \\ Thus,A^mS\cap SA^n &\subseteq A. \end{aligned}$$

Therefore, A is a (m,n) quasi-ideal. By similar arguement, $A=\{r_1a+r_2a^2+...+r_na^n+(a^mS\cap Sa^n)|r_i\in\mathbb{Z}^*, i=1\ to\ n\}$ when $n>m$, A is a (m,n) quasi-ideal. Suppose that B is a (m,n) quasi-ideal containing 'a', $a^k\in B$ for all $k=1$ to m. Now, $a\in B$ implies $a^mS\cap Sa^n\subseteq B$, then $A\subseteq B$. Therefore A is a (m,n) quasi-ideal generated by 'a'.

Theorem 3.7. Every (m,n) quasi-ideal is a (i,j) quasi-ideal for $i\ge m$ and $j\ge n$.

Proof: For a (m,n) quasi-ideal, $Q^mS\cap SQ^n\subseteq S$. Now,

$$Q^{m+1}S\cap SQ^n\subseteq Q^m(QS)\cap SQ^n$$

$$\begin{aligned} Q^mS\cap SQ^{n+1}\subseteq Q^mS\cap SQ^n &\subseteq Q \\ &\subseteq Q^mS\cap(SQ)Q^n \end{aligned}$$

$$\subseteq \quad Q^m S \cap SQ^n \subseteq Q$$

$$Q^{m+1} S \cap SQ^{n+1} \subseteq Q^m S \cap SQ^n \subseteq Q$$

Thus Q is a $(m+1,n)$ quasi-ideal, $(m,n+1)$ quasi-ideal. Therefore Q is a (i,j) quasi-ideal for $i \geq m$ and $j \geq n$.

Corollary 3.8. Every quasi-ideal is a (m,n)quasi-ideal for all
$m,n \geq 1$

Theorem 3.9. The intersection of (i,j) quasi-ideal and (k,l)quasi-ideal is a (m,n) quasi-ideal for all $m \geq max\{i,k\}$ and $n \geq \{j,l\}$.

Proof: Let B_1 be a (i,j) quasi-ideals, and B_2 be a (k,l) quasi-ideal. Then by Theorem 3.7, B_1 and B_2 are (m,n) quasi-ideals for $m \geq max\{i,k\}$ and $n \geq max\{j,l\}$. Therefore $(B_1 \cap B_2)^m S \cap S(B_1 \cap B_2)^n \subseteq B_i^m S \cap SB_i^n \subseteq B_i$, $i = 1,2$ imply $B_1 \cap B_2$ is a (m,n)quasi-ideal.

Corollary 3.10. If Q_i is a (m,n) quasi-ideal in S for all i, then $\bigcap\limits_{i=1}^{n} Q_i$ is a (m,n) quasi-ideal for any finite n.

Theorem 3.11. For a semiring S, the following statements are equivalent.

1. S is regular.
2. $G \cap Q \subseteq GSQ$ for any generalized (m,n) bi-ideal G and for any (m,n) quasi-ideal Q.
3. $B \cap Q \subseteq BSQ$ for any (m,n) bi-ideal B and for any (m,n) quasi-ideal Q.
4. $Q_1 \cap Q_2 \subseteq Q_1 S Q_2$ for any (m,n) quasi-ideal Q_1 and Q_2.
5. $I \cap Q \subseteq ISQ$ for any quasi-ideal I and for any (m,n) quasi-ideal Q.
6. $I_1 \cap I_2 \subseteq I_1 S I_2$ for any quasi-ideal I_1 and I_2.
7. $B \cap Q \subseteq BSQ$ for any bi-ideal B and for any (m,n) quasi-ideal Q.
8. $B \cap Q \subseteq BSQ$ for any bi-ideal B and for any quasi-ideal Q.
9. $G \cap Q \subseteq GSQ$ for any genaralized bi-ideal G and for any (m,n) quasi-ideal Q.
10. $G \cap Q \subseteq GSQ$ for any generalized bi-ideal G and for any quasi-ideal Q.
11. $Q_1 \cap Q_2 \subseteq Q_1 S Q_2$ for any quasi-ideal Q_1 and Q_2.
12. $Q \cap G \subseteq QSG$ for any (m,n) quasi-ideal Q and for any generalized (m,n) bi-ideal G.
13. $Q \cap B \subseteq QSB$ for any (m,n) quasi-ideal Q and for any (m,n) bi-ideal B.

14. $Q \cap I \subseteq QSI$ for any (m,n) quasi-ideal Q and for any quasi-ideal I.

15. $Q \cap B \subseteq QSB$ for any (m,n) quasi-ideal Q and for any bi-ideal B.

16. $Q \cap B \subseteq QSB$ for any quasi-ideal Q and for any bi-ideal B.

17. $Q \cap G \subseteq QSG$ for any (m,n) quasi-ideal Q and for any generalized bi-ideal G.

18. $Q \cap I \subseteq QSG$ for any quasi-ideal Q and for any generalized bi-ideal G.

19. $Q \cap L \subseteq QL$ for any quasi-ideal Q and for any left ideal L.

20. $R \cap Q \subseteq RQ$ for any right ideal R and for any quasi-ideal Q.

21. $R \cap L = RL$ for any right ideal R and for any left ideal L.

Proof: First we prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$, $(6) \Rightarrow (19) \Rightarrow (21) \Rightarrow (1)$, $(3) \Rightarrow (7) \Rightarrow (8) \Rightarrow (20) \Rightarrow (21)$, $(2) \Rightarrow (9) \Rightarrow (10) \Rightarrow (21)$, $(4) \Rightarrow (11) \Rightarrow (20)$, $(1) \Rightarrow (12) \Rightarrow (13) \Rightarrow (14) \Rightarrow (19)$, $(13) \Rightarrow (15) \Rightarrow (16) \Rightarrow (20)$, $(12) \Rightarrow (17) \Rightarrow (18) \Rightarrow (19)$.

$(1) \Rightarrow (2)$ Let $a \in G \cap Q$, then $a = axa \in GSQ$. Thus $G \cap Q \subseteq GSQ$.

$(2) \Rightarrow (3)$ Straight forward.

$(3) \Rightarrow (4)$ By Theorem 3.5, (4) holds.

$(4) \Rightarrow (5)$ By Theorem 3.7, (5)it follows.

$(5) \Rightarrow (6)$ By Theorem 3.7, (6)it follows.

$(6) \Rightarrow (19)$ By (6), $Q \cap L \subseteq QSL \subseteq QL$ for any quasi-ideal Q and left ideal L.

$(19) \Rightarrow (21)$ Now, $R \cap L \subseteq RL$ for any right ideal R and left ideal L $RL \subseteq R$ and $RL \subseteq L$ imply $R \cap L = RL$.

$(21) \Rightarrow (1)$ Now, $a \in \langle a \rangle_r \cap \langle a \rangle_l = \langle a \rangle_r \cdot \langle a \rangle_l$

$$\begin{aligned}
\text{Then, } \langle a \rangle_r . \langle a \rangle_l &= \{ma + as | m \in \mathbb{Z}^*, s \in S\}. \\
& \{na + as | m \in \mathbb{Z}^*, s \in S\} \\
&= \{na^2 | n \in Z^*\} + aSa + aSa + aSa \\
\text{If, } a \in \{na^2 | n \in Z^*\}, \text{ then, } a = na^2 &= (na)(na^2) \\
&= n^2 a^3 \\
&= a(n^2 a)a
\end{aligned}$$

Therefore S is regular.

$(3) \Rightarrow (7)$ Straightforward.

$(7) \Rightarrow (8)$ By Corollary 3.8 it follows.

$(8) \Rightarrow (20)$ For any right ideal R and by (20) holds.

$(20) \Rightarrow (21)$ By (20), for any right ideal R and left ideal L,

$R \cap L \subseteq RSL \subseteq RL$,$RL \subseteq RS \subseteq R$ and $RL \subseteq SL \subseteq L$ imply $RL \subseteq R \cap L$.Therefore $RL = R \cap L$.

$(2) \Rightarrow (9)$ Straightforward.

$(9) \Rightarrow (10)$ By Corollary 3.8 it follows.

$(10) \Rightarrow (21)$ By (10), for any right ideal R and left ideal L $R \cap L \subseteq RSL \subseteq RL$ but $RL \subseteq R \cap L$ imply $R \cap L = RL$.

$(4) \Rightarrow (11)$ By Corollary 3.8, (11) it follows.

$(11) \Rightarrow (20)$ Right ideal R is a quasi-ideal, then (20) follows.

$(1) \Rightarrow (12)$ Let $a \in Q \cap G$. Then $a = axa \in QSG$. Thus, $Q \cap G \subseteq QSG$.

$(12) \Rightarrow (13)$ Straight forward.

$(13) \Rightarrow (14)$ By Theorem 3.5 and Corollary 3.8, (11) it follows.

$(14) \Rightarrow (19)$ By Corollary 3.8, (19) it follows.

$(13) \Rightarrow (15)$ Straightforward.

$(15) \Rightarrow (16)$ Straight forward.

$(16) \Rightarrow (20)$ Right ideal R is a bi-ideal implies $R \cap Q \subseteq RSQ \subseteq RQ$.

$(12) \Rightarrow (17)$ Straight forward.

$(17) \Rightarrow (18)$ By Corollary 3.8, (18) it follows.

$(18) \Rightarrow (19)$ Straight forward.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] G. Mohanraj, K. Kalaiyarasi, On (m,n) bi-ideals in semirings, Submitted.

[2] G. Mohanraj, M. Vela, On T-fuzzy quasi-ideals of ternary semigroups, Math. sci. Int. Res. J. 7 (2018), 104-110.

[3] G. Mohanraj, M. Vela, On generalized fuzzy generalized fuzzy bi-ideals of ternary semigroups, Bull. Int. Math. Virt. Inst. 9 (2019), 441-453.

[4] G. Mohanraj, M. Palanikumar, On various quasi ideals in b-semirings, Malaya J. Mat. 8 (2020), 20-27.

[5] S. Lajos, F.A. Szasz, on the bi-ideals in associative rings, Proc. Japan Acad. 49 (1970), 505-507.

[6] M. Munir, M. Habib, Characterizing semirings using their quasi and bi-ideals, Proc. Pakistan Acad. Sci. A. Phys. Comput. Sci. 53 (2016), 469-475.

[7] O. Steinfeld, Uberdiequasiideale von Halbgruppen, Institutum Mathematicum Universitatis Debreceniensis. (1956).

[8] O. Steinfeld, quasi-ideals in rings and semigroups, Akad. Kiado, Budapest. (1978).

[9] R. Chinram, A note on (m,n)-quasi-ideals in semirings, Int. J. Pure Appl. Math. 49 (2008), 45-52.