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LIMIT CYCLES OF DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY LINEAR AND CUBIC ISOCHRONOUS CENTERS

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Abstract. One of the main problems in the qualitative theory of the planar differential systems is to control the existence and the number of their limit cycles. There are many researchers who tried to solve this problem for special classes of planar differential systems, see for instance [7, 16].

In this paper, we study the maximum number of limit cycles for discontinuous planar piecewise differential systems formed by four classes of isochronous cubic centers separated by irregular straight line. We provide a sharp upper bound for the maximum number of crossing limit cycles that these classes of discontinuous piecewise differential systems can exhibit. Therefore, we will solve the extended of the 16th Hilbert problem for these classes.

Keywords: limit cycles; cubic isochronous centers; linear differential center.

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1. Introduction

In this paper we deal with the piecewise differential systems defined by

$$F(\mathbf{x}) = \begin{cases} F^-(\mathbf{x}) = (F_1^-(x,y), F_2^-(x,y)) & (x,y) \in \Sigma^-, \\ F^+(\mathbf{x}) = (F_1^+(x,y), F_2^+(x,y)) & (x,y) \in \Sigma^+, \end{cases} \tag{1}$$

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where $\Sigma = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_1 = \{(x,y) \in \mathbb{R}^2 : x = 0 \ \ and \ \ y \geq 0\}$, $\Gamma_2 = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \ \ and \ \ y = 0\}$, when the separation curve $\Sigma^+ = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and $\Sigma^- = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y < 0\} \cup \{(x,y) \in \mathbb{R}^2 : x < 0\}$.

We can divide the discontinuity line Σ into important subsets:

(a) Crossing set $\Sigma^c = \{p \in \Sigma : \mathbf{F}^-(\mathbf{p})\ \Sigma(p)\ .\ \mathbf{F}^+(\mathbf{p})\ \Sigma(p) > 0\}$.

(b) Sliding set $\Sigma^s = \{p \in \Sigma : \mathbf{F}^-(\mathbf{p})\ \Sigma(p) > 0 \ \ and \ \ \mathbf{F}^+(\mathbf{p})\ \Sigma(p) < 0\}$.

(c) Escaping set $\Sigma^e = \{p \in \Sigma : \mathbf{F}^-(\mathbf{p})\ \Sigma(p) < 0 \ \ and \ \ \mathbf{F}^+(\mathbf{p})\ \Sigma(p) > 0\}$.

Follow the Filippov's convention for defining the discontinuous piecewise differential system, see [12]. That a limit cycle of system (1) is crossing if it shares no points with the sliding set of the system. In this paper, we work only with crossing limit cycles.

In 1920 Andronov, Vitt and Khaikin [1] started the study of the piecewise differential systems separated by a straight line and nowadays such systems have deserved the attention of many researchers. These differential systems are used extensively to model biological process as well as some electronics and mechanical applications see for instance [11, 19].

A limit cycle is a periodic orbit of a differential system isolated in the set of all periodic orbits of such system. This concept was defined by Poincaré [23, 24] at the end of the 19th century. The study of the existence and the number of limit cycles for the discontinuous piecewise differential systems is one of the main problem. Thus limit cycles have played and still playing an important role in physical phenomena, see for instance [20, 21, 22, 25].

In [13] Han and Zhang conjectured that discontinuous piecewise linear differential systems in the plane separated by a straight line have at most two crossing limit cycles, but in [14] Llibre and Ponce provided a negative answer to this conjecture by presenting an example with exactly three limit cycles. Many papers are dedicated to study the existence of limit cycles for the piecewise linear differential systems, when the curve of separation is either a straight line, or an algebraic curve, see [2, 4, 5, 6, 9, 10, 17, 18]. In the literature we find many papers interesting in solving the second part of the sixteenth Hilbert problem for linear discontinuous piecewise differential systems, but few papers devoted to solve this problem for nonlinear piecewise differential systems.

In 2020 Benterki and Llibre [3] studied the sixteenth Hilbert problem for discontinuous piecewise differential systems separated by a straight line, when these differential systems are linear centers or three families of cubic isochronous centers, and they proved that the maximum number of limit cycles varies from 0, 1 and 2 depending on the chosen class.

In this paper, we deal with the following four classes of isochronous linear and cubic centers.

Lemma 1. *After a linear change of variables and a rescaling of the independent variable every linear center in* $\mathbb{R}^2$ *can be written as*

$$\dot{x} = -Ax - (A^2+\omega^2)y + B, \qquad \dot{y} = x + Ay + C, \tag{2}$$

with $\omega > 0$, A, B, $C \in \mathbb{R}$ *and* $A \neq 0$.
The first integral of this system is

$$H(x,y) = (Ay+x)^2 + 2(Cx - By) + y^2\omega^2. \tag{3}$$

Or we can define the linear differential center as follows

$$\dot{x} = -Ax - (A^2+\omega^2)y, \qquad \dot{y} = x + Ay, \tag{4}$$

with $\omega > 0$, $A \in \mathbb{R} - \{0\}$, *and its corresponding first integral is*

$$H_1(x,y) = (Ay+x)^2 + y^2\omega^2. \tag{5}$$

For a proof of Lemma 1 see [15].

Now we give the three classes of isochronous cubic differential centers.

I) The first class is given by:

$$\dot{x} = y\left(2K_1x + 2K_2x^2 - 1\right), \qquad \dot{y} = K_1\left(y^2 - x^2\right) + 2K_2xy^2 + x,$$

which has the first integral

$$H_2(x,y) = \frac{x^2+y^2}{1-2x(K_2+K_2x)}.$$

(II) The second class is:

$$\dot{x} = y\left(\frac{8x}{3} - \frac{32y^2}{9} - 1\right), \qquad \dot{y} = x - \frac{4y^2}{3},$$

and its first integral is

$$H_3(x,y) = \left(3x - 4y^2\right)^2 + 9y^2.$$

(III) The last class is:

$$\dot{x} = (1-x)(1-2x)(-y), \qquad \dot{y} = 2x^3 - 2x^2 + x + y^2,$$

and its corresponding first integral is

$$H_4(x,y) = \frac{(x-1)^2\left(x^2+y^2\right)}{(2x-1)^2}.$$

For a proof, see [8].

2. Main Result

In this paper we study the existence and the upper bound of limit cycles that intersect with the irregular separation line Σ in two points, where we will find two possible configurations of limit cycles. The first configuration denoted by **conf 1** is when the limit cycles have two intersection points with Γ_1 or with Γ_2. But the study the limit cycles which intersect Γ_1 or Γ_2 in two point is equivalent to study the piecewise differential systems separated by one straight line. It was proved by Benterki and Llibre in Theorem 1 of [3] that the maximum number of limit cycles of this configuration varies from 0, 1 and 2. Then this configuration is not considered in our paper.

The second configuration denoted by **conf 2**, is when the limit cycles have two intersection points with the irregular line Σ, such that one point is situated in Γ_1 and the second point is located in Γ_2, i.e., the first point of intersection is $(x_1,0) \in \Gamma_1$ and the second point is $(0,y_2) \in \Gamma_2$. We notice that when we combine the two configurations **conf 1** and **conf 2** we obtain another configuration that have a combination between the two kinds of limit cycles and we will denoted it by **conf 3**.

We restrict our analysis to study the maximum number of limit cycles of **conf 2** and **conf 3**. The first main result of the present paper is the following.

Theorem 2. *Consider the discontinuous piecewise differential systems separated by the irregular line Σ and formed by an arbitrary linear differential center* (4) *in the regions Σ^- and an arbitrary cubic isochronous center in the regions Σ^+ after an affine change of variables. Then* *the maximum number of limit cycles with **conf 2** of these discontinuous piecewise differential systems is:*

(i) *at most two for systems of types* (4) *and (I), and there are systems with exactly two limit cycles, see Fig 1(a);*

(ii) *at most two for systems of types* (4) *and (II), and there are systems with exactly two limit cycles, see Fig 1(b);*

(iii) *at most three for systems of types* (4) *and (III), and there are systems with exactly two limit cycles, see Fig 2(c).*

Theorem 2 is proved in section 4.

Our second main result is given as follows.

Theorem 3. *Consider the discontinuous piecewise differential systems separated by the irregular line Σ and formed by an arbitrary linear differential center* (2) *in the regions Σ^- and an arbitrary cubic isochronous center in the regions Σ^+ after an affine change of variables. Then the maximum number of limit cycles with **conf 1** and **conf 2** simultaneously of these discontinuous piecewise differential systems is:*

(i) *at most three for systems of types* (2) *and (I), and there are systems with exactly three limit cycles, see Fig 3(a);*

(ii) *at most three for systems of types* (2) *and (II), and there are systems with exactly three limit cycles, see Fig 3(b);*

(iii) *at most five for systems of types* (2) *and (III), and there are systems with exactly five limit cycles, see Fig 4(c).*

Theorem 3 is proved in section 5.

3. Cubic Isochronous Systems After an Affine Change of Variables

In this section, we present the expression of the three families of the cubic isochronous centers (I), (II) and (III) after doing the general affine change of variables of the form $(x,y) \rightarrow (ax+$

$by+c, \alpha x+\beta y+\gamma$), with $b\alpha - a\beta \neq 0$. Thus, system (I) becomes

(6)
$$\begin{aligned}\dot{x} = &\ \frac{1}{\alpha b - a\beta}(b(-a^2K_1x^2 + d(-2aK_1x + 2K_2(\gamma+\alpha x+\beta y)(\gamma+\alpha x-\beta y)+1)+ax\\ &(2K_2(\gamma+\alpha x+\beta y)(\gamma+\alpha x-\beta y)+1) - d^2K_1 + K_1(\gamma+\alpha x+\beta y)(\gamma+\alpha x\\ &-\beta y)) + b^2y(-2aK_1x - 2dK_1 + 2K_2(\gamma+\alpha x)(\gamma+\alpha x+\beta y)+1) - \beta(2(ax\\ &+d)(K_2(ax+d)+K_1)-1)(\gamma+\alpha x+\beta y) - b^3K_1y^2),\\ \dot{y} = &\ \frac{1}{\alpha b - a\beta}(b(-a^2K_1x^2 + d(-2aK_1x + 2K_2(\gamma+\alpha x+\beta y)(\gamma+\alpha x-\beta y)+1)+ax\\ &(2K_2(\gamma+\alpha x+\beta y)(\gamma+\alpha x-\beta y)+1) - d^2K_1 + K_1(\gamma+\alpha x+\beta y)(\gamma+\alpha x\\ &-\beta y)) + b^2y(-2aK_1x - 2dK_1 + 2K_2(\gamma+\alpha x)(\gamma+\alpha x+\beta y)+1) - \beta(2(ax\\ &+d)(K_2(ax+d)+K_1)-1)(\gamma+\alpha x+\beta y) - b^3K_1y^2),\end{aligned}$$

with its first integral

$$\tilde{H}_2(x,y) = \frac{(ax+by+d)^2 + (\gamma+\alpha x+\beta y)^2}{1-2(ax+by+d)(K_2(ax+by+d)+K_1)}. \tag{7}$$

System (II) written as

(8)
$$\begin{aligned}\dot{x} = &\ \frac{1}{9\alpha b - 9a\beta}(3b(3ax+3d-4(\gamma+\alpha x+\beta y)(\gamma+\alpha x+3\beta y)) + \beta(\gamma+\alpha x+\beta y)(-24ax\\ &-24d + 32(\gamma+\alpha x+\beta y)^2+9) + 9b^2y),\\ \dot{y} = &\ \frac{1}{9a\beta - 9\alpha b}(9a^2x + 3a(3by+3d-4(\gamma+\alpha x+\beta y)(\gamma+3\alpha x+\beta y)) + \alpha(\gamma+\alpha x+\beta y)\\ &(-24by - 24d + 32(\gamma+\alpha x+\beta y)^2+9)),\end{aligned}$$

where its first integral is

$$\tilde{H}_3(x,y) = \big(3(ax+by+d) - 4(\gamma+\alpha x+\beta y)^2\big)^2 + 9(\gamma+\alpha x+\beta y)^2. \tag{9}$$

System (III) is given by

$$\begin{aligned}\dot{x} = &\ \frac{1}{\alpha b - a\beta}(b^2y(6a^2x^2 + 4d(3ax-1) - 4ax + 6d^2 + 2\beta y(\gamma+\alpha x+\beta y)+1) + b\\ &(2a^3x^3 + d(6a^2x^2 - 4ax + 4\beta y(\gamma+\alpha x+\beta y)+1) - 2a^2x^2 + d^2(6ax\\ &-2) + ax(4\beta y(\gamma+\alpha x+\beta y)+1) + 2d^3 + (\gamma+\alpha x+\beta y)(\gamma+\alpha x\\ &-2\beta y)) + 2b^3y^2(3ax+3d-1) + \beta(ax+d-1)(2ax+2d-1)(\gamma+\alpha x\\ &+\beta y) + 2b^4y^3),\end{aligned} \tag{10}$$

$$\begin{aligned}\dot{y} = & \frac{-1}{\alpha b - a\beta}(2a^4x^3 + 2a^3x^2(3by + 3d - 1) + a^2x(6b^2y^2 + 4d(3by - 1) - 4by + 6d^2 \\ & +2\alpha x(\gamma + \alpha x + \beta y) + 1) + a(2b^3y^3 + d(6b^2y^2 - 4by + 4\alpha x(\gamma + \alpha x \\ & +\beta y) + 1) - 2b^2y^2 + d^2(6by - 2) + by(4\alpha x(\gamma + \alpha x + \beta y) + 1) + 2d^3 \\ & -(-\gamma + 2\alpha x - \beta y)(\gamma + \alpha x + \beta y)) + \alpha(by + d - 1)(2by + 2d - 1)(\gamma \\ & +\alpha x + \beta y)),\end{aligned}$$

and its corresponding first integral is

$$\tilde{H}_4(x,y) = \frac{(ax+by+d-1)^2\left((ax+by+d)^2 + (\gamma+\alpha x+\beta y)^2\right)}{(2(ax+by+d)-1)^2}. \tag{11}$$

4. PROOF OF THEOREM 2

In the region Σ^- we consider the linear differential center (4) with its first integral $H_1(x,y)$ given by (5). In the region Σ^+ we consider one of the three families of cubic isochronous systems with its corresponding first integral $\tilde{H}_i(x,y)$ with $i = 2,3,4$. If the discontinuous piecewise differential system (4)–(2m), with $m \in \{3,4,5\}$ has a limit cycle, which intersects the separation line Σ in two distinct points $(0,y_1) \in \Gamma_1$ and $(x_1,0) \in \Gamma_2$. These two points must satisfy the system of equations

$$\begin{aligned} e_1 &= H_1(x_1,0) - H_1(0,y_1) = P_1(x_1,y_1) = 0, \\ e_2 &= \tilde{H}_i(x_1,0) - \tilde{H}_i(0,y_1) = P_i(x_1,y_1) = 0, \text{ with } i = 2,3,4. \end{aligned} \tag{12}$$

By solving $P_1(x_1,y_1) = 0$, we get $x_1 = g(y_1) = Dy_1$, with D=$\sqrt{A^2+\omega^2}$, and by substituting x_1 in $P_i(x_1,y_1) = 0$ we obtain an equation in the variable y_1, and we distinguish three cases according to the expression of the first integral $\tilde{H}_i(x,y)$.

Proof of statement (i) *of Theorem 2.* For $i = 2$, the corresponding isochronous cubic system is (3) with its first integral $\tilde{H}_2(x,y)$ given in (7), the solutions of the equation $P_2(g(y_1),y_1) = 0$ are equivalent to the solutions of the quartic equation $F_1(y_1) = 0$ such that

$$\begin{aligned}F_1(y_1) = \; & y_1(2b(K_1(D^2y_1^2(a^2+\alpha^2)+\gamma^2) - d^2K_1 + d(2K_2(\gamma^2+\alpha^2D^2y_1^2)+1)) - a^2D^2y_1 \\ & (-2dK_1 + 2K_2(\gamma+\beta y_1)^2 + 1) + b^2y_1(-2aDK_1y_1 - 2dK_1 + 2K_2(\gamma+\alpha Dy_1)^2 + 1) \\ & +2aD(d^2K_1 - d(2K_2(\gamma^2+\beta^2y_1^2)+1) - K_1(\gamma^2+\beta^2y_1^2)) + (2d(dK_2+K_1)-1)(\alpha D \\ & -\beta)(2\gamma+\alpha Dy_1+\beta y_1)) + 4\gamma Dy_1^2(2dK_2+K_1)(\alpha b - a\beta).\end{aligned}$$

This equation has at most four real solutions. Therefore system (12) has at most four real solutions, which can easily be proved that they are symmetric. These tow solutions provide at most two limit cycles for the discontinuous piecewise differential system (3)–(4).

Now we prove that the result of statement (i) is reached by giving an example of discontinuous piecewise differential system (4)–(3) with exactly two limit cycles.

In the region Σ^+ we consider the cubic isochronous differential center

$$\begin{aligned}\dot{x} = \; & x^2(0.29088.. - 0.234119..y) + x(y(0.389428.. - 0.210984..y) + 0.673442..)\\ & +0.2x^3 + (-0.594058..y - 1.39526..)y + 0.0801447..,\\ \dot{y} = \; & x(y(0.69824.. - 0.234119..y) + 0.788391..) + x^2(0.2y - 0.152264..)\\ & +y((-0.210984..y - 0.191014..)y - 0.289442..) + 0.251744..,\end{aligned} \tag{13}$$

with the first integral

$$\tilde{H}_2(x,y) = \frac{(x+0.596858..y+0.4..)^2 + (x-1.76745..y+0.2..)^2}{1-2(0.1..(x-1.76745..y+0.2..)+0.2..)(x-1.76745..y+0.2..)}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -\frac{1}{10}x - \frac{37}{100}, \qquad \dot{y} = x + \frac{1}{10}y, \tag{14}$$

which has the first integral

$$H_1(x,y) = (x+\frac{1}{10}y)^2 + \frac{9}{25}y^2.$$

The real solutions of system (12) are $(1.09545..., 1.8009...)$ and $(0.632456..., 1.03975...)$. Then the two crossing limit cycles of system (13)–(14) corresponding to these solutions are shown in Fig 1(a). □

Proof of statement (ii) *of Theorem 2.* For $i = 3$, the isochronous cubic system is (8) where its first integral is $\tilde{H}_3(x,y)$ given in (9). To obtain the number of real solutions of system (12) we have to solve the equation $P_3(g(y_1), y_1) = 0$ which has the same solutions as the quartic equation $F_2(y_1) = 0$, such that

$$\begin{aligned}F_2(y_1) = \; & 3y_1^2(-3a^2D^2 + 16a\alpha\gamma D^2 + 3b^2 - 16\beta b\gamma + (-32\gamma^2 + 8d - 3)(\alpha D - \beta)(\beta + \alpha D))\\ & -2y_1(3aD(3d-4\gamma^2) + b(12\gamma^2 - 9d) + \gamma(32\gamma^2 - 24d + 9)(\alpha D - \beta)) + 8y_1^3(3a\alpha^2 D^3\\ & -3b\beta^2 + 8\gamma(\beta^3 - \alpha^3 D^3)) + 16y_1^4(\beta^4 - \alpha^4 D^4).\end{aligned}$$

Its clear that this equation has at most four real solutions, and due to the fact that these solutions are symmetric, we know that system (12) has at most two distinct real solutions. Consequently, the discontinuous piecewise differential system (4)–(8) has at most two limit cycles.

To reach our result we shall give an example of discontinuous piecewise differential system (4)–(8) with exactly two limit cycles.

In the region Σ^+ we consider the cubic isochronous differential center

(15)
$$\begin{aligned}\dot{x} = {} & y(y(0.195657.. - 0.0114478..y) - 0.0714585..) + 0.0114478..x^3 + x^2(-0.0343434..y \\ & -0.0710098..) + x((0.0343434y - 0.124647)y - 1.44776) - 6.3691, \\ \dot{y} = {} & y(y(0.0623236.. - 0.0114478..y) + 1.44776..) + 0.0114478..x^3 + x^2(-0.0343434..y \\ & -0.204343..) + x((0.0343434..y + 0.14202..)y + 0.138904..) + 5.20374..,\end{aligned}$$

with the first integral

$$\begin{aligned}\tilde{H}_3(x,y) = {} & (3(0.1..x + 0.210589..y + 1.22182..) - 4(-0.1..x + 0.1..y + 0.22..)^2)^2 + 9(-0.1..x \\ & +0.1..y + 0.22..)^2.\end{aligned}$$

In region Σ^- we consider the linear differential center

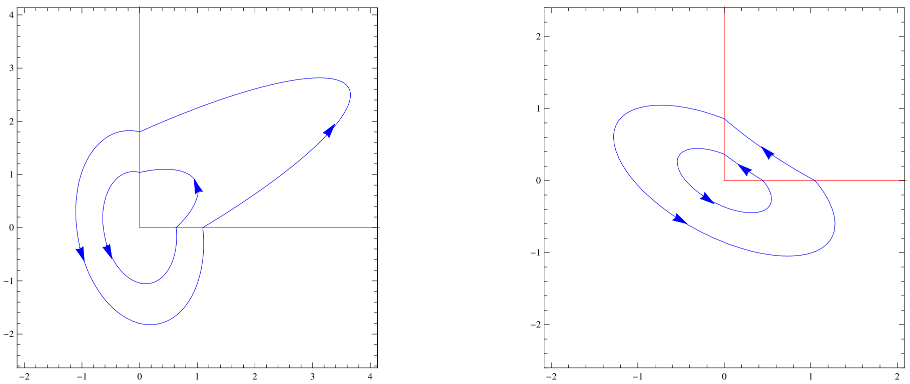
$$\dot{x} = -\frac{7}{10}x - \frac{149}{100}y, \qquad \dot{y} = x + \frac{7}{10}y, \tag{16}$$

which has the first integral

$$H_1(x,y) = \left(x + \frac{7}{10}y\right)^2 + y^2.$$

The discontinuous piecewise differential center (15)–(16) has exactly two crossing limit cycles, because system (12) has the two solutions $(1.04881...,0.859218..)$ and $(0.447214...,0.366372...)$. These limit cycles are shown in Fig 1(b). □

Proof of statement (iii) of Theorem 2. For $i = 4$, the first integral for the cubic isochronous system (10) is $\tilde{H}_4(x,y)$ given in (11). We are interesting in finding the solutions y_1 of the equation $P_4(g(y_1),y_1)) = 0$ which has the same solutions as the equation of degree six $F_3(y_1) = 0$ such that



(a) (b)

FIGURE 1. The two limit cycles of the discontinuous piecewise differential systems with **conf 2**, (a) for (13)–(14), and (b) for (15)–(16).

$$
\begin{aligned}
F_3(y_1) = \; & y_1^2(a^2D^2(3\gamma^2-2d(2\gamma^2+5d(2(d-2)d+3)-5)-1)-4a\gamma(d-1)(2d-1)D(\alpha(2d\\
& -1)D-2\beta(d-1))+b^2(-3\gamma^2+2d(2\gamma^2+5d(2(d-2)d+3)-5)+1)-4b\gamma(d-1)\\
& (2d-1)(\beta-2\beta d+2\alpha(d-1)D)+(d-1)^2(-(1-2d)^2)(\alpha D-\beta)(\beta+\alpha D))+4a^2\\
& b^2D^2y_1^6(-D^2(a^2+\alpha^2)+b^2+\beta^2)-2y_1^3(b(2a^2D^2(\gamma^2+(2d-1)^3)+8a\gamma(d-1)(2d\\
& -1)D(\alpha D-\beta)+(d-1)(2d-1)(\beta^2(1-2d)+2\alpha^2(d-1)D^2))+aD(a^2(2d-1)^3D^2\\
& +a\gamma D(\alpha(1-2d)^2D-4\beta(d-1)^2)+(d-1)(2d-1)(\alpha^2(2d-1)D^2-2\beta^2(d-1)))\\
& -b^2(2aD(\gamma^2+(2d-1)^3)+\gamma(\beta+4(d-1)(\beta d+D(\alpha-\alpha d))))-b^3(2d-1)^3)\\
& -4abDy_1^5(b(2a^2(2d-1)D^2+2a\gamma D(\alpha D-\beta)+\beta^2(1-2d)+2\alpha^2(d-1)D^2)+a(2d\\
& -1)D^3(a^2+\alpha^2)+2ab^2(1-2d)D-2a\beta^2(d-1)D+b^3(1-2d))+y_1^4(8abD(a\gamma D(2\beta\\
& (d-1)+D(\alpha-2\alpha d))-(2d-1)(a^2(2d-1)D^2+(d-1)(\alpha D-\beta)(\beta+\alpha D)))+a^2\\
& D^2(4\beta^2(d-1)^2-(1-2d)^2D^2(a^2+\alpha^2))+8ab^3(1-2d)^2D-b^2(\beta-2\beta d+2\alpha(d\\
& -1)D)(8a\gamma D+\beta(2d-1)+2\alpha(d-1)D)+b^4(1-2d)^2)+2(d-1)(2d-1)y_1(d(2(d\\
& -1)d+1)(b-aD)+\gamma^2(b-aD)-\gamma(d-1)(2d-1)(\alpha D-\beta)).
\end{aligned}
$$

The equation $F_3(y_1)=0$ has at most six real solutions. Therefore, system (12) has at most three real non symmetric solutions, which provide at most three limit cycles for the discontinuous piecewise differential system (4)–(10).

To complete the proof of this statement we shall provide an example of discontinuous piecewise differential system formed by an arbitrary linear center and a cubic isochronous center of type (10) with exactly three limit cycles.

In the region Σ^+, we consider the cubic isochronous differential center

$$
\begin{aligned}
\dot{x} = & -0.0333565..x^3 + x^2(0.0251109..y + 325.854..) + x((0.220905..y - 1633.64..)y \\
& +1.31696.. \times 10^6) + y((0.171752..y - 2564.66..)y + 1.1617.. \times 10^7) \\
& -1.43102.. \times 10^{10}, \\
\dot{y} = & \; x^2(1293.86.. - 0.139931..y) + x(y(1299.6.. - 0.0250305..y) - 7.36102.. \times 10^6) \\
& -0.0689903..x^3 + y((0.0526109..y - 409.335..)y - 1.31566.. \times 10^6) \\
& +1.13904.. \times 10^{10},
\end{aligned} \tag{17}
$$

which has the first integral

$$
\begin{aligned}
\tilde{H}_4(x,y) = & \frac{1}{(2(0.2x + 0.251176..y - 1624.97..) - 1)^2}((0.3x - 0.376965..y + 0.2)^2 + (0.2x \\
& +0.251176..y - 1624.97..)^2)(0.2x + 0.251176..y - 1625.97..)^2.
\end{aligned}
$$

In the region Σ^-, we consider the linear differential center

$$
\dot{x} = -\frac{3}{10}x - \frac{1973}{1250}y, \qquad \dot{y} = x + \frac{3}{10}y, \tag{18}
$$

and its corresponding first integral is

$$
H_1(x,y) = \left(x + \frac{3}{10}y\right)^2 + \frac{3721}{2500}y^2.
$$

The three solutions of system (12) for these systems are $(1.48324..., 1.1806...)$, $(1.18322..., 0.941793...)$ and $(0.774597..., 0.616548...)$. Then the three crossing limit cycles for the discontinuous piecewise differential system (4)–(18) are shown in Fig 2.

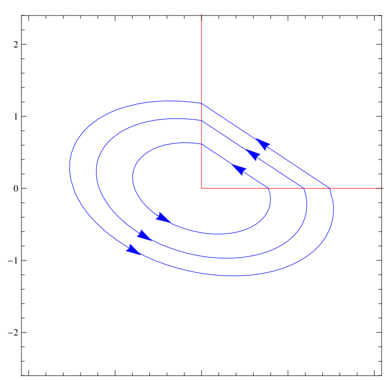


FIGURE 2. The three limit cycles of the discontinuous piecewise differential system (17)–(18) with **conf 2**.

□

5. PROOF OF THEOREM 3

In order to have limit cycles of **Conf 1** and **Conf 2** simultaneously, the limit cycles of **Conf 1** which intersect the separation line Γ_1 in two points must satisfy the equations

$$\begin{aligned} E_1 = H(0,y_1) - H(0,y_2) = &\quad (y_1 - y_2)(-2B + (y_1 + y_2)(A^2 + \omega^2)) = 0, \\ E_2 = \tilde{H}_i(0,y_1) - \tilde{H}_i(0,y_2) = &\quad P_i(y_1,y_2) = 0, \end{aligned} \tag{19}$$

where $\tilde{H}_i(x,y)$ for $i = 2,3,4$, are the first integrals given by $(7),(9),(11)$. On the other hand the two intersection points of limit cycles of **Conf 2** with the irregular separation line Σ must satisfy system (12). Then we have the following results.

Proof of statement (i) *of Theorem 3.* In what follows we give an example of discontinuous piecewise differential system formed by an arbitrary linear center (2) and the cubic isochronous center (3), which has one limit cycle of **Cnf 1** and two limit cycles of **Cnf 2**, i.e, has three limit cycles of **Cnf 3**.

In the region Σ^+ we consider the cubic isochronous center

(20)
$$\begin{aligned} \dot{x} = &\quad x^2(0.103711.. - 0.006568..y) - 0.00592118..x^3 + x((0.0124892..y + 0.189279..)y \\ &-0.755639..) + (-0.118324..y - 1.2791..)y + 2.88432.., \\ \dot{y} = &\quad x(y(0.112022.. - 0.006568..y) + 0.445381..) + x^2(-0.00592118..y - 0.0323263..) \\ &+y((0.0124892..y + 0.0949698..)y - 0.143181..) - 0.122118.., \end{aligned}$$

with the first integral

$$\tilde{H}_2(x,y) = \frac{(-0.1..x - 0.210924..y + 0.4..)^2 + (-0.1..x + 0.1..y - 0.2..)^2}{1 - 2(-0.296059..(-0.1..x + 0.1..y - 0.2..) - 0.680186..)(-0.1..x + 0.1..y - 0.2..)}.$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -\frac{1}{10}x - \frac{101}{100}y + 2, \qquad \dot{y} = \frac{5}{10} + x + \frac{1}{10}y, \tag{21}$$

which has the first integral

$$H(x,y) = \left(x + \frac{1}{10}y\right)^2 + 2\left(\frac{5}{10}x - 2y\right) + y^2.$$

For the discontinuous piecewise differential system (20)–(21), system (19) has the unique solution $(y_1,y_2) = (1.00506..,2.95533..)$, which provide one limit cycle intersecting Γ_1 in the two points $(0,y_1)$ and $(0,y_2)$, and system (12) has the two solutions $(x_3,y_3) = (1.56155..,4.78761..)$ and $(x_4,y_4) = (2.8541..,5.82887..)$, which provide the four intersecting points $(x_i,0),(0,y_i)$ with $i = 3,4$ of the two limit cycles with the separation irregular line Σ. Then the discontinuous piecewise differential system (20)–(21) has exactly three limit cycles, see Fig 3(a). $\square$

Proof of statement (ii) *of Theorem 3.* In what follows we give an example of discontinuous piecewise differential system formed by an arbitrary linear center (2) and the cubic isochronous center (8), which has one limit cycle of **Cnf 1** and two limit cycles of **Cnf 2**, i.e, has three limit cycles of **Cnf 3**.

In the region Σ^+ we consider the cubic isochronous center

$$
\begin{aligned}
\dot{x} = &\ x^2(0.169221.. - 0.246893..y) + x(y(1.03113.. - 0.292023..y) - 0.914334..)\\
&+y(y(0.982871.. - 0.115134..y) - 3.20465..) - 0.0695791..x^3 + 3.31428..,\\
\dot{y} = &\ 0.0588262..x^3 + x^2(0.208737..y + 0.082386..) + x((0.246893..y - 0.338442..)y\\
&+0.983531..) + y((0.0973408..y - 0.515564..)y + 0.914334..) - 0.716136..,
\end{aligned} \tag{22}
$$

and its first integral is

$$
\begin{aligned}
\tilde{H}_3(x,y) = &\ \big(0.16..x^2 + x(0.378493..y + 0.149387..) + (0.223839..y - 1.27391..)y + 1.48034..\big)^2\\
&+9(-0.2x - 0.236558..y + 0.469133..)^2.
\end{aligned}
$$

In the region Σ^- we consider the linear differential center

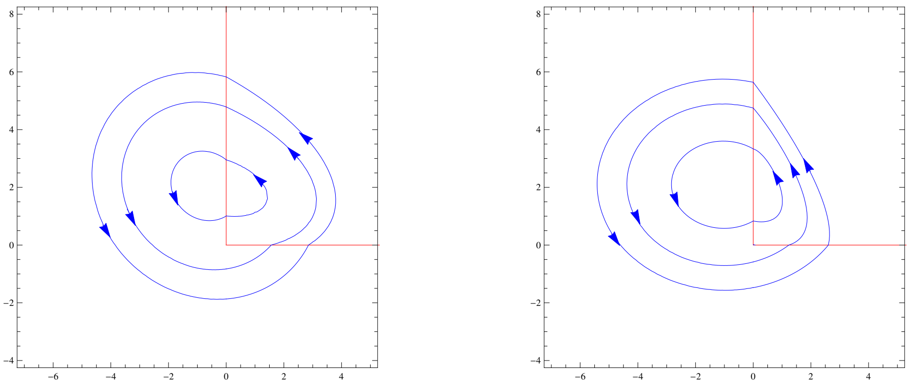
$$
\dot{x} = 3 - \frac{1}{100}x - \frac{14401}{10000}y, \qquad \dot{y} = 1 + x + \frac{1}{100}y, \tag{23}
$$

which has the first integral

$$
H(x,y) = \Big(x + \frac{1}{100}y\Big)^2 + 2\Big(x - 3y\Big) + \frac{36}{25}y^2.
$$

For the discontinuous piecewise differential system (22)–(23), system (19) has the unique solution $(y_1,y_2) = (0.833353..,3.33302..)$, which provide one limit cycle intersecting Γ_1 in the two points $(0,y_1)$ and $(0,y_2)$, and system (12) has the two solutions $(x_3,y_3) = (1.23607..,4.75101..)$ and $(x_4,y_4) = (2.60555..,5.64302..)$, which provide the four intersecting points $(x_i,0),(0,y_i)$ with $i = 3,4$ of the two limit cycles with the separation irregular line Σ. Then the discontinuous piecewise differential system (22)–(23) has exactly three limit cycles, see Fig 3(b).

$\square$



(a) (b)

FIGURE 3. The three limit cycles of the discontinuous piecewise differential system with **conf 3**, (a) for (20)–(21), and (b) for (22)–(23).

Proof of statement (iii) of Theorem 3. In what follows we give an example of discontinuous piecewise differential system formed by an arbitrary linear center (2) and the cubic isochronous center (10), which has two limit cycles of **Cnf 1** and three limit cycles of **Cnf 2**, i.e, has five limit cycles of **Cnf 3**.

In the region Σ^+ we consider the cubic isochronous center

(24)
$$\begin{aligned}\dot{x} = & -0.0481337..x^3 + x^2(-0.121939..y - 0.426587..) + x((-0.0950406..y - 0.972986...)y \\ & +0.995223..) + y((-0.0235266..y - 0.581449..)y + 1.46708..) + 1.15725.., \\ \dot{y} = & \ 0.0367839..x^3 + x^2(0.0930168..y + 0.59027..) + x((0.0724234..y + 1.42865..)y \\ & +0.128844..) + y((0.0179159..y + 0.892226..)y - 0.0374027..) + 0.794241..,\end{aligned}$$

with the first integral

$$\begin{aligned}\tilde{H}_4(x,y) = & \frac{1}{(2(0.0377968..x + 0.0230595..y + 0.77109..) - 1)^2}((0.0377968x + 0.0230595y \\ & +0.77109)^2 + (0.339871x + 0.447675y + 0.5)^2)(0.0377968x + 0.0230595y - 0.22891)^2.\end{aligned}$$

In the region Σ^- we consider the linear differential center

$$\dot{x} = -\frac{2}{10}x - \frac{26}{25}y, \qquad \dot{y} = \frac{5}{10} + x + \frac{2}{10}y, \tag{25}$$

and its first integral is

$$H(x,y) = \left(x + \frac{2}{10}y\right)^2 + 2\left(\frac{5}{10}x - 3y\right) + y^2.$$

For the discontinuous piecewics differential system (24)–(25), system (19) has the two solution $(y_1, y_2) = (2.09171.., 3.67752..)$ and $(y_3, y_4) = (0.553009.., 5.21622..)$, which provide the two limit cycles intersecting Γ_1 in the four points $(0, y_i)$ with $i = 1, 2, 3, 4$, and system (12) has the three solutions $(x_5, y_5) = (1, 6.08525..)$, $(x_6, y_6) = (2.19258.., 6.76428..)$ and $(x_7, y_7) = (3, 7.34101..)$, which provide the six intersecting points $(x_i, 0), (0, y_i)$ with $i = 5, 6, 7$ of the three limit cycles with the separation line Σ. Then the discontinuous piecewise differential system (24)–(25) has exactly five limit cycles of **Cnf 3** shown in Fig 4.

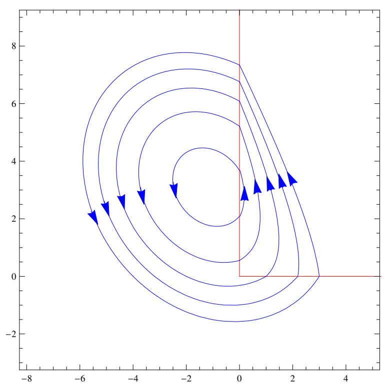


FIGURE 4. The five limit cycles of the discontinuous piecewise differential system (24)–(25) with **conf 3**.

□

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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