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SOLITON SOLUTIONS FOR THE BOUSSINESQ EQUATIONS

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Abstract. Based on three methods, exact solutions are obtained for cubic Boussinesq and modified Boussinesq equations. These are tanh method, sech method, and sine-cosine method. The obtained solutions contain solitary waves. The results reveal are very effective, convenient and quite accurate to such types of partial differential equations comparing with other methods.

Keywords: Tanh method, sech method, sine-cosine method, Nonlinear PDEs, Exact Solutions.

2000 AMS Subject Classification: 35G30, 35G25, 35Q80

1. Introduction

Nonlinear evolution equations (NLEEs) have come a long way through. These NLEEs appear in various areas of Physics, Engineering, Biological Sciences, Geological Sciences and many other places. These equations arise of necessity. Subsequently they are studied in these various scientific contexts. There are various aspects of these NLEEs that are studied by various scientists and engineers as the need arises. Some of the commonly studied aspects are integrability, conservation laws, stochasticity, numerical solutions and many other aspects.

The nonlinear wave phenomena observed in the above mentioned scientific fields, are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions, for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution. The investigation of exact solutions of NLPDEs plays an important role in the study of these phenomena. In the past several decades, many effective methods for obtaining exact solutions of NLPDEs have been presented. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions, such as Hirota's direct method [Hirota (2004)],

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tanh-sech method [Malfliet (1993), Wazwaz (2006)], extended tanh method [Ma and Fuchssteiner (1996), El-Wakil et al. (2007), Fan (2000), Wazwaz (2005)], hyperbolic function method [Xia and Zhang (2001)], sine-cosine method[Wazwaz (2004),Yusufoglu and Bekir (2006)], F-expansion method [Zhang (2006)], and the transformed rational function method [Ma and Lee (2009)].

The NLEEs equations that are going to be studied in this paper are the cubic Boussinesq and modified Boussinesq equations. The various analytical techniques that are going to be applied to these equations are tanh, sech, and sine-cosine methods. Subsequently, these solutions are supported by numerical simulations. Comparison of the present solutions is made with HPM and Padé technique. Excellent agreement is noted.

2. Outline of Tanh and Sech Methods

The tanh method will be introduced as presented by [Malfliet (1992), and by Wazwaz (2006)]. The tanh method is based on a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function to solve the coupled KdV equations.

The tanh method is developed by [Malfliet (1992)]. The method is applied to find out exact solutions of a coupled system of nonlinear differential equations with two unknowns:

$$p_1(u, v, u_t, v_t, u_x, v_x, u_{xx}, v_{xy}, \dots \dots \dots .) = 0$$

$$p_2(u, v, u_t, v_t, u_x, v_x, u_{xx}, v_{xy}, \dots \dots \dots .) = 0 \qquad (1)$$

Where p_1, p_2 are polynomials of the variable u, v and its derivatives. If we consider $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$, and let:

$$\xi = kx + \alpha y + \omega t \qquad (2)$$

so that $u(x,t) = U(\xi)$, $v(x,t) = V(\xi)$we can use the following changes:

$$\frac{\partial u}{\partial x} = k\,\frac{du}{d\xi},\ \frac{\partial u}{\partial y} = \alpha\,\frac{du}{d\xi},\frac{\partial u}{\partial t} = \omega\,\frac{du}{d\xi}\,,\frac{\partial^2 u}{\partial x^2} = k^2\,\frac{d^2u}{d\xi^2} \qquad (3)$$

and so on, then Eq. (1) becomes an ordinary differential equation

$$Q_1(u, v, u', v', u'', v'', \dots \dots .) = 0$$

$$Q_2(u, v, u', v', u'', v'', \dots \dots .) = 0 \qquad (4)$$

WithQ_1, Q_2being another polynomials form of there argument, which will be called the reduced ordinary differential equations of Eq. (4). Integrating Eq. (4) as long as all terms contain derivatives, the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well. Now finding the traveling wave solutions to Eq. (1) is equivalent to obtaining the solution to the reduced ordinary differential equation (4). For the tanh method, we introduce the new independent variable [Malfliet (1992)]:

3. Tanh Method

We now describe the tanh method for the given partial differential equations (4). To use this method, consider nonlinear equation of form:

$$Y = \tanh(\xi) \tag{5}$$

that leads to the change of variables:

$$\frac{du}{d\xi} = (1 - Y^2)\frac{du}{dY}$$

$$\frac{d^2u}{d\xi^2} = -2Y(1 - Y^2)\frac{du}{dY} + (1 - Y^2)^2\frac{d^2u}{dY^2},$$

$$\frac{d^2u}{d\xi^2} = 2(1 - Y^2)(3Y^2 - 1)\frac{du}{dY} - 6Y(1 - Y^2)^2\frac{d^2u}{dY^2} + (1 - Y^2)^3\frac{d^3u}{dY^3} \tag{6}$$

4. Sech Method

The Sech method consider nonlinear equation of form: [Davodi et al (2009)]

$$Y(x, t) = \mathrm{sech}(\xi) \tag{7}$$

that leads to the change of variables:

$$\frac{d}{d\xi} = -Y\sqrt{(1 - Y^2)}\frac{d}{dY}$$

$$\frac{d^2}{d\xi^2}\ Y(1 - 2Y^2)\frac{d}{dY} + Y^2(1 - Y^2)\frac{d^2}{dY^2}$$

$$\frac{d^3}{d\xi^3} = -Y\sqrt{(1 - Y^2)}\{(1 - 6Y^2)\frac{d}{dY} + 3Y(1 - 2Y^2)\quad \frac{d^2}{dY^2} + Y^2(1 - Y^2)\quad \frac{d^3}{dY^3}\} \tag{8}$$

The next crucial step is that the solution we are looking for is expressed in the form

$$u(x, t) = \sum_{i=0}^{m} a^i Y^i, \quad v(x, t) = \sum_{i=0}^{m} b^i Y^i \tag{9}$$

where the parameters m, and n can be found by balancing the highest-order linear term with the nonlinear terms in Eq. (4), and $k, \lambda, a_0, a_1, \dots, a_m$, $b_0, b_1, \dots, b_m$ are to be determined. Substituting (9) into (4) will yield a set of algebraic equations for $k, \lambda, a_0, a_1, \dots, a_m$, $b_0, b_1, \dots, b_m$ because all coefficients of Y^i have to vanish. From these relations, $k, \lambda, a_0, a_1, \dots, a_m$, $b_0, b_1, \dots, b_m$ can be obtained. Having determined these parameters, knowing that m , n, s are positive integers in most cases, and using (9) we obtain analytic solutions $u(x, t)$, $v(x, t)$ in a closed form [Malfliet (1992)]. The tanh method seems to be powerful tool in dealing with coupled nonlinear physical models.

5. The Sine-Cosine Function Method

Consider the nonlinear partial differential equation in the form: see [Mitchell(1980), Parkes (1998), Khater (2002)]

$$F\left(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{yy}, \dots \dots \dots \dots\right) = 0 \tag{10}$$

where u(x, y, t) is a traveling wave solution of nonlinear partial differential equation Eq. (10). We use the transformations,

$$\xi = x + y - \lambda t \tag{11}$$

This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -\lambda \frac{d}{d\xi}(.),\ \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.),\frac{\partial}{\partial y}(.) = \frac{d}{d\xi}(.) \tag{12}$$

Using (11) to transfer the nonlinear partial differential equation Eq. (10) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \dots\dots\dots\dots\dots) = 0 \tag{13}$$

The ordinary differential equation (13) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form: [Ali et al (2007), Wazwaz (2004)]

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi) \quad , \quad \text{or} \quad f(\xi) = \alpha \cos^{\beta}(\mu\xi) \quad , \qquad |\xi| \le \frac{\pi}{2\mu} \tag{14}$$

Where α , μ, and β are parameters to be determined, μ and λ are the wave number and the wave speed, respectively. Parks (1998) use

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi)$$
$$f'(\xi) = \alpha\,\beta\,\mu\ \sin^{\beta-1}(\mu\xi)\ \cos(\mu\xi) \tag{15}$$
$$f''(\xi) = \alpha\,\beta(\beta-1)\,\mu^2\ \sin^{\beta-2}(\mu\xi) - \alpha\,\beta^2\mu^2\ \sin^{\beta}(\mu\xi)$$
$$f'''(\xi) = \alpha\,\beta(\beta-1)(\beta-2)\mu^3\ \sin^{\beta-3}(\mu\xi)\cos(\mu\xi) - \alpha\,\beta^3\mu^3\ \sin^{\beta-1}(\mu\xi)\cos(\mu\xi)$$

and their derivative. Or use

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi)$$
$$f'(\xi) = -\,\alpha\,\beta\,\mu\ \cos^{\beta-1}(\mu\xi)\ \sin(\mu\xi) \tag{16}$$
$$f''(\xi) = \alpha\,\beta(\beta-1)\,\mu^2\ \cos^{\beta-2}(\mu\xi) - \alpha\,\beta^2\mu^2\ \cos^{\beta}(\mu\xi)$$
$$f'''(\xi) = -\,\alpha\,\beta(\beta-1)(\beta-2)\mu^3\ \cos^{\beta-3}(\mu\xi)\sin(\mu\xi) + \alpha\,\beta^3\mu^3\ \cos^{\beta-1}(\mu\xi)\sin(\mu\xi)$$

and so on. Substitute (15) or (16) into the reduced equation (13), balance the terms of the sine functions when (15) are used, or balance the terms of the cosine functions when (16) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\sin^{k}(\mu\xi)$ or $\cos^{k}(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown'sα , μ and β, and solve the subsequent system.

6. Applications:

The tanh , sech , and sine-cosine methods are generalized on equations.

6.1. Example 1: Consider the cubic modified Boussinesq equation,

$$u_{tt} + u_{xxt} + \frac{2}{9}u_{xxxx} - (u^3)_{xx} = 0 \tag{17}$$

[Mohamed et al (2009)] tried to solve Eq.(17) by applied the Homotopy Perturbation method and Padé approximants. Eq.(17) has an exact solution[Mohamed et al (2009)],

$$u(x,t) = 1 + \tanh\frac{3}{2}(x - 2t) \tag{18}$$

The traveling wave hypothesis as given by

$$\xi = kx - \lambda t \tag{19}$$

The nonlinear partial differential equation (17) is carried to an ordinary differential equation

$$\lambda^2 \; U'' - k^2\,\lambda\,U''' + \frac{2}{9}k^4\,U'''' - 3\,k^2(U^2U')' = 0 \tag{20}$$

Integrating Eq.(20) twice with zero constant and we postulate the tanh series, Eq(20) reduces to

$$\lambda^2 \; U - k^2\,\lambda\,U' + \frac{2}{9}k^4\,U'' - k^2U^3 = 0 \tag{21}$$

6.1.1. Tanh Method : Applying tanh method with the transformaion in (5), Eq.(21) becomes:

$$\lambda^2 \; U - k^2\,\lambda\,(1 - Y^2)\frac{dU}{dY} + \frac{2}{9}k^4\,[-2Y(1 - Y^2)\frac{dU}{dY} + (1 - Y^2)^2\frac{d^2U}{dY^2}] - k^2U^3 = 0 \tag{22}$$

Now to determine the parameters m, we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq. (22) we balance U'' with U^3, to obtain

2 + m = 3m , then m = 1

The tanh method admits the use of the finite expansion for both:

$$u(x,t) = U(Y) = a_0 + a_1Y \quad , \quad a_1 \neq 0 \tag{23}$$

Substituting $\frac{dU}{dY}, \frac{d^2U}{dY^2}$ in Eq. (22) to get:

$$\lambda^2[a_0 + a_1Y] - k^2\,\lambda\,(1 - Y^2)a_1 + \frac{2}{9}k^4[-2Y(1 - Y^2)a_1] - k^2[a_0{}^3 + 3a_0{}^2a_1Y + 3a_0a_1{}^2Y^2 + a_1{}^3Y^3] = 0 \tag{24}$$

then equating the coefficient of Y^i i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations:

$$Y^0 \;:\; \lambda^2[a_0] - k^2\,\lambda\,a_1 - k^2[a_0{}^3] = 0$$

$$Y^1 \;:\; \lambda^2[a_1] + \frac{2}{9}k^4[-2\,a_1] - k^2[\,3a_0{}^2a_1] = 0$$

$$Y^2 \;:\; k^2\,\lambda\,a_1 - k^2[\,3a_0a_1{}^2\,] = 0$$

$$Y^3 \;:\; \frac{4}{9}k^4a_1 - k^2a_1{}^3 = 0 \tag{25}$$

Solving the nonlinear system of algebraic equations (25) to get the following cases:

Case 1

$a_0 = -\frac{2}{3}k\ ,\ a_1 = \frac{2}{3}k\ ,\ \lambda = -\frac{4}{3}k^2$

$$\mathrm{u}_1(\mathrm{x},\mathrm{t}) = \frac{2}{3}k\left[-1 + \tanh\left\{k\left(x + \frac{4}{3}k\ t\right)\right\}\right] \quad (26)$$

Case 2

$a_0 = \frac{2}{3}k\ ,\ a_1 = \frac{2}{3}k\ ,\ \lambda = \frac{4}{3}k^2$

$$\mathrm{u}_2(\mathrm{x},\mathrm{t}) = \frac{2}{3}k\left[1 + \tanh\left\{k\left(x - \frac{4}{3}k\ t\right)\right\}\right] \quad (27)$$

Case 3

$a_0 = -\frac{2}{3}k,\ a_1 = -\frac{2}{3}k\ ,\ \lambda = \frac{4}{3}k^2$

$$\mathrm{u}_3(\mathrm{x},\mathrm{t}) = -\frac{2}{3}k\left[1 + \tanh\left\{k\left(x - \frac{4}{3}k\ t\right)\right\}\right] \quad (28)$$

Case 4

$a_0 = \frac{2}{3}k, a_1 = -\frac{2}{3}k\ ,\ \lambda = -\frac{4}{3}k^2$

$$\mathrm{u}_4(\mathrm{x},\mathrm{t}) = \frac{2}{3}k\left[1 - \tanh\left\{k\left(x + \frac{4}{3}k\ t\right)\right\}\right] \quad (29)$$

Remark:the exact solution given in Eq.(18) seems the same soliton solution in case 2, for $\mathrm{k} = \frac{3}{2}$ then:

$$\mathrm{u}_2(\mathrm{x},\mathrm{t}) = 1 + \tanh\frac{3}{2}(\mathrm{x} - 2\mathrm{t}) \quad (30)$$

Figures (1),(2),(3), and (4) represent the soliatry solution in cases (1),(2),(3), and (4) respectively at $\mathrm{k} = \frac{3}{2}$ and $-10 < x < 10$, and $0 < t < 1$.

6.1.2. Sine-Cosine Method

By applying Sine-cosine method to solve Eq.(21), and seeking the solution in (16) then

$$\lambda^2\alpha\cos^{\beta}(\mu\xi) + \mathrm{k}^2\lambda\ \alpha\,\beta\,\mu\ \cos^{\beta-1}(\mu\xi)\ \sin(\mu\xi)\ +$$
$$\frac{2}{9}\mathrm{k}^4\alpha\,\beta(\beta-1)\,\mu^2\ \cos^{\beta-2}(\mu\xi) - \alpha\,\beta^2\mu^2\ \cos^{\beta}(\mu\xi) - \mathrm{k}^2\alpha^3\cos^{3\beta}(\mu\xi) = 0 \quad (31)$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$\beta - 2 = 3\beta$, then $\beta = -1$

$$\lambda^2\alpha - \alpha\,\beta^2\mu^2 = 0$$
$$\frac{2}{9}\mathrm{k}^4\alpha\,\beta(\beta-1)\,\mu^2 - \mathrm{k}^2\alpha^3 = 0 \quad (32)$$

By solving the algebraic system (32), we get,

$$\lambda = \mp\mu\ ,\ \alpha = \frac{2}{3}\ \mathrm{k}\mu \quad (33)$$

Then by substituting (33) into Eq. (16) then, the exact soliton solution of equation (17) can be written in the form:

$$u_2(x,t) = \frac{2}{3}\, k\mu\ sec(\ \mu\ (kx \mp\ \mu t)) \qquad (34)$$

for $k = \frac{3}{2}$, $\mu = 1$, then:

$$u_2(x,t) = \ sec\left(\frac{3}{2}x \mp t\right) \qquad (35)$$

Figure (5) represents the soliatry of the solution $u_2(x,t) = \ sec\left(\frac{3}{2}x - t\right)$ at $-10 < x < 10$, and $0 < t < 1$.

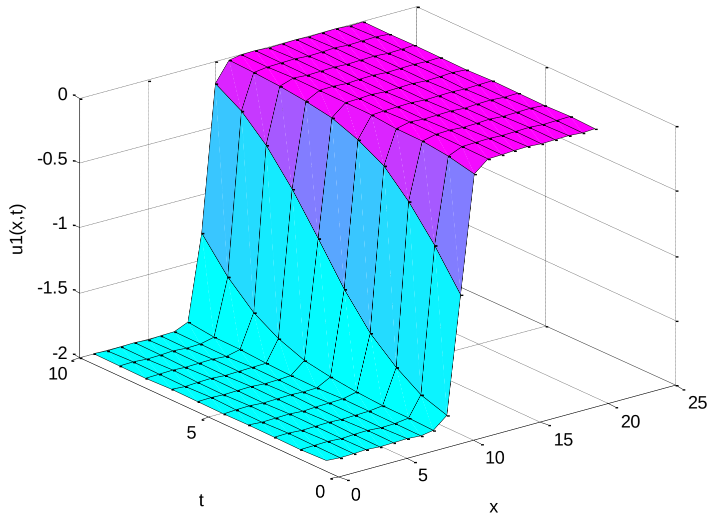

Figure (1)

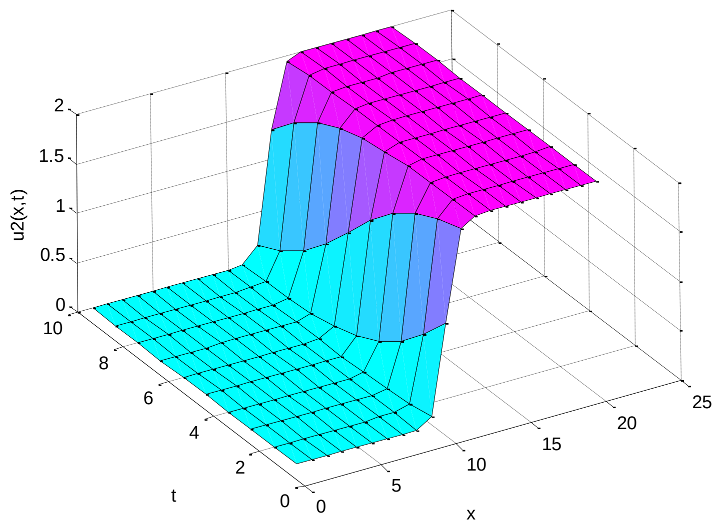

Figure (2)

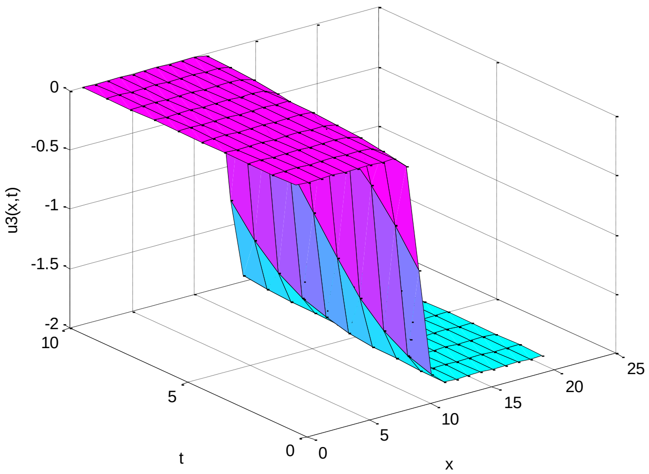

Figure (3)

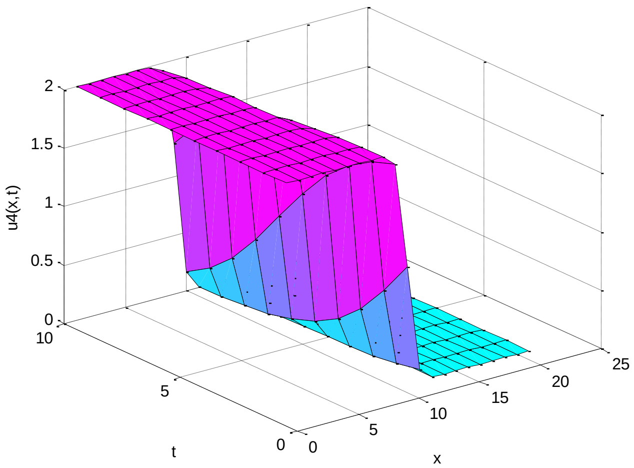

Figure (4)

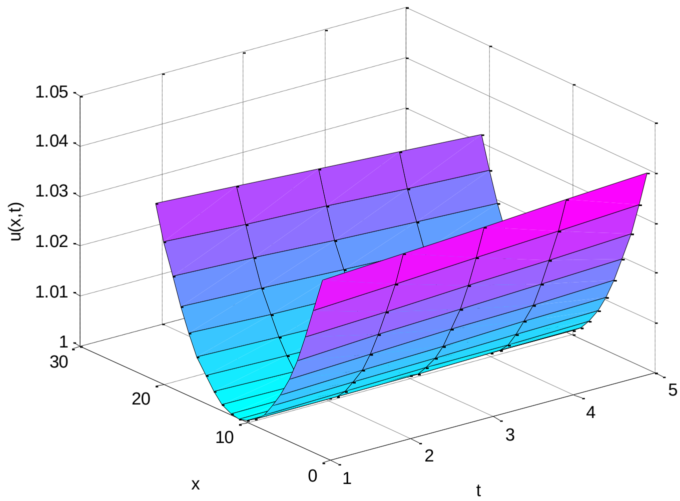

Figure (5)

6.2. Example 2. Cubic modified Boussinesq equation

Consider the cubic modified Boussinesq equation,

$$u_{tt} - u_{xxxx} - (u^3)_{xx} = 0 \quad (36)$$

[Mohamed et al (2009)] tried to solve Eq.(36) by applied the Homotopy Perturbation method and Padé approximants. The exact solution of Eq.(36) is;

$$u(x,t) = \sqrt{2}\,\mathrm{sech}(x-t) \quad (37)$$

The nonlinear partial differential equation (36) is carried to an ordinary differential equation using the transformation

$$\xi = kx - \lambda t \quad (38)$$

then

$$\lambda^2 U'' - k^4 U'''' - 3\,k^2\,(U^2 U')' = 0 \quad (39)$$

Integrating Eq.(39) twice and assuming the constant of integration equal to zero, then

$$\lambda^2\ U - k^4 U'' - k^2 U^3 = 0 \quad (40)$$

6.2.1. Sech method

We postulate the following sech series Eq. (40) reduces to

$$\lambda^2\ U\ - k^4\{\,Y(1-2Y^2)\frac{dU}{dY} + Y^2(1-Y^2)\frac{d^2U}{dY^2}\} - k^2U^3 = 0 \quad (41)$$

Now, to determine the parameter m , we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq. (41) we balance U'' with U^3 , to obtain m=1.

The sech method admits the use of the finite expansion for :

$$u(x,t) = U(Y) = a_0 + a_1 Y\ , a_1 \neq 0 \quad (42)$$

Substituting U, U', U'' from Eq. (42) into Eq. (41), then equating the coefficient of Y^i, i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$\lambda^2\,[a_0\,] - k^2[a_0{}^3\,] = 0$$
$$\lambda^2[a_1] - k^4 a_1 - k^2[\,3a_0{}^2 a_1\,] = 0$$
$$- k^2[\,3a_0 a_1{}^2\ \,] = 0$$
$$-k^4(-2)a_1 - k^2[\,a_1{}^3\ \,] = 0 \quad (43)$$

Solving the nonlinear systems of equations (43) we can get:

$$\lambda = \mp k^2,\ a_0 = 0\ ,\ a_1 = \mp\sqrt{2}\ \ k \quad (44)$$

<u>Case 1</u>

$$u_1(x,t) = \sqrt{2}\,k\ \ sech[k\,(x - kt)] \quad (45)$$

<u>Case 2</u>

$$u_2(x,t) = \sqrt{2}\,k\ \ sech[k\,(x + kt)] \quad (46)$$

Case 3

$$u_3(x,t) = -\sqrt{2}\,k \;\; sech[k\,(x - k\,t)] \tag{47}$$

Case 4

$$u_4(x,t) = -\sqrt{2}\,k \;\; sech[k\,(x + kt)] \tag{48}$$

Remark: the exact solution given in Eq.(37) compatible with soliton solution in case 1, for $\mathrm{k} = 1$ then,

$$\mathrm{u_1(x,t)} = \sqrt{2}\ \ \mathrm{sech}\ \ (\mathrm{x} - \mathrm{t}) \tag{49}$$

Figure (6) represents the soliatry of the solution in (49) at $-10 < x < 10$, and $0 < t < 1$.

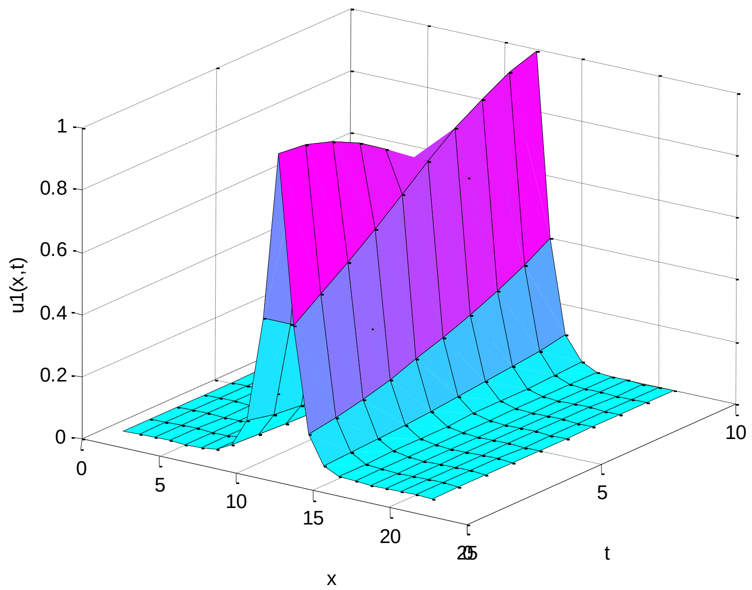

Figure (6)

6.2.2. Sine-Cosine Method

By applying Sine-cosine method to solve Eq.(40), and seeking the solution in (16) then

$$-\lambda^2\,\alpha\,\beta\,\mu\,\cos^{\beta-1}(\mu\xi)\sin(\mu\xi) + k^4\,[\,\alpha\,\beta(\beta-1)(\beta-2)\mu^3\,\cos^{\beta-3}(\mu\xi)\sin(\mu\xi) - \alpha\,\beta^3\mu^3\,\cos^{\beta-1}(\mu\xi)\sin(\mu\xi)] + 3\,k^2\alpha^3\beta\,\mu\,\cos^{3\beta-1}(\mu\xi)\,\sin(\mu\xi) = 0 \tag{50}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following algebraic system:

$\beta - 3 = 3\beta - 1$, then $\beta = -1$

$$-\lambda^2\,\alpha\,\beta\,\mu\; - \alpha\;\beta^3\mu^3k^4\;= 0$$

$$k^4\,\alpha\,\beta(\beta-1)(\beta-2)\mu^3 + 3\,k^2\alpha^3\,\beta\,\mu\;= 0 \tag{51}$$

By solving the algebraic system (51), we get,

$\lambda = \mp i\, k^2 \mu\,, \quad \alpha = \mp i\sqrt{2}\, k\mu$ (52)

Then by substituting (52) into Eq. (15) then, the exact soliton solution of equation (36) can be written in the form:

$u\,(x,t) = \mp i\sqrt{2}\, k\mu\, sec(\,\mu\, k(x \pm i\, k\, \mu\, t))$ (53)

or

$u\,(x,t) = \mp \sqrt{2}\, k\mu \mathrm{sech}(\mu\, k(ix \mp k\, \mu\, t))$ (54)

for $k = \mu = 1$, Eq.(54) becomes

$u\,(x,t) = \mp \sqrt{2}\mathrm{sech}(ix \mp\ t)$ ion

(55)

Remark

Eq.(55) is compatable with Eq.(49), that means sech method, and cosine method gave the same solution.

7. Conclusion

In this Letter, the tanh, sech, and sine-cosine function methods have been successfully applied to find the solution for nonlinear partial differential equations. We can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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