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NONTRIVIAL SOLUTIONS FOR NONLINEAR SYSTEM INVOLVING NON-COMPACT RESOLVENT OPERATORS

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Abstract. This article establishes the existence of distributional solutions to a nonlinear system involving non-compact resolvent. By means of the Leray-Schauder degree theory, with suitable assumptions on the nonlinearities, the authors prove the existence of distributional solutions.

Keywords: topological degree; non-compact resolvent operators; homotopy.

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1. Introduction

After the pioneer work by Landesman and Lazer in 1970 [16], many researchers were interested in nonlinear problems at resonance (see [1, 2, 3, 4, 5, 9, 10, 11, 12, 14, 21, 25, 26]). In their article, Landesman and Lazer provided sufficient conditions (which in certain circumstances are also necessary) for the existence of solutions for the smooth semilinear Dirichlet problems. Recently, Lakhal, and Khodja [15] treated an elliptic system at resonance for jumping non-linearities in the compact case, employing the Leray-Schauder degree theory (see [20]) . The scalar case considered in [12] shows the existence of solutions to the problem

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$Au = \alpha u^+ - \beta u^- + f(x,u) + h$, where A is a self-adjoint operator and $f(\cdot,\cdot)$ maps $\Omega \times \mathbb{R}$ into $\mathbb{R}$, such that $\lim_{s\to\infty} \frac{f(x,s)}{s} = 0$ and $\alpha < \beta$ satisfy $[\alpha,\beta] \cap Sp(A) = \lambda$, ($\lambda$ is a simple eigenvalue of A).

In our work, we concentrate our efforts to extend the results obtained in [12, 15] to the system for the non-compact case. This type of problems has been extensively studied by many authors. In 2021, Zhang and Liu [27] obtained the existence of nontrivial solutions for a quasilinear system. In [17], the authors studied the existence of weak solutions of a quasilinear system of partial differential equations which are a combination of the Perona-Malik equation and the heat equation. Their study is mainly based on the use of the compactness method and the motonocity arguments. Far from being complete, we refer the interested readers to [18, 19, 22, 24] and the references therein.

The aim of this article is to investigate the existence of weak solutions to a nonlinear system at resonance, when the resolvent of our operator is non-compact. We consider the following problem

$$
\begin{aligned}
&Au = g_1(x,u,v) + h_1(x) \quad \text{in } \Omega,\\
&Av = g_2(x,u,v) + h_2(x) \quad \text{in } \Omega, \qquad (1)\\
&u = v = 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^{\mathbb{N}} (N \geq 2)$ with smooth boundary $\partial\Omega$ and $h = (h_1,h_2)$ is in $(L^2(\Omega))^2$, A is a linear self-adjoint operator with non-compact resolvent, $R(A)$ is closed in $L^2(\Omega)$, $D(A) \subset L^2(\Omega)$, and the inclusion of $D(A) \cap R(A)$ (equipped with the graph norm) in $L^2(\Omega)$ is compact, the functions g_i are semi-linear at infinity such that

$$g_1(x,s,\xi) = \alpha_1 s^+ - \beta_1 s^- + f_1(x,s,\xi),$$

$$g_2(x,s,\xi) = \alpha_2 \xi^+ - \beta_2 \xi^- + f_2(x,s,\xi),$$

where $s^+ = \max(s,0)$ (resp $\xi^+ = \max(\xi,0)$) and $s^- = \max(-s,0)$ (resp $\xi^- = \max(-\xi,0)$), f maps $\Omega \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. The difficulty here lays in the fact that for a regular value r of A, $(A - rI)^{-1}$ is not compact on $L^2(\Omega) \times L^2(\Omega)$ while the restriction of $(A - rI)^{-1}$ to $R(A)$ is compact. We can write (1) in the form

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}^+ - \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}^- + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

We put $w = (u,v)$, $f = (f_1, f_2)$, $h = (h_1, h_2)$ and

$$M = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \, and \quad \beta = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}.$$

Then

$$Mw = \alpha w^+ - \beta w^- + If + Ih \quad \text{in } \Omega.$$

In what follows, we need a strong motonocity hypotheses on $g_i, i = 1,2$, i.e.

$$\exists \delta_i > 0, i = 1,2, \quad \forall\, s, \hat{s}, \xi, \hat{\xi} \in \mathbb{R},$$

(2)
$$(g_1(x,s,\xi) - g_1(x,\hat{s},\hat{\xi}))(s-\hat{s}) \geq \delta |s-\hat{s}|^2, \text{ a.e. } \in \Omega,$$

$$(g_2(x,s,\xi) - g_2(x,\hat{s},\hat{\xi}))(\xi-\hat{\xi}) \geq \delta |\xi-\hat{\xi}|^2, \text{ a.e. } \in \Omega.$$

From now on, we suppose that $\alpha_i, \beta_i \in \left]\underline{\lambda}, \overline{\lambda}\right[= I_\lambda$ satisfy

(3)
$$[\alpha_i, \beta_i] \cap Sp(A) = \{\lambda\}, i = 1,2$$

where $\overline{\lambda}$ and $\underline{\lambda}$ are defined as follows

$$\overline{\lambda} = \inf\{\lambda_k : \lambda_k > \lambda, k \in \mathbb{N}^*\}, \underline{\lambda} = \sup\{\lambda_k : \lambda_k < \lambda, k \in \mathbb{N}^*\}.$$

For every $\alpha_i, \beta_i \in I_\lambda \times I_\lambda$, we define the function $C(\cdot,\cdot)$ on $I_\lambda \times I_\lambda$ satisfying

$$Au = \alpha u^+ - \beta u^- + C(\alpha,\beta)\varphi, \text{ and } \int_\Omega u\varphi = 1,$$

where φ is a normalized eigenfunction corresponding to λ.

The function $C(.,.)$ is defined on $I_\lambda \times I_\lambda$, with values in $\mathbb{R}$ and satisfies the following properties

(1) for every $\alpha \in I_\lambda, C(\alpha,\alpha) = \lambda - \alpha$,

(2) if $\varphi \geq 0$, then $C(\alpha,\beta) = \lambda - \alpha$,

(3) if $\varphi^- \neq 0$ and $\varphi^+ \neq 0$, $C(\cdot,\cdot)$ is decreasing in each variable,

(4) the curve

$$\Upsilon = \{(\alpha,\beta) \in I_\lambda \times I_\lambda, C(\alpha,\beta) = \{0\}\},$$

is continuous, passing through the point (λ,λ) of $I_\lambda \times I_\lambda$.

In the present paper we study the case where $C(\alpha_i,\beta_i)\cdot C(\beta_i,\alpha_i) = 0, \quad i = 1,2$. For the case of a system, the interested reader is referred to [6, 7] and [8]. The main idea in [23] is to present a priori bounds for the solutions of (1) where $C(\alpha,\beta)\cdot C(\beta,\alpha) \neq 0$. Let $N(\alpha,\beta)$ be defined as follows

$$N(\alpha,\beta) = \left\{u \in D(A), Au = \alpha u^+ - \beta u^-\right\},$$

then $N(\alpha,\beta) = \{0\}$ if and only if $C(\alpha,\beta)\cdot C(\beta,\alpha) \neq 0$ noting that $N(\lambda,\lambda) = N_\lambda = \ker(A - \lambda I)$. The equation of existence of solutions for (1) when $N(\alpha,\beta) = \{0\}$ has been studied in [11, 23]. The main idea of the present paper is to study the existence of solutions for a system with non compact resolvent operators of the form (1) in the case where $N(\alpha,\beta) \neq \{0\}$:

▷ If $C(\beta,\alpha) = C(\alpha,\beta) = 0$, we have resonance;

▷ If $C(\alpha,\beta) = 0 \neq C(\beta,\alpha)$, or $C(\beta,\alpha) = 0 \neq C(\alpha,\beta)$, we have semi resonance.

We assume that $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the conditions:

$$\begin{aligned} &|f_1(x,s,\xi)| \leq c'(1+|s|+|\xi|), \\ &|f_2(x,s,\xi)| \leq c''(1+|s|+|\xi|), \end{aligned} \tag{4}$$

where c', c'' are real positive constants;

$$\begin{gathered} \lim_{s,|\xi|\to\infty} f_1(x,s,\xi) = \zeta_1^+, \lim_{-s,|\xi|\to\infty} f_1(x,s,\xi) = \zeta_1^-, \\ \zeta_1^-, \zeta_1^+ \in L^2(\Omega) \text{ and } \zeta_1^- \leq f_1(x,s,\xi) \leq \zeta_1^+ \end{gathered} \tag{5}$$

and

$$\begin{gathered} \lim_{\xi,|s|\to\infty} f_2(x,s,\xi) = \zeta_2^+, \lim_{-\xi,|s|\to\infty} f_2(x,s,\xi) = \zeta_2^-, \\ \zeta_2^-, \zeta_2^+ \in L^2(\Omega) \text{ and } \zeta_2^- \leq f_2(x,s,\xi) \leq \zeta_2^+. \end{gathered} \tag{6}$$

Let $\theta_1 = (\nu_1,\nu_2)$ and $\theta_2 = (\nu_3,\nu_4)$ be defined as follows

$$
(7) \qquad \begin{aligned}
A v_i = \alpha_i v_i^+ - \beta_i v_i^-, \int_\Omega v_i \varphi dx = 1 \text{ when } C(\alpha_i, \beta_i) = 0, i = 1, 2, \\
A v_{i+2} = \alpha_i v_{i+2}^+ - \beta_i v_{i+2}^-, \int_\Omega v_{i+2} \varphi dx = -1 \text{ when } C(\beta_i, \alpha_i) = 0, i = 1, 2.
\end{aligned}
$$

Our main theorem is

Theorem 1. *Assume that* (3), (4), (5), (6) *and* (7) *are fulfilled. For each* $(h_1, h_2) \in (L^2(\Omega))^2$, *we define* $T_i(h_i)$ *and* $T_{i+2}(h_i)$ *as follows*

$$
\begin{aligned}
T_i(h_i) &= \int_\Omega h_i v_i dx + \int_\Omega \zeta_i^+ v_i^+ dx - \int_\Omega \zeta_i^- v_i^- dx, \quad i = 1, 2 \\
T_{i+2}(h_i) &= \int_\Omega h_i v_{i+2} dx + \int_\Omega \zeta_i^+ v_{i+2}^+ dx - \int_\Omega \zeta_i^- v_{i+2}^- dx, \quad i = 1, 2
\end{aligned}
$$

(1) If $C(\alpha_i, \beta_i) = C(\beta_i, \alpha_i) = 0$, (1) *has at least one solution for every* $h_i \in L^2(\Omega)$ *such that* $T_i(h_i) T_{i+2}(h_i) > 0$, $i = 1, 2$.

(2) If $C(\alpha_i, \beta_i) = 0 \neq C(\beta_i, \alpha_i)$ *(resp* $C(\beta_i, \alpha_i) = 0 \neq C(\alpha_i, \beta_i)$*)*, (1) *has at least one solution for every* $h_i \in L^2(\Omega)$ *such that*
$C(\beta_i, \alpha_i) T_i(h_i) < 0$ *(resp* $C(\alpha_i, \beta_i) T_{i+2}(h_i) < 0$*)*, $i = 1, 2$.

2. PRELIMINARIES

Let us consider the space

$$
\mathrm{V} = D(A) \times D(A),
$$

endowed with the norm

$$
\|(u, v)\|_{\mathrm{V}}^2 = \|u\|_{D(A)}^2 + \|v\|_{D(A)}^2,
$$

and let $\widetilde{\mathrm{V}} = L^2(\Omega) \times L^2(\Omega)$. In the sequel, $\|\cdot\|_{L^2(\Omega)}$ denote the usual norm on $L^2(\Omega)$. Throughout this paper, we denote by λ a simple eigenvalue of A, φ is an eigenfunction associated to λ normalized in $L^2(\Omega)$, P designates the orthogonal projection of $\widetilde{V}$ on $(\varphi^\perp)^2$ ($\varphi^\perp$ is the orthogonal of φ in $L^2(\Omega)$). We recall the following proposition proved by Gallouet and Kavian (see [11]).

Proposition 1. *For all $\alpha, \beta \in \left]\underline{\lambda}, \overline{\lambda}\right[$ there exists a unique $C(\alpha,\beta) \in \mathbb{R}$ and a unique $u \in D(A)$, such that*

$$Au = \alpha u^+ - \beta u^- + C(\alpha,\beta)\varphi,$$
$$\int_\Omega u\varphi dx = 1.$$

The next result is given in a general framework.

Proposition 2. *Let $G(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable function on $x \in \Omega$ and continuous on $s \in \mathbb{R}$, verifying*

(i) *There exist $\alpha, \beta \in \mathbb{R}$ such that $\underline{\lambda} < \alpha \leq \frac{G(x,s)-G(x,t)}{s-t} \leq \beta < \overline{\lambda}$,for all $s,t \in \mathbb{R}, a.e.in\ \Omega$,*
(ii) $\lim\limits_{|s| \to +\infty} \frac{G(x,s)}{s} = l$ *a.e. in* Ω,
(iii) $G(x,0) = 0$ *a.e. in* Ω.

Then, for all $s \in \mathbb{R}$, and all $h_0 \in \varphi^\perp$, there exists a unique $v \in D(A) \cap \varphi^\perp$ such that

$$Av = PG(.,v+s\varphi) + h_0.$$

The proof of the above proposition can also be found in [11]. The following lemma is proved in [12]

Lemma 1. *Let (2), (4), (5) and (6) be satisfied, and let F be the orthogonal projection of $L^2(\Omega)$ on $N(A)$ $(\neq \{0\})$ and $B = (A - rI)^{-1}$ for $r \in \mathbb{R} - Sp(A)$. For $\tau \in [0,1]$ and $u \in L^2(\Omega)$, we define*

$$D_\tau u = (I-F)u + \frac{1}{r} F g_\tau(x,u,v).$$

Then, $D_\tau : L^2(\Omega) \to L^2(\Omega)$ is inversible, $D_\tau^{-1} : L^2(\Omega) \to L^2(\Omega)$ is continuous, and bounded on bounded sets.

Then, for $(\tau,u,v) \in [0,1] \times \widetilde{V}$, we define

$$S_1(\tau,u,v) = B(I-F)g_{1,\tau}(x,u,v) - rB(I-F)u,$$

$$S_2(\tau,u,v) = B(I-F)g_{2,\tau}(x,u,v) - rB(I-F)v,$$

and

$$H(\tau,u,v) = \begin{pmatrix} H_1(\tau,u,v) \\ H_2(\tau,u,v) \end{pmatrix} = \begin{pmatrix} D_{1,\tau}^{-1} & 0 \\ 0 & D_{2,\tau}^{-1} \end{pmatrix} \begin{pmatrix} S_1(\tau,u,v) + Bh_1 \\ S_2(\tau,u,v) + Bh_2 \end{pmatrix};$$

$B(I-F)$, is compact and so $H:[0,1]\times\widetilde{V}\to\widetilde{V}$ is compact.

Clearly, the following problems are equivalent:

$$(u,v)\in V,$$

$$Au=\alpha_1u^+-\beta_1u^-+\tau f_1(x,u,v)+(1-\tau)(\beta_1-\alpha_1)u^-+h_1(x), \tag{8}$$

$$Av=\alpha_2v^+-\beta_2v^-+\tau f_2(x,u,v)+(1-\tau)(\beta_2-\alpha_2)v^-+h_2(x),$$

and

$$(u,v)\in\widetilde{V},\quad (u,v)=H(\tau,u,v). \tag{9}$$

3. A Priori Bounds for Solutions of (1)

Lemma 2. *Under the assumptions of theorem* (1)*, and assuming that*

$$0\le\underline{\lambda}<\alpha_i<\lambda<\beta_i<\overline{\lambda}$$

and $T_i(h_i)<0$, $T_{i+2}(h_i)<0, i=1,2$, there exists $R>0$ such that for all $\tau\in[0,1]$ and all $(u,v)\in V$

$$(u,v)=H(\tau,u,v)\Longrightarrow\|(u,v)\|_V<R.$$

Proof. To prove this lemma we assume by contradiction, that for all $R>0$ there exists $(\tau,u,v)\in[0,1]\times V$ such that

$$(u,v)=H(\tau,u,v)\text{ and }\|(u,v)\|_V>R.$$

In other words, we can find a sequence $(\tau_n,u_n,v_n)\in[0,1]\times V$ such that

$$(u_n,v_n)=H(\tau_n,u_n,v_n)\ \text{ and }\ a_n=\|(u_n,v_n)\|_V>n. \tag{10}$$

Taking

$$(\hat{u}_n,\hat{v}_n)=\left(\frac{u_n}{\|(u_n,v_n)\|_V},\frac{v_n}{\|(u_n,v_n)\|_V}\right),$$

then it follows with this choice of $(\hat{u}_n,\hat{v}_n)$ that

$$(\hat{u}_n,\hat{v}_n)\in V\text{ and }\|(\hat{u}_n,\hat{v}_n)\|_V=1.$$

Indeed, it is easy to see that $(\hat{u}_n,\hat{v}_n)=\left(\frac{u_n}{a_n},\frac{v_n}{a_n}\right)\to(\hat{u},\hat{v})$ in V and $\|(\hat{u}_n,\hat{v}_n)\|_V=1$. Let us show that $(\hat{u}_n,\hat{v}_n)\in V$.

$$
\begin{aligned}
&(\hat{u}_{1,n},\hat{v}_{1,n}) = (I-F)\left(\frac{u_n}{a_n},\frac{v_n}{a_n}\right) \in (D(A)\cap R(A))^2, \quad (\hat{u}_{1,n},\hat{v}_{1,n}) \to (\hat{u}_1,\hat{v}_1) \text{ in } \widetilde{V}\\
&(\hat{u}_{2,n},\hat{v}_{2,n}) = F\left(\frac{u_n}{a_n},\frac{v_n}{a_n}\right) \in (N(A))^2, \quad (\hat{u}_{2,n},\hat{u}_{2,n}) \rightharpoonup (\hat{u}_2,\hat{v}_2) \text{ in } \widetilde{V} \text{ weak.}
\end{aligned}
\tag{11}
$$

It is easy to see that

$$
\begin{aligned}
&\forall u,v,\widetilde{u},\widetilde{v}\in\mathbb{R}, \forall \tau\in[0,1],\\
&(g_{1,\tau}(x,u,v)-g_{1,\tau}(x,\widetilde{u},\widetilde{v}))(u-\widetilde{u}) \geq \delta_1|u-\widetilde{u}|^2,\\
&(g_{2,\tau}(x,u,v)-g_{2,\tau}(x,\widetilde{u},\widetilde{v}))(v-\widetilde{v}) \geq \delta_2|v-\widetilde{v}|^2,
\end{aligned}
\tag{12}
$$

and $(\hat{u}_n,\hat{v}_n)$ satisfies

$$
A\hat{u}_n = \frac{1}{a_n}[g_{1,\tau_n}(x,a_n\hat{u}_n,a_n\hat{v}_n)+h_1(x)] = A\hat{u}_{1,n}, \tag{13}
$$

$$
A\hat{v}_n = \frac{1}{a_n}[g_{2,\tau_n}(x,a_n\hat{u}_n,a_n\hat{v}_n)+h_2(x)] = A\hat{v}_{1,n}, \tag{14}
$$

such that

$$
\hat{u}_n = \hat{u}_{1,n}+\hat{u}_{2,n}, \quad \hat{u}_{2,n}\in N(A), \quad \hat{v}_n = \hat{v}_{1,n}+\hat{v}_{2,n}, \quad \hat{v}_{2,n}\in N(A).
$$

Using (12), we have for $(w,\widetilde{w})\in\widetilde{V}$:

$$
\frac{1}{a_n}\int_\Omega (g_{1\tau_n}(x,a_n\hat{u}_n,a_n\hat{v}_n)-g_{1,\tau_n}(x,a_nw,a_n\widetilde{w}))(\hat{u}_n-w)\,\mathrm{d}x \geq \delta_1\|\hat{u}_n-w\|_{L^2}^2, \tag{15}
$$

and

$$
\frac{1}{a_n}\int_\Omega (g_{2,\tau_n}(x,a_n\hat{u}_n,a_n\hat{v}_n)-g_{2,\tau_n}(x,a_nw,a_n\widetilde{w}))(\hat{v}_n-\widetilde{w})\,\mathrm{d}x \geq \delta_2\|\hat{v}_n-\widetilde{w}\|_{L^2}^2. \tag{16}
$$

But

$$
\begin{aligned}
\frac{1}{a_n}g_{1,\tau_n}(x,a_n\hat{u}_n,a_n\hat{v}_n) &= A\hat{u}_{1,n}-\frac{h_1(x)}{a_n},\\
\frac{1}{a_n}g_{2,\tau_n}(x,a_n\hat{u}_n,a_n\hat{v}_n) &= A\hat{v}_{1,n}-\frac{h_2(x)}{a_n},
\end{aligned}
\tag{17}
$$

hence, using (15), (16) and (17), we have

$$
\int_\Omega A\hat{u}_{1,n}(\hat{u}_n-w)\,\mathrm{d}x - \int_\Omega\left(\frac{h_1(x)}{a_n}+\frac{1}{a_n}g_{1,\tau_n}(x,a_nw,a_n\widetilde{w})\right)(\hat{u}_n-w)\,\mathrm{d}x \geq 0, \tag{18}
$$

and

$$\int_\Omega A\hat{v}_{1,n}(\hat{v}_n-\widetilde{w})\,\mathrm{d}x-\int_\Omega\left(\frac{h_2(x)}{a_n}+\frac{1}{a_n}g_{2,\tau_n}(x,a_nw,a_n\widetilde{w})\right)(\hat{v}_n-\widetilde{w})\,\mathrm{d}x\geq 0. \tag{19}$$

We write

$$w=w_1+w_2 \text{ with } w_1\in R(A),\ w_2\in N(A),$$

and

$$\widetilde{w}=\widetilde{w}_1+\widetilde{w}_2 \text{ with } \widetilde{w}_1\in R(A),\ \widetilde{w}_2\in N(A).$$

By (11), we have

$$\int_\Omega A\hat{u}_{1,n}(\hat{u}_n-w_1)\,\mathrm{d}x=\int_\Omega A\hat{u}_{1,n}(\hat{u}_{1,n}-w_1)\,\mathrm{d}x\to\int_\Omega A\hat{u}_1(\hat{u}_1-w_1)\,\mathrm{d}x,$$
$$\int_\Omega A\hat{v}_{1,n}(\hat{v}_n-\widetilde{w}_1)\,\mathrm{d}x=\int_\Omega A\hat{v}_{1,n}(\hat{v}_{1,n}-\widetilde{w}_1)\,\mathrm{d}x\to\int_\Omega A\hat{v}_1(\hat{v}_1-\widetilde{w}_1)\,\mathrm{d}x,$$

and (we put $y_{1,n}=a_nw_1$, $y_{2,n}=a_n\widetilde{w}_1$)

$$\begin{aligned}\frac{1}{a_n}[h_1(x)+g_{1,\tau_n}(x,a_nw,a_n\widetilde{w})] &= \frac{1}{a_n}[h_1(x)+g_{1,\tau_n}(x,a_nw_1,a_n\widetilde{w}_1)]\\ &= \frac{1}{a_n}[h_1(x)+\alpha_1y_{1,n}^+-\beta_1y_{1,n}^-\\ &\quad+\tau_nf_1(x,y_{1,n},y_{2,n})+(1-\tau_n)(\beta_1-\alpha_1)y_{1,n}^-],\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{1}{a_n}[h_2(x)+g_{2,\tau_n}(x,a_nw,a_n\widetilde{w})] &= \frac{1}{a_n}[h_2(x)+g_{2,\tau_n}(x,a_nw_1,a_n\widetilde{w}_1)]\\ &= \frac{1}{a_n}[h_2(x)+\alpha_2y_{2,n}^+-\beta_2y_{2,n}^-\\ &\quad+\tau_nf_2(x,y_{1,n},y_{2,n})+(1-\tau_n)(\beta_2-\alpha_2)y_{2,n}^-].\end{aligned} \tag{21}$$

From (4) and noticing that $(a+b)^2\leq 2(a^2+b^2)$, we obtain the following estimate

$$\begin{aligned}\int_\Omega|f_1(x,y_{1,n},y_{2,n})|^2dx &\leq \int_\Omega c'^2(1+|y_{1,n}|+|y_{2,n}|)^2dx\\ &\leq 2c'^2\int_\Omega((1+|y_{1,n}|)^2+|y_{2,n}|^2)dx\leq k_1(1+\|y_{1,n}\|^2+\|y_{2,n}\|^2).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\int_\Omega |f_2(x,y_{1,n},y_{2,n})|^2 dx &\leq \int_\Omega c''^2(1+|y_{1,n}|+|y_{2,n}|)^2 dx\\ &\leq 2c''^2\int_\Omega ((1+|y_{1,n}|)^2+|y_{2,n}|^2)dx \leq k_2(1+\|y_{1,n}\|^2+\|y_{2,n}\|^2),\end{aligned}$$

where k_1, k_2 are positive constants. Therefore

$$\int_Q \frac{|f_1(x,y_{1,n},y_{2,n})|^2}{a_n^2}dx \leq k_1\left(\frac{1}{a_n^2}+\frac{\|y_{1,n}\|^2}{a_n^2}+\frac{\|y_{2,n}\|^2}{a_n^2}\right),$$

and

$$\int_Q \frac{|f_2(x,y_{1,n},y_{2,n})|^2}{a_n^2}dx \leq k_2\left(\frac{1}{a_n^2}+\frac{\|y_{1,n}\|^2}{a_n^2}+\frac{\|y_{2,n}\|^2}{a_n^2}\right).$$

Then

$$\int_\Omega \frac{|f_1(x,y_{1,n},y_{2,n})|^2}{a_n^2}dx \leq k_1(1+\|w_1\|^2_{L^2(\Omega)}+\|\widetilde{w}_1\|^2_{L^2(\Omega)}),$$

$$\int_\Omega \frac{|f_2(x,y_{1,n},y_{2,n})|^2}{a_n^2}dx \leq k_2(1+\|w_1\|^2_{L^2(\Omega)}+\|\widetilde{w}_1\|^2_{L^2(\Omega)}),$$

that is, $\frac{f_1(x,y_{1,n},y_{2,n})}{a_n}$ and $\frac{f_2(x,y_{1,n},y_{2,n})}{a_n}$ are bounded in $L^2(\Omega)$. Moreover, by (10) we have

$$\frac{\|h_1\|_{L^2(\Omega)}}{a_n} \leq \frac{\|h_1\|_{L^2(\Omega)}}{n} \leq \|h_1\|_{L^2(\Omega)},$$

and

$$\frac{\|h_2\|_{L^2(\Omega)}}{a_n} \leq \frac{\|h_2\|_{L^2(\Omega)}}{n} \leq \|h_2\|_{L^2(\Omega)};$$

then the right hand side of (20) and (21) are bounded in $L^2(\Omega)$ for all n, thus

$$\frac{1}{a_n}\left[h_1(x)+\alpha_1 y_{1,n}^+-\beta_1 y_{1,n}^-+\tau_n f_1(x,y_{1,n},y_{2,n})+(1-\tau_n)(\beta_1-\alpha_1)y_{1,n}^-\right]\in L^2(\Omega),$$

and

$$\frac{1}{a_n}\left[h_2(x)(x)+\alpha_2 y_{2,n}^+-\beta_2 y_{2,n}^-+\tau_n f_2(x,y_{1,n},y_{2,n})+(1-\tau_n)(\beta_2-\alpha_2)y_{2,n}^-\right]\in L^2(\Omega).$$

Since $(w_1, \widetilde{w}_1) \in (R(A))^2$ and as $R(A)$ is closed in $L^2(\Omega)$, there exists $(\tau_n, w_{1n}, \widetilde{w}_{1n}) \in [0,1] \times (R(A))^2$ which converges to $(\tau, w, \widetilde{w})$. Consequently $(w_{1n}, \widetilde{w}_{1n})$ are bounded in $L^2(\Omega)$, then

$$\frac{f_1(x, y_{1n}, y_{2n})}{a_n} \leq c^{'}(1 + |w_{1n}| + |\widetilde{w}_{1n}|) \leq k' \text{ a.e. in } \Omega,$$

and

$$\frac{f_2(x, y_{1n}, y_{2n})}{a_n} \leq c^{''}(1 + |w_{1n}| + |\widetilde{w}_{1n}|) \leq k'' \text{ a.e. in } \Omega,$$

where k', k'' are real positive constants.

And from the hypotheses (5) and (6), it follows that

$$\begin{aligned} \frac{f_1(x, y_{1,n}, y_{2,n})}{a_n} &= \frac{y_{1,n}}{a_n} \frac{f_1(x, y_{1,n}, y_{2,n})}{y_{1,n}} \\ &= w_{1,n} \frac{f_1(x, a_n w_{1,n}, y_{2,n})}{a_n w_{1,n}} \underset{n \to \infty}{\to} 0 \text{ a.e. in } \Omega, \end{aligned}$$

and

$$\begin{aligned} \frac{f_2(x, y_{1,n}, y_{2,n})}{a_n} &= \frac{y_{2,n}}{a_n} \frac{f_2(x, y_{1,n}, y_{2,n})}{y_{2,n}} \\ &= \widetilde{w}_{1,n} \frac{f_2(x, y_{1,n}, a_n \widetilde{w}_{1,n})}{a_n \widetilde{w}_{1,n}} \underset{n \to \infty}{\to} 0 \text{ a.e. in } \Omega, \end{aligned}$$

Whereupon, using Lebesgue's convergence theorem,

$$\frac{f_1(x, y_{1,n}, y_{2,n})}{a_n} \to 0 \text{ in } L^2(\Omega), n \to \infty,$$

and

$$\frac{f_2(x, y_{1,n}, y_{2,n})}{a_n} \to 0 \text{ in } L^2(\Omega), n \to \infty.$$

Consequently

$$\frac{h_1(x)}{a_n} + \frac{1}{a_n} g_{1,\tau_n}(x, a_n w, a_n \widetilde{w}) \to \alpha_1 w^+ - \beta_1 w^- + (1-\tau)(\beta_1 - \alpha_1) w^-, \text{ in } L^2(\Omega),$$

and

$$\frac{h_2(x)}{a_n} + \frac{1}{a_n} g_{2,\tau_n}(x, a_n w, a_n \widetilde{w}) \to \alpha_2 \widetilde{w}^+ - \beta_2 \widetilde{w}^- + (1-\tau)(\beta_2 - \alpha_2) \widetilde{w}^-, \text{ in } L^2(\Omega),$$

and hence

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega\left(\frac{h_1(x)}{a_n}+\frac{1}{a_n}g_{1,\tau_n}(x,a_nw,a_n\widetilde{w})\right)(\hat{u}_n-w)dx \\ = \int_\Omega(\alpha_1w^+-\beta_1w^-+(1-\tau)(\beta_1-\alpha_1)w^-)(\hat{u}_n-w)dx,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega\left(\frac{h_2(x)}{a_n}+\frac{1}{a_n}g_{2,\tau_n}(x,a_nw,a_n\widetilde{w})\right)(\hat{v}_n-\widetilde{w})dx \\ = \int_\Omega(\alpha_2\widetilde{w}^+-\beta_1\widetilde{w}^-+(1-\tau)(\beta_2-\alpha_2)\widetilde{w}^-)(\hat{v}_n-\widetilde{w})dx.\end{aligned}$$

So, passing to the limit in (18) and (19), we have

$$\int_\Omega(A\hat{u}_1-(\alpha_1w^+-\beta_1w^-+(1-\tau)(\beta_1-\alpha_1)w^-)(\hat{u}-w)\,\mathrm{d}x\geq 0,$$

and

$$\int_\Omega(A\hat{v}_1-(\alpha_2\widetilde{w}^+-\beta_2\widetilde{w}^-+(1-\tau)(\beta_2-\alpha_2)\widetilde{w}^-)(\hat{v}-\widetilde{w})\,\mathrm{d}x\geq 0.$$

Using Minty's trick we replace w by $\hat{u}+\varepsilon_1w$ and $\widetilde{w}$ by $\hat{v}+\varepsilon_2\widetilde{w}$ and let $\varepsilon_i\to 0, i=1,2,$

$$\begin{aligned}A\hat{u}&=\alpha_1\hat{u}^+-\beta_1\hat{u}^-+(1-\tau)(\beta_1-\alpha_1)\hat{u}^-,\\ A\hat{v}&=\alpha_2\hat{v}^+-\beta_2\hat{v}^-+(1-\tau)(\beta_2-\alpha_2)\hat{v}^-.\end{aligned}\tag{22}$$

Now, coming back to (15) and (16), we put $w=\hat{u}$ in the first equation, and $\widetilde{w}=\hat{v}$ in the second equation and pass to the limit, we find

$$\overline{\lim_{n\to\infty}}\,\delta_1\|\hat{u}_n-\hat{u}\|^2\leq 0,$$

$$\overline{\lim_{n\to\infty}}\,\delta_2\|\hat{v}_n-\hat{v}\|^2\leq 0,$$

i.e.

$$\hat{u}_n\underset{n\to\infty}{\longrightarrow}\hat{u}\text{ in }L^2(\Omega),\quad \hat{v}_n\underset{n\to\infty}{\longrightarrow}\hat{v}\text{ in }L^2(\Omega),$$

$$A\hat{u}_n=\frac{1}{a_n}[g_{1,\tau_n}(x,a_n\hat{u}_n,a_n\hat{v}_n)+h_1(x)]\underset{n\to\infty}{\longrightarrow}\alpha_1\hat{u}^+-\beta_1\hat{u}^-+(1-\tau)(\beta_1-\alpha_1)\hat{u}^-\text{ in }L^2(\Omega),$$

$$A\hat{v}_n=\frac{1}{a_n}[g_{2,\tau_n}(x,a_n\hat{u}_n,a_n\hat{v}_n)+h_2(x)]\underset{n\to\infty}{\longrightarrow}\alpha_2\hat{v}^+-\beta_2\hat{v}^-+(1-\tau)(\beta_2-\alpha_2)\hat{v}^-\text{ in }L^2(\Omega),$$

so that $(\hat{u}_n, \hat{v}_n) \underset{n\to\infty}{\longrightarrow} (\hat{u}, \hat{v})$ in V and $\|(\hat{u}, \hat{v})\|_V = 1$.

Case I: $\int_\Omega \hat{u}\varphi dx = \int_\Omega \hat{v}\varphi dx = 0$.

By projecting on $\varphi^\perp$, we have

$$A\hat{u} = P[\alpha_1 \hat{u}^+ - \beta \hat{u}^- + (1-\tau)(\beta_1 - \alpha_1)\hat{u}^-],$$

$$A\hat{v} = P[\alpha_2 \hat{v}^+ - \beta_2 \hat{v}^- + (1-\tau)(\beta_2 - \alpha_2)\hat{v}^-],$$

and from proposition 2 ($s = 0, h_0 = 0$), we have $\hat{u} = \hat{v} = 0$, this is in contradiction with $\|(\hat{u}, \hat{v})\|_V = 1$.

Case II: $\int_\Omega \hat{u}\varphi dx = \theta_1 > 0$, and $\int_\Omega \hat{v}\varphi dx = \theta_2 > 0$.

Then $v_1 = \dfrac{\hat{u}}{\theta_1}, v_2 = \dfrac{\hat{v}}{\theta_2}$ verifies

$$Av_1 = \alpha_1 v_1^+ - \beta_1 v_1^- + (1-\tau)(\beta_1 - \alpha_1)v_1^-, \quad \int_\Omega v_1 \varphi dx = 1,$$

$$Av_2 = \alpha_2 v_2^+ - \beta_2 v_2^- + (1-\tau)(\beta_2 - \alpha_2)v_2^-, \quad \int_\Omega v_2 \varphi dx = 1.$$

From proposition 1, we deduce that

$$C(\alpha_1, \beta_1 + (1-\tau)(\alpha_1 - \beta_1) = 0,$$

$$C(\alpha_2, \beta_2 + (1-\tau)(\alpha_2 - \beta_2) = 0.$$

As the function $C(.,.)$ is strictly decreasing with respect to each variable, with $\alpha_i < \beta_i, i = 1,2$ and $\tau < 1$, we have

$$C(\alpha_1, \beta_1 + (1-\tau)(\alpha_1 - \beta_1) > C(\alpha_1, \beta_1) = 0,$$

$$C(\alpha_2, \beta_2 + (1-\tau)(\alpha_2 - \beta_2) > C(\alpha_2, \beta_2) = 0,$$

which is a contradiction.

Case III: $\int_\Omega \hat{u}\varphi dx = \theta_1 < 0$, and $\int_\Omega \hat{v}\varphi dx = \theta_2 < 0$.

Then $v_1 = \dfrac{\hat{u}}{\theta_1}, v_2 = \dfrac{\hat{v}}{\theta_2}$, we obtain a contradiction with the similar argument as in the above step. Hence we have $\tau = 1$ and

$$\begin{aligned} &A\hat{u} = \alpha_1 \hat{u}^+ - \beta_1 \hat{u}^-, (\hat{u}, \hat{v}) \in N(\alpha_1, \beta_1), \\ &A\hat{v} = \alpha_2 \hat{v}^+ - \beta_2 \hat{v}^-, (\hat{u}, \hat{v}) \in N(\alpha_2, \beta_2). \end{aligned} \tag{23}$$

Then, we can write

$$\hat{u} = c_1 \nu_1 \quad \text{if} \quad c_1 = \int_\Omega \hat{u}\varphi \,\mathrm{d}x > 0,$$
$$\hat{v} = c_2 \nu_2 \quad \text{if} \quad c_2 = \int_\Omega \hat{v}\varphi \,\mathrm{d}x > 0,$$

and

$$\hat{u} = c_1 \nu_3 \quad \text{if} \quad -c_1 = \int_\Omega \hat{u}\varphi \,\mathrm{d}x < 0,$$
$$\hat{v} = c_2 \nu_4 \quad \text{if} \quad -c_2 = \int_\Omega \hat{v}\varphi \,\mathrm{d}x < 0.$$

We assume first that

$$\int_\Omega \hat{u}\varphi \,\mathrm{d}x < 0, \text{ and } \int_\Omega \hat{v}\varphi \,\mathrm{d}x < 0,$$

and we define

$$c_{1,n} \in \mathbb{R}, \quad X_{1,n} \in D(A), \quad c_{1,n} = -\int_\Omega \hat{u}_n \varphi \,\mathrm{d}x, \quad X_{1,n} = \hat{u}_n - c_{1,n}\nu_3,$$

$$c_{2,n} \in \mathbb{R}, \quad X_{2,n} \in D(A), \quad c_{2,n} = -\int_\Omega \hat{v}_n \varphi \,\mathrm{d}x, \quad X_{2,n} = \hat{v}_n - c_{2,n}\nu_4,$$

in such a way that

$$\hat{u}_n = c_{1,n}\nu_3 + X_{1,n}, \quad c_{1,n} \to c_1 > 0, \quad \|X_{1,n}\|_{D(A)} \to 0, \quad X_{1,n} \in \varphi^\perp,$$

$$\hat{v}_n = c_{2,n}\nu_4 + X_{2,n}, \quad c_{2,n} \to c_2 > 0, \quad \|X_{2,n}\|_{D(A)} \to 0, \quad X_{2,n} \in \varphi^\perp.$$

We claim that

$$\exists K > 0 \text{ such that for any } n \geq 1, \quad a_n \|X_{i,n}\|_{D(A)} \leq K, \quad i = 1,2. \tag{24}$$

Supposing that (24) is established and multiplying (13) on both sides by ν_3, we obtain

$$\begin{aligned} a_n \int_\Omega A\hat{u}_n \nu_3 dx &= a_n \int_\Omega \left(\alpha_1 \hat{u}_n^+ - \beta_1 \hat{u}_n^-\right) \nu_3 dx + (1-\tau_n)(\beta_1 - \alpha_1) a_n \int_\Omega \hat{u}_n^- \nu_3 dx \\ &+ \tau_n \int_\Omega f_1(x, a_n\hat{u}_n, a_n\hat{v}_n)\nu_3 dx + \int_\Omega h_1(x)\nu_3 dx. \end{aligned}$$

For n large enough, $\int_\Omega \hat{u}_n^- \nu_3 \leq 0$, because $\hat{u}_n^- \to c_1 \nu_3^-$ in $L^2, c_1 > 0$, hence

$$\begin{aligned} &\int_\Omega h_1(x)\nu_3 \,\mathrm{d}x + \tau_n \int_\Omega f_1(x, a_n\hat{u}_n, a_n\hat{u}_n)\nu_3 \,\mathrm{d}x \\ &\geq a_n \int_\Omega A\hat{u}_n \nu_3 dx - a_n \int_\Omega (\alpha_1 \hat{u}_n^+ - \beta_1 \hat{u}_n^-)\nu_3 dx. \end{aligned} \tag{25}$$

Using the same arguments with (14), we obtain

$$\int_\Omega h_2(x)v_4dx+\tau_n\int_\Omega f_2(x,a_n\hat{u}_n,a_n\hat{v}_n)v_4dx \geq a_n\int_\Omega A\hat{v}_nv_4dx-a_n\int_\Omega(\alpha_2\hat{v}_n^+-\beta_2\hat{v}_n^-v_4)dx. \tag{26}$$

Noticing that

$$\begin{aligned}\mathscr{E}_{1,n}&=\int_\Omega A\hat{u}_nv_3\,\mathrm{d}x-\int_\Omega\left(\alpha_1\hat{u}_n^+-\beta_1\hat{u}_n^-\right)v_3\,\mathrm{d}x,\\ \mathscr{E}_{2,n}&=\int_\Omega A\hat{v}_nv_4\,\mathrm{d}x-\int_\Omega\left(\alpha_2\hat{v}_n^+-\beta_2\hat{v}_n^-\right)v_4\,\mathrm{d}x,\end{aligned}\tag{27}$$

and because $(A=A^*)$

$$\mathscr{E}_{1,n}=\int_\Omega\hat{u}_n(Av_3)\,\mathrm{d}x-\int_\Omega\left(\alpha_1\hat{u}_n^+-\beta_1\hat{u}_n^-\right)v_3\,\mathrm{d}x,$$
$$\mathscr{E}_{2,n}=\int_\Omega\hat{v}_n(Av_4)\,\mathrm{d}x-\int_\Omega\left(\alpha_2\hat{v}_n^+-\beta_2\hat{v}_n^-\right)v_4\,\mathrm{d}x.$$

Then

$$\mathscr{E}_{1,n}=\int_\Omega\hat{u}_n(\alpha_1v_3^+-\beta_1v_3^-)\,\mathrm{d}x-\int_\Omega\left(\alpha_1\hat{u}_n^+-\beta_1\hat{u}_n^-\right)v_3\,\mathrm{d}x,$$
$$\mathscr{E}_{2,n}=\int_\Omega\hat{v}_n(\alpha_2v_4^+-\beta_2v_4^-)\,\mathrm{d}x-\int_\Omega\left(\alpha_2\hat{v}_n^+-\beta_2\hat{v}_n^-\right)v_4\,\mathrm{d}x,$$

that is

$$\mathscr{E}_{1,n}=\alpha_1\int_\Omega(\hat{u}_n^+v_3^--\hat{u}_n^-v_3^+)\,\mathrm{d}x-\beta_1\int_\Omega(\hat{u}_n^+v_3^--\hat{u}_n^-v_3^+)\,\mathrm{d}x,$$
$$\mathscr{E}_{2,n}=\alpha_2\int_\Omega(\hat{v}_n^+v_4^--\hat{v}_n^-v_4^+)\,\mathrm{d}x-\beta_2\int_\Omega(\hat{v}_n^+v_4^--\hat{v}_n^-v_4^+)\,\mathrm{d}x,$$

and we have

$$\left|\mathscr{E}_{1,n}\right|\leq|\beta_1-\alpha_1|\left(\int_\Omega\hat{u}_n^+v_3^-\,\mathrm{d}x+\int_\Omega\hat{u}_n^-v_3^+\,\mathrm{d}x\right),\tag{28}$$

$$\left|\mathscr{E}_{2,n}\right|\leq|\beta_2-\alpha_2|\left(\int_\Omega\hat{v}_n^+v_4^-\,\mathrm{d}x+\int_\Omega\hat{v}_n^-v_4^+\,\mathrm{d}x\right).\tag{29}$$

If $x\in\Omega$ is such that

$$v_3(x)\geq0\text{ and }\hat{u}_n(x)=c_{1,n}v_3(x)+X_{1,n}(x)\leq0,$$

$$v_4(x)\geq0\text{ and }\hat{v}_n(x)=c_{2,n}v_4(x)+X_{2,n}(x)\leq0,$$

then

$$X_{1,n}(x) \leq \hat{u}_n(x) \leq 0 \quad \text{and} \quad 0 \leq v_3(x) = \frac{\hat{u}_n(x) - X_{1,n}(x)}{c_{1,n}} \leq \frac{|X_{1,n}(x)|}{c_{1,n}},$$

$$X_{2,n}(x) \leq \hat{v}_n(x) \leq 0 \quad \text{and} \quad 0 \leq v_4(x) = \frac{\hat{v}_n(x) - X_{2,n}(x)}{c_{2,n}} \leq \frac{|X_{2,n}(x)|}{c_{2,n}},$$

we obtain

$$\hat{u}_n^-(x) v_3^+(x) \leq \frac{|X_{1,n}(x)|^2}{c_{1,n}} \text{ a.e. in } \Omega, \tag{30}$$

$$\hat{v}_n^-(x) v_4^+(x) \leq \frac{|X_{2,n}(x)|^2}{c_{2,n}} \text{ a.e. in } \Omega. \tag{31}$$

Using the same arguments, one can see that

$$\hat{u}_n^+(x) v_3^-(x) \leq \frac{|X_{1,n}(x)|^2}{c_{1,n}} \text{ a.e. in } \Omega, \tag{32}$$

$$\hat{v}_n^+(x) v_4^-(x) \leq \frac{|X_{2,n}(x)|^2}{c_{2,n}} \text{ a.e. in } \Omega. \tag{33}$$

From (28), (29), (30), (31), (32) and (33) we deduce

$$|\mathscr{E}_{1,n}| \leq \frac{2}{c_{1,n}} |\beta_1 - \alpha_1| \|X_{1,n}\|^2_{L^2(\Omega)}, \tag{34}$$

$$|\mathscr{E}_{2,n}| \leq \frac{2}{c_{2,n}} |\beta_2 - \alpha_2| \|X_{2,n}\|^2_{L^2(\Omega)}. \tag{35}$$

Hence, (24) implies that

$$a_n |\mathscr{E}_{1,n}| \leq \frac{2K}{c_{1,n}} |\beta_1 - \alpha_1| \|X_{1,n}\|_{D(A)},$$

$$a_n |\mathscr{E}_{2,n}| \leq \frac{2K}{c_{2,n}} |\beta_2 - \alpha_2| \|X_{2,n}\|_{D(A)},$$

therefore $\lim_{n\to\infty} a_n |\mathscr{E}_{1,n}| = 0$, and $\lim_{n\to\infty} a_n |\mathscr{E}_{2,n}| = 0$.
Now, coming back to (25), (26), we have by (5) and (6)

$$a_n \mathscr{E}_{1,n} \leq \tau_n \int_\Omega \zeta_1^+ v_3^+ \,\mathrm{d}x - \int_\Omega \zeta_1^- v_3^- \,\mathrm{d}x + h_1(x) v_3 \,\mathrm{d}x,$$

$$a_n \mathscr{E}_{2n} \leq \tau_n \int_\Omega \zeta_2^+ v_4^+ \,\mathrm{d}x - \int_\Omega \zeta_2^- v_4^- \,\mathrm{d}x + h_2(x) v_4 \,\mathrm{d}x,$$

and passing to the limit, we find

$$0=\lim_{n\to\infty}a_n\mathscr{E}_{1,n}\le\int_\Omega\zeta_1^+\nu_3^+\,\mathrm{d}x-\int_\Omega\zeta_1^-\nu_3^-\,\mathrm{d}x+\int_\Omega h_1(x)\nu_3\,\mathrm{d}x=T_3(h_1),$$
$$0=\lim_{n\to\infty}a_n\mathscr{E}_{2,n}\le\int_\Omega\zeta_2^+\nu_4^+\,\mathrm{d}x-\int_\Omega\zeta_2^-\nu_4^-\,\mathrm{d}x+\int_\Omega h_2(x)\nu_4\,\mathrm{d}x=T_4(h_2),$$

which is in contradiction with $T_3(h_1)<0$ and $T_4(h_2)<0$.

Now, we assume that,

$$\int_\Omega\hat{u}\varphi\,\mathrm{d}x>0\text{ and }\int_\Omega\hat{v}\varphi\,\mathrm{d}x>0,$$

defining

$$c_{1,n}\in\mathbb{R},\quad X_{1,n}\in D(A),\quad c_{1,n}=\int_\Omega\hat{u}_n\varphi\,\mathrm{d}x,\quad X_{1,n}=\hat{u}_n-c_{1,n}\nu_1,$$
$$c_{2,n}\in\mathbb{R},\quad X_{2,n}\in D(A),\quad c_{2,n}=\int_\Omega\hat{v}_n\varphi\,\mathrm{d}x,\quad X_{2,n}=\hat{v}_n-c_{2,n}\nu_2,$$

in such a way that

$$\hat{u}_n=c_{1,n}\nu_1+X_{1,n},\quad c_{1,n}\to c_1>0,\quad \|X_{1,n}\|_{D(A)}\to 0,\quad X_{1,n}\in\varphi^\perp,$$
$$\hat{v}_n=c_{2,n}\nu_2+X_{2,n},\quad c_{2,n}\to c_2>0,\quad \|X_{2,n}\|_{D(A)}\to 0,\quad X_{2,n}\in\varphi^\perp,$$

multiplying (13) on both sides by ν_1 and (14) by ν_2 and by the same arguments used in the previous step, we obtain

$$\begin{aligned}\int_\Omega h_1(x)\nu_1dx+\tau_n\int_\Omega f_1(x,a_n\hat{u}_n,a_n\hat{v}_n)\nu_1dx\\ \ge\ a_n\int_\Omega A\hat{u}_n\nu_1-a_n\int_\Omega(\alpha_1\hat{u}_n^+-\beta_1\hat{u}_n^-)\nu_1dx,\end{aligned}\tag{36}$$

and

$$\begin{aligned}\int_\Omega h_2(x)\nu_2dx+\tau_n\int_\Omega f_2(x,a_n\hat{u}_n,a_n\hat{v}_n)\nu_2dx\\ \ge\ a_n\int_\Omega A\hat{u}_n\nu_2-a_n\int_\Omega(\alpha_2\hat{v}_n^+-\beta_2\hat{v}_n^-)\nu_2dx,\end{aligned}\tag{37}$$

leading to a contradiction as $T_1(h_1)<0$ and $T_2(h_2)<0$.

Now, if (24) does not hold, then there exists a subsequence denoted by $a_n\|X_n\|_V$ such that $\lim\limits_{n\to\infty}a_n\|X_n\|_V\to\infty$. Let

$$b_n=\|X_n\|_V,\quad z_n=(\widetilde{z}_n,\hat{z}_n)=\left(\frac{X_{1,n}}{\|X_n\|_V},\frac{X_{2,n}}{\|X_n\|_V}\right),$$

$z_n = (\widetilde{z}_n, \hat{z}_n) \in V, \quad \|(\widetilde{z}_n, \hat{z}_n)\|_V = 1$. We notice that

$$z_{1,n} = (\widetilde{z}_{1,n}, \hat{z}_{1,n}) = (I - F)\big(\frac{X_{1,n}}{b_n}, \frac{X_{2,n}}{b_n}\big) \in (D(A) \cap R(A))^2,$$

and

$$z_{2,n} = (\widetilde{z}_{2,n}, \hat{z}_{2,n}) = F\big(\frac{X_{1,n}}{b_n}, \frac{X_{2,n}}{b_n}\big) \in (N(A))^2.$$

The inclusion $D(A) \cap R(A) \hookrightarrow L^2(\Omega)$ being compact, then there is a subsequence still denoted by $(\widetilde{z}_{1,n}, \hat{z}_{1,n})$ such that

$$(\widetilde{z}_{1,n}, \hat{z}_{1,n}) \to (\widetilde{z}_1, \hat{z}_1) \text{ in } \widetilde{V}, \quad Az_{1,n} \to Az_1 \text{ in } \widetilde{V} \text{ weak }, \quad z \in (\varphi^{\perp})^2 \tag{38}$$

$$z_{1,n}(x) \to z_1(x) \quad \text{a.e. in } \Omega \text{ and there exists } (c_1, c_2) \in \widetilde{V} \text{ such that}$$

$$|\widetilde{z}_{1,n}(x)| \leq c_1(x) \text{ a.e.} \quad |\hat{z}_{1,n}(x)| \leq c_2(x) \text{ a.e.}$$

On the other hand $\hat{u}_n = c_{1,n}v_3 + X_{1,n}, \quad \hat{v}_n = c_{2,n}v_4 + X_{2,n}$, satisfy

$$A\hat{u}_n = \alpha_1 \hat{u}_n^+ - \beta_1 \hat{u}_n^- + \tau_n \frac{f_1(x, a_n\hat{u}_n, a_n\hat{v}_n)}{a_n} + (1-\tau_n)(\beta_1 - \alpha_1)\hat{u}_n^- + \frac{h_1}{a_n},$$
$$A\hat{v}_n = \alpha_2 \hat{v}_n^+ - \beta_2 \hat{v}_n^- + \tau_n \frac{f_2(x, a_n\hat{u}_n, a_n\hat{v}_n)}{a_n} + (1-\tau_n)(\beta_2 - \alpha_2)\hat{v}_n^- + \frac{h_2}{a_n}.$$

Multiplying the first equation by v_3/b_n, and the second equation by v_4/b_n, we have

$$\begin{aligned}
\frac{1}{b_n}\int_\Omega A\hat{u}_n v_3 \, dx &= \frac{1}{b_n}\int_\Omega (\alpha_1 \hat{u}_n^+ - \beta_1 \hat{u}_n^+) v_3 \, dx + \tau_n \int_\Omega \frac{f_1(x, a_n\hat{u}_n, a_n\hat{v}_n)}{a_n b_n} v_3 \, dx \\
&\quad + \frac{(1-\tau_n)}{b_n}(\beta_1 - \alpha_1)\int_\Omega \hat{u}_n^- v_3 \, dx + \int_\Omega \frac{h_1}{a_n b_n} v_3 \, dx, \\
\frac{1}{b_n}\int_\Omega A\hat{v}_n v_4 \, dx &= \frac{1}{b_n}\int_\Omega (\alpha_2 \hat{v}_n^+ - \beta_2 \hat{v}_n^+) v_4 \, dx + \tau_n \int_\Omega \frac{f_2(x, a_n\hat{u}_n, a_n\hat{v}_n)}{a_n b_n} v_4 \, dx \\
&\quad + \frac{(1-\tau_n)}{b_n}(\beta_2 - \alpha_2)\int_\Omega \hat{v}_n^- v_4 \, dx + \int_\Omega \frac{h_2}{a_n b_n} v_4 \, dx.
\end{aligned}$$

Using the fact that $A = A^*$ and 27, we get

$$\frac{1}{b_n}\mathscr{E}_{1,n} = \frac{(1-\tau_n)}{b_n}(\beta_1 - \alpha_1)\int_\Omega \hat{u}_n^- v_3 \, dx + \tau_n \int_\Omega \frac{f_1(x, a_n\hat{u}_n, a_n\hat{v}_n)}{a_n b_n} v_3 \, dx + \int_\Omega \frac{h_1}{a_n b_n} v_3 \, dx,$$

and

$$\frac{1}{b_n}\mathscr{E}_{2,n} = \frac{(1-\tau_n)}{b_n}(\beta_2 - \alpha_2)\int_\Omega \hat{v}_n^- v_4 \, dx + \tau_n \int_\Omega \frac{f_2(x, a_n\hat{u}_n, a_n\hat{v}_n)}{a_n b_n} v_4 \, dx + \int_\Omega \frac{h_2}{a_n b_n} v_4 \, dx.$$

Then

$$\frac{(1-\tau_n)}{b_n}(\beta_1-\alpha_1)\int_\Omega \hat{u}_n^- v_3\,\mathrm{d}x=\frac{1}{b_n}\mathscr{E}_{1,n}-\tau_n\int_\Omega \frac{f_1(x,a_n\hat{u}_n,a_n\hat{v}_n)}{a_nb_n}v_3\,\mathrm{d}x-\int_\Omega \frac{h_1}{a_nb_n}v_3\,\mathrm{d}x,$$

and

$$\frac{(1-\tau_n)}{b_n}(\beta_2-\alpha_2)\int_\Omega \hat{v}_n^- v_4\,\mathrm{d}x=\frac{1}{b_n}\mathscr{E}_{2,n}-\tau_n\int_\Omega \frac{f_2(x,a_n\hat{u}_n,a_n\hat{v}_n)}{a_nb_n}v_4\,\mathrm{d}x-\int_\Omega \frac{h_2}{a_nb_n}v_4\,\mathrm{d}x.$$

From (4), (5), (6), (34) and (35), we conclude that

$$(\beta_1-\alpha_1)\lim_{n\to\infty}\frac{(1-\tau_n)}{b_n}\int_\Omega \hat{u}_n^- v_3\,\mathrm{d}x=0,$$

$$(\beta_2-\alpha_2)\lim_{n\to\infty}\frac{(1-\tau_n)}{b_n}\int_\Omega \hat{v}_n^- v_4\,\mathrm{d}x=0,$$

such that

$$\lim_{n\to\infty}\int_\Omega \hat{u}_n^- v_3\,\mathrm{d}x=-c_1\int_\Omega |v_3^-|^2\,\mathrm{d}x,$$

$$\lim_{n\to\infty}\int_\Omega \hat{v}_n^- v_4\,\mathrm{d}x=-c_2\int_\Omega |v_4^-|^2\,\mathrm{d}x.$$

Since v_i satisfies (7) and $\alpha_i\notin Sp(A)$, then $v_i\neq 0$. As $\beta_i-\alpha_i\neq 0$, we find that

$$\lim_{n\to\infty}\frac{(1-\tau_n)}{b_n}=0. \tag{39}$$

From (7), (13), (14) and

$$\hat{u}_n=c_{1,n}v_3+X_{1,n},\quad \hat{v}_n=c_{2,n}v_4+X_{2,n},$$

we obtain

$$\begin{aligned}A\widetilde{z}_{1,n}&=\alpha_1\Big(\big(\frac{c_{1,n}}{b_n}v_3+\widetilde{z}_{1n}\big)^+-\frac{c_{1,n}}{b_n}v_3^+\Big)-\beta_1\Big(\big(\frac{c_{1,n}}{b_n}v_3+\widetilde{z}_{1,n}\big)^--\frac{c_{1,n}}{b_n}v_3^-\Big)\\&\quad+\tau_n\frac{f_1(x,a_n\hat{u}_n,a_n\hat{v}_n)}{a_nb_n}+\frac{1-\tau_n}{b_n}(\beta_1-\alpha_1)\hat{u}_n^-+\frac{h_1}{a_nb_n},\end{aligned} \tag{40}$$

$$\begin{aligned}A\hat{z}_{1,n}&=\alpha_2\Big(\big(\frac{c_{2,n}}{b_n}v_4+\hat{z}_{1,n}\big)^+-\frac{c_{2,n}}{b_n}v_4^+\Big)-\beta_2\Big(\big(\frac{c_{2,n}}{b_n}v_4+\hat{z}_{1,n}\big)^--\frac{c_{2,n}}{b_n}v_4^-\Big)\\&\quad+\tau_n\frac{f_2(x,a_n\hat{u}_n,a_n\hat{v}_n)}{a_nb_n}+\frac{1-\tau_n}{b_n}(\beta_2-\alpha_2)\hat{v}_n^-+\frac{h_2}{a_nb_n},\end{aligned} \tag{41}$$

when $n \to \infty$, $a_n b_n$ goes to infinity, and the last three terms of (40) and (41) converge to zero in $L^2(\Omega)$, it is easy to see that

$$
\begin{aligned}
&|\big(\frac{c_{1,n}}{b_n}v_3+\widetilde{z}_{1,n}\big)^+ - \frac{c_{1,n}}{b_n}v_3^+| \le |\widetilde{z}_{1,n}| \le c_1 \text{ a.e.},\\
&|\big(\frac{c_{1,n}}{b_n}v_3+\widetilde{z}_{1,n}\big)^- - \frac{c_{1,n}}{b_n}v_3^-| \le |\widetilde{z}_{1,n}| \le c_1 \text{ a.e.},\\
&|\big(\frac{c_{2,n}}{b_n}v_4+\hat{z}_{1,n}\big)^+ - \frac{c_{2,n}}{b_n}v_4^+| \le |\hat{z}_{1,n}| \le c_2 \text{ a.e.},\\
&|\big(\frac{c_{2,n}}{b_n}v_4+\hat{z}_{1,n}\big)^- - \frac{c_{2,n}}{b_n}v_4^-| \le |\hat{z}_{1,n}| \le c_2 \text{ a.e.}
\end{aligned} \tag{42}
$$

Moreover, extracting a subsequence, we may assume that the last three terms of the two equations (40) and (41) go to zero a.e. in Ω and there exists $(c_1', c_2') \in L^2(\Omega)\times L^2(\Omega)$ such that

$$|\tau_n \frac{f_1(x,a_n\hat{u}_n,a_n\hat{v}_n)}{a_nb_n} + \frac{1-\tau_n}{b_n}(\beta_1-\alpha_1)\hat{u}_n^- + \frac{h_1}{a_nb_n}| \le c_1' \text{ a.e. in } \Omega,$$

and

$$|\tau_n \frac{f_2(x,a_n\hat{u}_n,a_n\hat{v}_n)}{a_nb_n} + \frac{1-\tau_n}{b_n}(\beta_2-\alpha_2)\hat{v}_n^- + \frac{h_2}{a_nb_n}| \le c_2' \text{ a.e. in } \Omega.$$

Then applying (40), (41), (42) and the above inequalities, we obtain

$$
\begin{aligned}
&|A\widetilde{z}_{1,n}(x)| \le 2\max(|\alpha_1|,|\beta_1|)c_1(x) + c_1'(x),\\
&|A\hat{z}_{1,n}(x)| \le 2\max(|\alpha_2|,|\beta_2|)c_2(x) + c_2'(x).
\end{aligned} \tag{43}
$$

Let $\rho(x)$ be defined a.e. in Ω as follows

$$\rho(x) = \begin{cases} \alpha_i & \text{if } v_i > 0, \quad \text{or if } v_i = 0 \text{ and } z_i(x) \ge 0,\\ \beta_i & \text{if } v_i < 0, \quad \text{or if } v_i = 0 \text{ and } z_i(x) < 0.\end{cases}$$

From (40), (41) and the fact that $b_n \to 0$ one can see that

$$A\widetilde{z}_{1,n}(x) \to \rho(x)\widetilde{z}_1(x) \text{ a.e. in } \Omega,$$

$$A\hat{z}_{1,n}(x) \to \rho(x)\hat{z}_1(x) \text{ a.e. in } \Omega.$$

Using Lebesgue dominated convergence theorem and (43), we conclude that

$$A\widetilde{z}_{1,n} \overset{L^2(\Omega)}{\longrightarrow} \rho\widetilde{z}_1, \quad A\hat{z}_{1,n} \overset{L^2(\Omega)}{\longrightarrow} \rho\hat{z}_1,$$

$$Az_n \overset{(L^2(\Omega))^2}{\longrightarrow} \rho z, \quad z_n \overset{(L^2(\Omega))^2}{\longrightarrow} z.$$

The operator M being closed, we have

$$Az = \rho z, \quad z \in \varphi^{\perp}, \quad \|z\|_V = 1.$$

Since ρ satisfies: $\overline{\lambda} < \alpha_i \leq \rho \leq \beta_i < \underline{\lambda}$ thanks to proposition 2, we conclude that $z = 0$, this is in contradiction with $\|z\|_V = 1$, and hence, (24) is established. □

Now, we give the proof of our main result.

Proof of Theorem (1). Let

$$B(0,R) = \{(u,v) \in V, \|(u,v)\|_V < R\}.$$

By invariance of the topological degree, for $t \in [0,1]$, $\deg(H(t,\cdot,\cdot), B(0,R), 0)$ is constant. In particular, if $t = 0$ we have

$$H(0,u,v) = \begin{pmatrix} H_1(0,u,v) \\ H_2(0,u,v) \end{pmatrix} = \begin{pmatrix} D_{1,0}^{-1} & 0 \\ 0 & D_{2,0}^{-1} \end{pmatrix} \begin{pmatrix} S_1(0,u,v) + Bh_1 \\ S_2(0,u,v) + Bh_2 \end{pmatrix}.$$

On the other hand, for $t = 0$, the linear problem

$$Au = \alpha_1 u + h_1 \text{ in } \Omega,$$

$$Av = \alpha_2 v + h_2 \text{ in } \Omega,$$

$$u = v = 0 \text{ on } \partial\Omega.$$

admits a unique solution $(u,v) \in V$.

By the homotopy invariance property, we have

$$\deg(H(0,\cdot,\cdot), B(0,R), D^{-1}Bh)$$

$$= \deg(H(1,\cdot,\cdot), B(0,R), D^{-1}Bh) = \pm 1.$$

This completes the proof. □

4. Conclusions

We considered the existence of nontrivial solutions for nonlinear system with non-compact resolvent operators at resonance for jumping non-linearities. By using the Leray-Schauder degree theory, we obtained the existence results of our problem when the resolvent of our operator is non-compact. Some future works include similar problems with variable growth, fractional derivatives models and an application in image processing.

Conflict of Interests

The authors declare that there is no conflict of interests.

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