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A METHOD PROJECTIVE FOR SOLUTION OF A QUADRATIC SEMIDEFINITE PROBLEM WITH A NEW POTENTIAL FUNCTION

MOUNIA LAOUAR*, MAHMOUD BRAHIMI, EL-AMIR DJEFFAL

Laboratory of partial differential equations, Department of Mathematics,Faculty of Mathematics and Computer Science, University of Batna2, Batna-05000, Algeria

Abstract. In this paper, we are interested in a primal-dual algorithmic study of a projective interior point method of a semidefinite quadratic problem *(QSDP)*. In the first instance, we suggested a new projective function to have a set of simplex like constraints and a linearization of the objectif function, in the second time we have defined a new potential function to obtain a new polynomial complexity, such that the convergence is obtained after $O(L(n+1))$ iterations bound.

Keywords: quadratic semidefinite problem; projective method; primal-dual algorithm; potential function.

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1. Introduction

After the appearance of the Karmarkar's algorithm [4], most researchers interested in the optimality conditions of the positive semidefinite problems *(SDP)* based on the potential reduction methods for linear programming, which are extensions of the interior point methods for linear programming. Indeed, in 1994, [1] was the first researcher to conduct a study primal-dual of the point interior type projective on *(SDP)*. The choice of the starting point greatly affects the performance of interior point methods *(SDP)*, in 2007, [2] proposed a feasible *(SDP)* to solve this

*Corresponding author
E-mail address: m.laouar@univ-batna2.dz
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kind of problem. The interior point methods are considered to be among the various approximations that are effective in solving semidefinite problems. Several authors have proposed some extensions of interior point methods (IPM) for solving positive semidefinite programming from linear problem (LP). On the other hand, a primal-dual solution has been obtained by [6], this method is based on (IPM) with polynomial like complexity.

[3] proposed a quadratic linearization problem so that every iteration has x^k and [5] suggested that every iteration has a point x^0, these technics are applied in many problems such as economics, automation, engineering...etc.

The main objectif of this paper is inspired by [1, 4, 5, 9] it aims to construct an improved primal-dual polynomial feasible interior point algorithm for solving *(QSDP)*. The first step of this research is to transform a nonlinear problem to a linear programming (LP), by using a new projective function that generates a new set of simplex constraints and by transforming algebraically the objective function using a linearization in [5]. Based on a few properties of the potential function proposed by [7], we have defined a new potential function showing to obtain a theoretical convergence of primal-dual algorithm with a complexity of the polynomial type.

This paper is organised as follows: In section 2, we present a description of the method. In section 3, we give the statement of the algorithm and its theoretical convergence. Finally, a conclusion is drawn in section 4.

2. DESCRIPTION OF METHOD

General overview.

In the sequel, some notations used throughout of our work are as follows: we denotes the nonnegative and positive orthants respectively by $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$. $\mathbb{R}^{n\times n}$ denotes the set of $n\times n$ real matrices. $\|\cdot\|_2$ denotes the matrices spectral norm. Let S^n, S^n_+, S^n_{++} denote the sets of $n\times n$ symmetric, symmetric positive semidefinite and symmetric positive definite $n\times n$ matrices , respectively endowed with the standard trace inner product $Trace(AB) = A\bullet B$ for $A,B\in S^n_+$, and the lowner partial order $\succeq$ (or $\succ$) on positive semidefinite (or positive definite) matrices means $A\succeq B$ (or $A\succ B$) if $A-B$ is positive semidefinite(or positive definite). We consider the primal problem $(\mathscr{P})$ of the quadratic semidefinite program $(QSDP)$ and its dual $(\mathscr{D})$ in the following forms:

$$(\mathscr{P}) \qquad \begin{cases} z^* = \min F(X), \\ A_i \bullet X = b_i \text{ for } i = 1, \ldots, m \ , \\ X \succeq 0 \ , \end{cases}$$

where: $F(X) = C \bullet X + \frac{1}{2} X \bullet \Omega(X)$,

$$(\mathscr{D}) \qquad \begin{cases} r^* = \max b^T w - \frac{1}{2} X \bullet \Omega(X), \\ \sum\limits_{i=1}^{m} w_i A_i - \Omega(X) + S = C, \\ S \succeq 0 \ . \end{cases}$$

In which C and A_i for $i = 1, \ldots, m$ are matrices in S^n, $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$, Ω is a self-adjoint positive semidefinite linear operator acting on S^n_+ and X in S^n_+.

Without loss of generality, the following assumptions are given throughout this article:

(1) The matrices $A_i \in S^n, i = 1, \ldots, m$ are linearly independent.
(2) The optimal value z^* of the objective function is unknown from the beginning.
(3) The problems $(\mathscr{P})$ and $(\mathscr{D})$ satisfies the interior point conditions (IPC): i.e. there exists $(X_0, S_0) \succeq 0$ such that:

$$A_i \bullet X_0 = b_i \text{ for } i = 1, \ldots, m \ ,$$

and

$$\sum_{i=1}^{m} w_i A_i - \Omega(X_0) + S_0 = C.$$

Remark 2.1. *The objective function is convex and twice continuously differentiable function, therefore, the problem* $(\mathscr{P})$ *is convex.*

2.1. Transforming $(\mathscr{P})$ to a simplified Karmarkar problem.

This subsection is divided into two parts:

(1) The first part is an extension of the projective method obtained from [1], to have a new set of so-called simplex constraints.
(2) The objective of the second part is to have a new type of convex problem.

2.1.1. *Projective function.* We describe the process of passage from iteration X_k to iteration X_{k+1} by a projective type algorithm, at each current iteration k, the iterative solution X_k is strictly feasible of $(\mathscr{P})$, such that $X_k \in S^n_{++}$ is given as $X_k = L_k L_k^T$ is a Cholesky's decomposition where L_k is a lower triangular matrix. We use the projective transformation proposed in our work to bring the current iteration to the center of the simplex.

We define a projective transformation T_k as an extension of the one proposed by [1], such that the current iteration reduces to the center $A = \frac{1}{n+1} I_{n+1}$ of the simplex Sp^{n+1}_+, where:

$$T_k : S^n_+ \to Sp^{n+1}_+$$

$$X \mapsto T_k(X) = Y,$$

and:

$$Sp^{n+1}_+ = \left\{ Y \in S^{n+1}_+ : I_{n+1} \bullet Y = 1 \right\},$$

with:

$$\begin{cases} y_{i,j} = \frac{(L_k^{-1} X L_k^{-T})_{i,j}}{1 + I_n \bullet (X_k^{-1} X)}, \text{ for } i,j = 1,...,n , \\ y_{n+1,n+1} = 1 - \sum_{i=1}^{n} y_{i,i}, \\ y_{n+1,n+1} \neq 0, \\ y_{i,n+1} = y_{n+1,j} = 0, \; for \; i,j = 1,...,n . \end{cases} \tag{2.1}$$

Thus, the transformation projective:

$$Y = T_k(X) = \begin{bmatrix} Y[n] & 0_n \\ 0_n^T & y_{n+1,n+1} \end{bmatrix}, \tag{2.2}$$

and its inverse:

$$X = T_k^{-1}(Y) = \frac{1}{y_{n+1,n+1}} (L_k Y[n] L_k^T), \tag{2.3}$$

where:

(1) $Y[n] = y_{n+1,n+1}(L_k^{-1} X L_k^{-T})$ is the first $n \times n$ elements of the matrix Y,

(2) $y_{n+1,n+1} = \frac{1}{1+I_n \bullet (X_k^{-1} X)}$,

(3) the components of $Y[n]$ are defined as follows:

$$y_{i,j} = y_{n+1,n+1}(L_k^{-1} X L_k^{-T})_{i,j}, \quad i,j = 1,...,n.$$

Lemma 2.1. *We have Y is a semidefinite symmetric matrix.*

Proof. We know that, $\forall t = (t_1,...,t_{n+1})^T \in \mathbb{R}^n$:

$$\begin{aligned} \langle Yt,\ t\rangle &= \langle Y[n]t[n],\ t[n]\rangle + y_{n+1,n+1}\ t_{n+1}^2 \\ &= y_{n+1,n+1}\left\langle X\ \ L_k^T t[n],\ L_k^T t[n]\right\rangle + y_{n+1,n+1}\ t_{n+1}^2. \end{aligned}$$

we have: $y_{n+1,n+1} > 0$ and $X \in S_+^n$, we conclude that: $Y \in S_+^{n+1}$ □

2.1.2. *The new constraints set.*

We use the projective transformation (2.2) and its inverse (2.3) in the problem ($\mathscr{P}$), we obtain:

$$(2.4)\qquad \begin{cases} \min G(Y,z^*) = \frac{1}{y_{n+1,n+1}}\left(C_k(z^*) \bullet Y + \frac{1}{2} Y \bullet \Omega_k(Y)\right), \\ B_i^k \bullet Y = 0 \ \text{ for } \ i = 1,...,m \ , \\ I_{n+1} \bullet Y = 0, \\ Y[n] \succeq 0 \ \text{ and } \ y_{n+1,n+1} > 0, \end{cases}$$

with:

(1) $C_k(z^*) = \begin{pmatrix} L_k^T C L_k & 0_n \\ & \\ 0_n^T & -z^* \end{pmatrix}$,

(2) $\Omega_k(Y) = \begin{pmatrix} \Omega_k(X) & 0_n \\ & \\ 0_n^T & 0 \end{pmatrix}$, where: $\Omega_k(X) = L_k^T \Omega(X) L_k$,

(3) $B_i^k = \begin{pmatrix} -L_k^T A_i L_k & 0_n \\ & \\ 0_n^T & b_i \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \;\; i = 1,...,m,$

(4) $Y = \begin{pmatrix} Y[n] & 0_n \\ & \\ 0_n^T & y_{n+1,n+1} \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}.$

To ensure the convexity of the objective function of the last problem, we define the objective function under the simplex set of constraints as follows:

$$\text{(2.5)} \qquad \begin{cases} \min H(Y,z^*) = y_{n+1,n+1} G(Y,z^*), \\ \\ B_i^k \bullet Y = 0 \quad \text{for} \quad i = 1,...,m \;\; , \\ \\ Y \in Sp_+^{n+1} \;. \end{cases}$$

Lemma 2.2. *The function $H(Y,z^*)$ is a convex function into a feasible set $\hat{F}$ of the problem (2.5), where $\hat{F} = \{Y \in Sp_+^{n+1} : \;\; B_i^k \bullet Y = 0 \, for \;\; i = 1,...,m\}$.*

Proof. It is enough to verify the following inequality:

$$\forall Y, \hat{Y} \in \hat{F} : \;\; \forall t \in [0,1] : H(t \;\; Y + (t-1) \;\; \hat{Y}, z^*) \leq \;\; t \;\; H(Y,z^*) + (1-t) \;\; H(\hat{Y},z^*).$$

We have:

$$\forall \;\; Y, \hat{Y} \in \hat{F}, \;\; \exists \;\; X, \hat{X} \in \mathbb{R}_+^n : T_k^{-1}(Y) = X \quad \text{and} \quad T_k^{-1}(\hat{Y}) = \hat{X},$$

we know that:

$$H(Y,z^*) = y_{n+1,n+1}[F(T_k^{-1}(Y)) - z^*],$$

therefore:

$$H(tY + (1-t)\hat{Y}, z^*) = (ty_{n+1,n+1} + (1-t)\hat{y}_{n+1,n+1}) \left[F\left(\frac{L_k(tY[n] + (1-t)\hat{Y}[n])L_k^T}{ty_{n+1} + (1-t)\hat{y}_{n+1}}\right) - z^* \right].$$

From the convexity of the function F and we get our result. □

Remark 2.2. *$A = \frac{1}{n+1} I_{n+1}$ is a feasible strictly solution of the problem (2.5), seeing that:*

(1) *On the one hand*

$$B_i^k \bullet A = \frac{1}{n+1} B_i^k \bullet I_{n+1} = \frac{1}{n+1}(-A_i^k \bullet I_n + b_i) = \frac{1}{n+1}(-A_i \bullet X_k + b_i) = 0,$$

(2) *and on the other hand*

$$I_{n+1} \bullet A = \frac{1}{n+1} I_{n+1} \bullet I_{n+1} = 1.$$

2.2. Approximation of the optimal value z^*.

Since, the optimal value z^* is unknown, an upper bound on z^* is used, taking at each iteration k,

$$z_k = C \bullet X_k + \frac{1}{2} X_k \bullet \Omega(X_k),$$

we establish that:

$$C \bullet X_k + \frac{1}{2} X_k \bullet \Omega(X_k) = C_k \bullet I_n + \frac{1}{2} I_n \bullet \Omega_k(X_k), \tag{2.6}$$

wherein:

(1) $z_k > z^*$,
(2) $C_k = L_k^T C L_k$,

In the following, we relax the problem (2.5) into a semidefinite problem such that we obtain a linear objective function under a spherical constraint set.

2.3. Linearization of the objective function.

We have:

$$\begin{cases} \min H(Y, z^*), \\ B_i^k \bullet Y = 0 \quad \text{for} \quad i = 1, \ldots, m, \\ Y \in Sp_+^{n+1}. \end{cases} \tag{2.7}$$

According to the properties of the cost function of the problem (2.5), this is what will give us the right to use the Taylor series development in the neighbourhood of the point $Y_0 = \frac{1}{n+1} I_{n+1}$ to obtain the linearization of H and to have the following result:

$$H(Y, z_k) = H(Y_0, z_k) + \nabla H(Y_0, z_k) \bullet (Y - Y_0) \;\; for \; Y \in \left\{ Y \in S_+^{n+1} : \; \|Y - Y_0\|^2 \leq \alpha^2 \right\}.$$

Thus, we get the following sub-problem:

$$
(2.8)\qquad \begin{cases}
\mathfrak{r}_k(\alpha) = \min \nabla H(Y_0, z_k) \bullet (Y - Y_0), \\
B_i^k \bullet (Y - Y_0) = 0 \quad \text{for } i = 1, \ldots, m \;\;, \\
I_{n+1} \bullet (Y - Y_0) = 0, \\
\|Y - Y_0\|^2 \leq \alpha^2,
\end{cases}
$$

in which: $\nabla H(Y_0, z_k) = C_k(z_k) + \Omega_k(Y_0)$.

2.4. The iterative solutions.

Lemma 2.3. *The optimal solution of the problem (2.8) is:*

$$
(2.9)\qquad Y^*(\alpha) = Y_0 - \alpha D_k,
$$

such that:

$$
D_k = \frac{\nabla H(Y_0, z_k) + \sum_{i=1}^m w_i B_i^k + w_{m+1} I_{n+1}}{\left\| \nabla H(Y_0, z_k) + \sum_{i=1}^m w_i B_i^k + w_{m+1} I_{n+1} \right\|}.
$$

Proof. We put $D = Y - Y^0$, so:

$$
(2.10)\qquad \begin{cases}
\min_{(D,\alpha)} \nabla H(Y_0, z_k) \bullet D = \mathfrak{r}_k(\alpha), \\
B_i'^k \bullet D = 0 \;\; \text{for} \quad i = 1, \ldots, m+1 \;\;, \\
\|D\|^2 \leq \alpha^2 \;\; \text{and} \quad D \in S_{++}^{n+1},
\end{cases}
$$

where:

$$
B_i'^k = \begin{cases}
B_i^k & \text{if } \; i = 1, \ldots, m \;\;, \\
I_{n+1} & \text{if } \; i = m+1.
\end{cases}
$$

Since the problem (2.10) is convex and $D \in S_{++}^{n+1}$ is an optimal solution, there exists $w \in \mathbb{R}^m$ and $t \in \mathbb{R}_+^{n+1}$, such that:

$$
(2.11)\qquad \nabla H(Y_0, z_k) + \sum_{i=1}^{m+1} w_i B_i'^k + tD = 0,
$$

$$B_i^{\prime k} \bullet D = 0 \ \text{ for } \ i = 1, ..., m+1 \ , \tag{2.12}$$

$$t(\|D\|^2 - \alpha^2) = 0, \tag{2.13}$$

$$\|D\|^2 \leq \alpha^2. \tag{2.14}$$

From (2.11), we obtain:

$$D = -t^{-1} P_k, \tag{2.15}$$

with:

$$P_k = \nabla H(Y_0, z_k) + \sum_{i=1}^{m+1} w_i B_i^{\prime k},$$

and $w = (w_1, ..., w_{m+1})^T$ is a solution of the following linear system:

$$Mw = d,$$

we have, from the equation (2.12):

$$\forall i = 1, ..., m+1 : B_i^{\prime k} \bullet P_k = 0,$$

implies that:

$$\forall i = 1, ..., m+1 : B_i^{\prime k} \bullet \nabla H(Y_0, z_k) + \sum_{j=1}^{m+1} w_j B_i^{\prime k} \bullet B_j^{\prime k} = 0.$$

So:

(1) $d_i = -\nabla H(Y_0, z_k) \bullet B_i^{\prime k}, \ \ i = 1, ..., m+1 \ ,$
(2) $M_{i,j} = B_i^{\prime k} \bullet B_j^{\prime k}, \ \ i, j = 1, ..., m+1.$

Then, the solution of (2.10) is $D = -\alpha \frac{P_k}{\|P_k\|}$, we conclude the solution of (2.8), by:

$$Y_k(\alpha) = Y^0 - \alpha \frac{P_k}{\|P_k\|}.$$

□

In the following, at each iteration, we need to choose α so that $Y_k(\alpha)$ is in S_{++}^{n+1} and subsequently this will ensure that X_{k+1} is in S_{++}^n.

2.5. The step value. By definition

$$X = T_k^{-1}(Y) = \frac{1}{y_{n+1,n+1}}(L_k Y\,[n]\,L_k^T),$$

so that, if Y_k is a positive semidefinite matrix, so is X_{k+1}.

We conclude that, the primal new iteration is given by

$$X_{k+1} = T_k^{-1}(Y_k) = \frac{1}{(y_k)_{n+1,n+1}} L_k Y_k\,[n]\,L_k^T$$

and its dual is

$$S_{k+1} = C - \sum_{i=1}^{m} w_i A_i + \Omega(X_{k+1})\ .$$

Proposition 2.1. *The value of α is defined as follows $\alpha \in (0, \alpha_k)$, where:*

$$\alpha_k = \left[max \left(\frac{(n+1)(m+\frac{s}{\sqrt{n}})}{\rho(D_k)}, \frac{(n+1)(m-s\sqrt{n})}{\rho(D_k)}, \frac{(n+1)(d_k)_{n+1,n+1}}{\rho(D_k)} \right) \right]^{-1},$$

and:

$$m = \frac{1}{n+1}\sum_{i=1}^{n+1}(D_k)_{i,i}\ ,\ \ s^2 = \frac{1}{n+1}\sum_{i,j=1}^{n+1}(D_k)_{i,j}^2 - m^2.$$

Proof.

From the theorem 2.1 (see [8]), we have:

$$\begin{cases} m - s\sqrt{n} \le min\lambda_i(D_k) \le m - \frac{s}{\sqrt{n}}, \\ m + \frac{s}{\sqrt{n}} \le max\lambda_i(D_k) \le m + s\sqrt{n}, \end{cases} \tag{2.16}$$

where $(\lambda_i)_{i=1}^{n}$ are an eigenvalues of D_k,

and by lemma 2.3, we have:

$$Y^*(\alpha) = Y_0 - \alpha D_k = Y_0 - \alpha\frac{P_k}{\|P_k\|},$$

we obtain the following inequality:

$$\frac{1}{n+1} - \alpha\frac{\max_i \lambda_i(D_k)}{\rho(D_k)} > 0,$$

where: $\rho(D_k)$ is a the spectral radius of D_k,

so:

$$\frac{1}{\alpha} > \frac{(n+1)max\lambda_i(D_k)}{\rho(D_k)}. \tag{2.17}$$

We are based on (2.16) and (2.17), we have:

$$\begin{cases} \frac{1}{\alpha} > \frac{(n+1)max\lambda_i(D[n])_k}{\rho(D_k)} > \frac{(n+1)}{\rho(D_k)}(m+\frac{s}{\sqrt{n}}), \\ \\ \frac{1}{\alpha} > \frac{(n+1)max\lambda_i(D_k)}{\rho(D_k)} > \frac{(n+1)min\lambda_i(D_k)}{\rho(D_k)} > \frac{(n+1)}{\rho(D_k)}(m-s\sqrt{n}), \\ \\ \frac{1}{\alpha} > \frac{(n+1)(d_k)_{n+1,n+1}}{\rho(D_k)}. \end{cases}$$

Summing up, X_{k+1} is a strictly feasible solution when $0 < \alpha < \alpha_k$.

□

In the following proposition, we will obtain a reduction of objective function of the problem $(\mathscr{P})$ when $\alpha \in (0, \alpha_k)$.

Proposition 2.2. *For to $\alpha \in (0, \alpha_k)$, we have:*

$$X_{k+1} = X_k - \frac{\alpha}{\frac{1}{n+1} - \alpha(d_k)_{n+1,n+1}} \left[L_k D_k L_k^T - (d_k)_{n+1,n+1} X_k\right],$$

$$X_{k+1} = X_k - \frac{\alpha}{\frac{\|P_k\|}{n+1} - \alpha(p_k)_{n+1,n+1}} L_k \left[P_k - (p_k)_{n+1,n+1} I_n\right] L_k^T,$$

and

$$F(X_{k+1}) - F(X_k) = \frac{1}{\frac{1}{n+1} - \alpha(D_k)_{n+1,n+1}} \mathfrak{r}_k(\alpha),$$

with:

$$\mathfrak{r}_k(\alpha) = -\alpha\left(C_k \bullet D_k[n] + \frac{1}{2} D_k[n] \bullet \Omega_k(X_k) - z_k(d_k)_{n+1,n+1}\right) < 0.$$

Proof. We have:

$$X = T_k^{-1}(Y(\alpha)) = \frac{1}{y_{n+1,n+1}(\alpha)} L_k Y(\alpha) L_k^T,$$

and:

$$Y(\alpha) = \frac{1}{n+1} I_{n+1} - \alpha D_k,$$

then

$$\begin{cases} Y(\alpha)[n] = \frac{1}{n+1} I_n - \alpha D_k[n], \\ \\ y_{n+1,n+1}(\alpha) = \frac{1}{n+1} - \alpha(d_k)_{n+1,n+1}. \end{cases}$$

Therefore:

$$X_{k+1}=X_k-\frac{\alpha}{\frac{1}{n+1}-\alpha(d_k)_{n+1,n+1}}\left[L_kD_k[n]L_k^T-(d_k)_{n+1,n+1}X_k\right].$$

On the one hand:

$$\begin{aligned}C\bullet X_{k+1}&=C\bullet X_k-\frac{\alpha}{\frac{1}{n+1}-\alpha(d_k)_{n+1,n+1}}\left[C\bullet L_kD_k[n]L_k^T-(d_k)_{n+1,n+1}C\bullet X_k\right]\\&=C\bullet X_k-\frac{\alpha}{\frac{1}{n+1}-\alpha(d_k)_{n+1,n+1}}\left[C_k\bullet D_k[n]-(d_k)_{n+1,n+1}C_k\bullet I_n\right],\end{aligned}$$

so:

$$-\alpha\left[C_k\bullet D_k[n]-(d_k)_{n+1,n+1}C_k\bullet I_n\right]=C_k\bullet Y[n]-y_{n+1,n+1}C_k\bullet I_n. \tag{2.18}$$

And on the other hand:

$$\begin{aligned}X_{k+1}\bullet\Omega(X_{k+1}))&=X_k\bullet\Omega(X_k)-\frac{\alpha}{\frac{1}{n+1}-\alpha(d_k)_{n+1,n+1}}\left[L_kD_k[n]L_k^T-(d_k)_{n+1,n+1}X_k\right]\bullet\Omega(X_k)\\&=X_k\bullet\Omega(X_k)-\frac{\alpha}{\frac{1}{n+1}-\alpha(d_k)_{n+1,n+1}}\left[D_k[n]-(d_k)_{n+1,n+1}I_n\right]\bullet\Omega_k(X_k).\end{aligned}$$

Then:

$$-\alpha(D_k[n]-(d_k)_{n+1,n+1}I_n)\bullet\Omega_k(X_k)=(Y[n]-y_{n+1,n+1}I_n)\bullet\Omega_k(X_k). \tag{2.19}$$

From (2.6), (2.18) and (2.19), we obtain the result. □

2.6. The initial feasible solution.

We have noticed since the beginning that the initial strictly feasible solution is unknown in the problem ($\mathscr{P}$), so in what follows our objective is to find an $n\times n$ matrix X so that:

$$\left\{X\in S^n_{++},A_i\bullet X=b_i,i=1,...,n\right\}. \tag{2.20}$$

To treat the problem, we present the following linear semidefinite program:

$$\begin{cases}\min\lambda,\\ A_i\bullet X+\lambda(b_i-A_i\bullet F^0)=b_i\ \text{ for }\ i=1,...,m\ ,\\ X\in S^n_{++},\ \lambda\geq 0,\end{cases} \tag{2.21}$$

where:

(1) F^0 is a fixed arbitrary symmetric semidefinite matrix,

(2) $\begin{pmatrix} F^0 & 0 \\ 0 & \lambda \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$ is a strictly feasible solution of problem (2.21).

We can be re-written as follows:

$$\begin{cases} \min C' \bullet X' + \frac{1}{2} X' \bullet \Omega'(X'), \\ A'_i \bullet X' = b'_i \quad \text{for } i = 1, ..., m \ , \\ X' \succeq 0, \end{cases} \tag{2.22}$$

with:

(1) $C'[i,j] = \begin{cases} \frac{n+1}{2} \ \text{ if } \ i = j = n+1 \\ \\ 0 \ \text{ otherwise} \end{cases},$

(2) $\Omega'(X')[i,j] = \begin{cases} n+1 \text{ if } i = j = n+1 \\ \\ 0 \ \text{ otherwise} \end{cases},$

(3) $A'_i = \begin{pmatrix} A_i & 0 \\ \\ 0 & (n+1)(b_i - A_i \bullet F^0) \end{pmatrix} \in S^n_{++}$, for $i = 1, ..., m$.

Theorem 2.1. *(See* [2]*) X^* is a solution of the problem (2.20) if and only if (X^*, λ^*) is an optimal solution of the problem (2.21) and $X^* \in S^n_{++}$.*

According to remark 2.2, we conclude that $\begin{pmatrix} F^0 & 0_n \\ 0_n^T & \frac{1}{n+1} \end{pmatrix}$ is a strictly feasible solution of (2.22), so it is enough to take $F^0 = \frac{1}{n+1} I_n$.

3. Statement of Algorithm and Its Convergence

Now, we summarize the primal-dual algorithm for solving the problems ($\mathscr{P}$) and ($\mathscr{D}$).

3.1. Primal-dual algorithm.

Algorithm 1 Solving a positive semidefinite quadratic problem

Initialize $X_0 = \frac{1}{n+1} I_n$ is a strictly feasible solution of $(\mathscr{P})$ and $Y_0 = \frac{1}{n+1} I_{n+1}$ is a strictly feasible solution of (2.5)

Choose $\varepsilon > 0$

while $(|\mathfrak{r}_k| \geq \varepsilon)$ or $(X_{k+1} \bullet S_{k+1} \geq \varepsilon)$ **do**

  Define:

  $C_k = L_k^T C L_k$

  $\Omega_k(X_k) = L_k^T \Omega(X_k) L_k$

  $z_k = C_k \bullet I_n + \frac{1}{2} I_n \bullet \Omega_k(X_k)$

  $C_k(z_k) = \begin{pmatrix} C_k & 0_n \\ & \\ 0_n^T & -z_k \end{pmatrix}$

  $\Omega_k(Y_0) = \begin{pmatrix} \Omega_k(X_k) & 0_n \\ 0_n^T & 0 \end{pmatrix}$

  $\nabla H(Y_0, z_k) = C_k(z_k) + \Omega_k(Y_0)$

  $A_i^k = L_k^T A_i L_k \quad i = 1, \ldots, m$

  $B_i^k = \begin{pmatrix} -A_i^k & 0_n \\ 0_n^T & b_i \end{pmatrix} \quad i = 1, \ldots, m$

  $B_i'^k = \begin{cases} B_i^k & \text{if } i = 1, \ldots, m \\ I_{n+1} & \text{if } i = m+1 \end{cases}$

  Solve the linear system $Mw = d$

  Calculate the matrix M and the vector d as follows:

  $d_i = -\nabla H(Y_0, z_k) \bullet B_i'^k \quad i = 1, \ldots, m+1,$

  $M_{i,j} = B_i'^k \bullet B_j'^k, \quad i, j = 1, \ldots, m+1$

  Compute:

  $\mathfrak{r}_k = -\alpha_k \left(C_k \bullet D_k[n] + \frac{1}{2} D_k[n] \bullet \Omega_k(X_k) - z_k (d_k)_{n+1,n+1} \right)$

  $P_k = \nabla H(Y_0, z_k) + \sum_{i=1}^{m+1} w_i B_i'^k$

  $D_k = \frac{P_k}{\|P_k\|}$

  $m = \frac{1}{n+1} \sum_{i=1}^{n+1} (D_k)_{i,i}$

  $s^2 = \frac{1}{n+1} \sum_{i,j=1}^{n+1} (D_k)_{i,j}^2 - m^2$

  Calculate the primal-dual solutions:

  $X_{k+1} = X_k - \frac{\alpha_k}{\frac{\|P_k\|}{n+1} - \alpha_k (p_k)_{n+1,n+1}} L_k \left[P_k - (p_k)_{n+1,n+1} I_n \right] L_k^T$

  $w_k \leftarrow w_{k+1}$

  $S_{k+1} = C - \sum_{i=1}^{m} (w_i)_{k+1} A_i + \Omega(X_{k+1})$

  Let $k \leftarrow k+1$

end while

3.2. Theoretical convergence of the algorithm.

In order to establish the convergence of our algorithm, we introduce a potential function similar to the one in [7] associated with the problem ($\mathscr{P}$) defined by:

$$\Phi(X,z_k)=\frac{[F(X)-z_k]}{(\det X)^{\frac{1}{n+1}}}. \tag{3.1}$$

Lemma 3.1. *For each iteration, we get a reduction of H i.e:*

$$H(Y,z_k)-H(Y_0,z_k)\leq 0.$$

Proof. From lemma 2.3:

$$Y=Y_0-\alpha\frac{P_k}{\|P_k\|}\ ,$$

then:

$$\begin{aligned}\nabla H(Y_0,z_k)\bullet(Y-Y_0) &= -\alpha\frac{\nabla H(Y_0,z_k)\circ P_k}{\|P_k\|}\\ &= -\alpha\,\|P_k\|<0.\end{aligned}$$

□

Lemma 3.2. $H(Y_k)\leq(1-\frac{\alpha}{n+1})H(Y_0)$ *where Y_k is the optimal solution of (2.10).*

Proof. See [4]. □

Lemma 3.3. *For all Y*

a symmetric matrix, with $0\prec Y\prec\frac{1}{n+1}I_{n+1}$, we have:

$$\ln det(Y)\geq -n-\frac{\alpha^2}{2(1-\alpha)}.$$

Proof. See [1]. □

The reduction of $\Phi(X)$ brings us to the reduction $F(X)-z_k$.

Lemma 3.4. *Let us X is a feasible solution of the problem ($\mathscr{P}$) if we have:*

$$\frac{\Phi(X_k,z_k)}{\Phi(X_0,z_k)}\leq\gamma,$$

then:

$$\frac{F(X_k)-z_k}{F(X_0)-z_k}\leq\gamma\ \ v(X_k).$$

such as:

$$v(X_k) = \left(\prod_{i=1}^{n} \frac{\lambda_i(X_k)}{\lambda_i(X_0)} \right)^{\frac{1}{n+1}}.$$

Proof. We have:

$$\frac{\Phi(X_k, z_k)}{\Phi(X_0, z_k)} = \frac{(F(X_k) - z_k)}{F(X_0) - z_k} \ \frac{(\det X_0)^{\frac{1}{n+1}}}{(\det X_k)^{\frac{1}{n+1}}}.$$

We obtain:

$$\frac{F(X_k) - z_k}{F(X_0) - z_k} = v(X_k) \ \frac{\Phi(X_k, z_k)}{\Phi(X_0, z_k)},$$

with: $v(X_k) = \left(\prod_{i=1}^{n} \frac{\lambda_i(X_k)}{\lambda_i(X_0)} \right)^{\frac{1}{n+1}}$. □

In the next theorem, we will show the reduction value of potential function Φ on any iteration.

Theorem 3.1. *The potential function is reduced by a constant value γ where:*

$$\frac{\Phi(X_k, z_k)}{\Phi(X_0, z_k)} \leq \gamma^k \ \text{with} \ \gamma = \exp\left[(k+1) \left(1 - \frac{1}{n+1} + \frac{\alpha^2}{2(1-\alpha)(n+1)} \right) \right].$$

Proof. We have:

$$\begin{aligned}
\frac{\Phi(X_{k+1}, z_k)}{\Phi(X_k, z_k)} &= \frac{[F(X_{k+1}) - z_k]}{(\det X_{k+1})^{\frac{1}{n+1}}} \ \frac{(\det X_k)^{\frac{1}{n+1}}}{[F(X_k) - z_k]} \\
&= \frac{H(Y_k, z_k)}{H(Y_0, z_k)} \frac{1}{(n+1)(det(Y_k))^{\frac{1}{n+1}}}.
\end{aligned}$$

From lemma 3.2 and lemma 3.3 we obtain:

$$\frac{\Phi(X_{k+1}, z_k)}{\Phi(X_k, z_k)} \leq \exp\left(1 - \frac{1}{n+1} + \frac{\alpha^2}{2(1-\alpha)(n+1)} \right),$$

by reccurence, we have:

$$\frac{\Phi(X_{k+1}, z_k)}{\Phi(X_0, z_k)} = \frac{\Phi(X_{k+1}, z_k)}{\Phi(X_k, z_k)} \frac{\Phi(X_k, z_k)}{\Phi(X_{k-1}, z_k)} \cdots \frac{\Phi(X_1, z_k)}{\Phi(X_0, z_k)} \leq \exp\left[(k+1) \left(1 - \frac{1}{n+1} + \frac{\alpha^2}{2(1-\alpha)(n+1)} \right) \right].$$

□

Theorem 3.2. *If we are satisfied with the following hypotheses:*

(1) *The initial feasible solution X_0 verifies: $X_0 \geq 2^{-2L} I_n$, where L is considered as the number of bits.*

(2) *The optimal solution X^* verifies: $X^* \leq 2^{-2L} I_n$, for any solution X we have: $-2^{3L} \leq F(X^*) \leq z_k \leq 2^{3L}$.*

Then, the number of iterations for each of the iterations to find the optimal solution is $O((n+1)L)$.

Proof. We have:

$$\frac{\Phi(X_k,z_k)}{\Phi(X_0,z_k)} \leq \exp\left[k\left(1-\frac{1}{n+1}+\frac{\alpha^2}{2(1-\alpha)(n+1)}\right)\right],$$

from lemma 3.4, theorem 3.1 and under hypotheses 1, 2, we will obtain:

$$\frac{F(X_k)-z_k}{F(X_0)-z_k} \leq \left(\prod_{i=1}^{n}\frac{\lambda_i(X_k)}{\lambda_i(X_0)}\right)^{\frac{1}{n+1}} \exp\left[k\left(1-\frac{1}{n+1}+\frac{\alpha^2}{2(1-\alpha)(n+1)}\right)\right].$$

Then: $K \geq hL(n+1)$, where $h \in \mathbb{R}_+^*$. □

4. Conclusions

In this paper, we have extended the results of [1, 4, 5, 9] to obtain a dual primal type algorithm for the solution of the positive semidefinite quadratic problem and for the convergence we have used a potential function with characteristics similar to the one in [7] where the polynomial complexity of the algorithm has been proved.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] F. Alizadeh, Interior point methods in semidefinite programming with applications to combinatorial optimization, SIAM J. Optim. 5 (1995), 13–51.

[2] D. Benterki, J.P. Crouzeix, B. Merikhi, A numerical feasible interior point method for linear semidefinite programs, RAIRO-Oper. Res. 41 (2007), 49–59.

[3] M. Franck, P. Wolfe, An algorithm for quadratic programing, Naval Res. Logist Quat. 3 (1956), 95-120.

[4] N. Karmarkar, A new polynomial-time algorithm for linear programming the method for linearly constrained convex programming, Combinatorica. 4 (1984), 373-395.

[5] Z. Kebbiche, A. Keraghel. A. Yassine, Extension of a projective interior point method for linearly constrained convex pro gramming, Appl. Math. Comput. 193 (2007), 553-559.

[6] Y. Nesterov, A. Nemirovskii, Conic formulation of a convex programming problem and duality, Optim. Methods Softw. 1 (1992), 95–115.

[7] M. Padberg, A different convergence proof of the projective method for linear programming, Oper. Res. Lett. 4 (1986), 253–257.

[8] H. Wolkowicz, G.P.H. Styan, Bounds for eigenvalues using traces, Linear Algebra Appl. 29 (1980), 471–506.

[9] A. Zerari, D. Benterki, A new efficient primal-dual projective interior point method for semidefinite programming, 2019 (2019), Article ID 12.