Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:200
https://doi.org/10.28919/jmcs/7686
ISSN: 1927-5307

INVERSE PROBLEM FOR DIFFERENTIAL OPERATORS WITH BOUNDARY CONDITIONS DEPENDENT ON EIGENPARAMETER

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Abstract. In this paper, we give a reconstruction formula for the potential q for a second order differential equation with boundary condition which contains spectral parameter. For this as methodology, we use Prüfer substitution that has an advantage different from other methods. Because in this method, we do not need any information of eigenfunctions.

Keywords: Sturm-Liouville; Inverse problems; Eigenvalues; Prüfer substitutions; Nodal points.

2010 AMS Subject Classification: 34L05, 34B24.

1. Introduction

The spectral theory of operators is widely used in various fields of mathematics, physics, and mechanics. The main sources of the spectral theory of linear operators are the problems of the theory of vibration like wire vibration, membrane vibration, etc [1, 2].

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Received August 19, 2022

There are two types of differential operators as regular and singular. If coefficients are continuous functions in the definition region and also the interval is finite, the problem is regular; differential operators whose definition region is infinite or whose coefficients are not a summable problem is called singular. Spectral theory for second-order regular operators is known today as Sturm-Liouville theory. The distribution of eigenvalues of operators with a discrete spectrum and defined throughout the space is particularly interesting for Quantum mechanics.In later years, regular systems of two first order equations are discussed.

Inverse problems of spectral analysis have an important role in the theory of linear differential operators. The inverse differential operator problem is defined as below.

1) According to which spectral data is the differential operator defined?

2) Is the operator defined exactly according to spectral data?

3) By which method it is possible to define operators according to data?

First result of the inverse problem was found by [3]. In this study, Ambarzumyan showed that the potential function is identical to zero if the eigenvalues have a certain formula. In the following years, many results have been obtained using this first step of the inverse problem. Also, the inverse problem the solution is very popular for the problems that contain the eigenvalue parameter under the boundary condition[4]-[22].

A different approach to inverse problems was given by Mclaughlin in 1988 [23]. Mclaughlin showed the uniqueness of the potential function using zeros of eigenfunctions. This theory is known in the literature as inverse nodal problems. In the following years, the theory was handled for different types of operators, and many results were obtained in this subject [24]-[38]. For example; Law and colleagues have provided a formula for potential function by the nodal points, which has been an inspiration for many people [31]. In addition to all of these, this problem was applied to diffusion, singular Sturm-Liouville, Dirac, and other operators, and successful results were obtained. We should note that inverse nodal problems were first studied by Koyunbakan in Turkey . In this first study, the uniqueness of the potential function has been created by using nodal points for the diffusion operator [39, 40, 41, 42, 43, 44]. In this way, inverse nodal problems have been studied by many researchers from different universities.

In this study, the solution of the inverse nodal problem for the Sturm-Liouville problem, which contains the eigenvalue parameter in the boundary condition is examined. For this, Prüfer transformation [2, 45, 46] is used which is more advantageous than the others.

2. PRELIMINARIES

We determine some formulas and a reconstruction for the potential q which was obtained as a solution of an inverse problem. We are dealing with inverse problem for a second order differential operator by taking a new type of spectral data in this boundary condition. Let us consider the following Sturm-Liouville problem on the interval $0 \leq x \leq \pi$

$$-y''(x) + q(x)y(x) = \lambda^2 y(x) \tag{1.1}$$

where $q(x)$ is a real-valued functions which is integrable in the interval $[0,\pi]$, the spectrum of the problem uniquely determines $q(x)$, almost everywhere then the boundary conditions are given by

$$y(0) = 0 \tag{1.2}$$

$$(a_1\lambda + b_1)y'(\pi) + (a_2\lambda + b_2)y(\pi) = 0. \tag{1.3}$$

We can rewrite the equation (1.3) as

$$\frac{y'(\pi)}{y(\pi)} = \frac{-(a_2\lambda + b_2)}{a_1\lambda + b_1} \tag{1.4}$$

where a_i,b_i are real constants $(i = 1,2)$, and we introduce a Prüfer substitution [45, 46]for the solution y_k of (1.1) as follow

$$\begin{aligned} y(x) &= s(x)\sin(\lambda\theta(x)), \\ y'(x) &= s(x)\lambda\cos(\lambda\theta(x)), \end{aligned} \tag{1.5}$$

or

$$\frac{y'}{y} = \lambda\cot(\lambda\theta(x)) \tag{1.6}$$

where $s(x)$ is amplitude and $\theta(x,\lambda)$ is Prüfer's variable.

Taking derivative to both sides of (1.6)

$$\left(\frac{y'}{y}\right)' = \frac{-\lambda^2\theta'(x)}{\sin^2(\lambda\theta(x))} \tag{1.7}$$

the left hand side of (1.7) can be written as follow

$$\frac{y''}{y} = \left(\frac{y'}{y}\right)' + \left(\frac{y'}{y}\right)^2 \tag{1.8}$$

Putting (1.6) and (1.7) into (1.8), we have determined

$$-\frac{y''}{y} = \left(\frac{\lambda^2\theta'(x)}{\sin^2(\lambda\theta(x)} - \lambda^2 cot^2(\lambda\theta(x)\right) \tag{1.9}$$

now we plug (1.9) into the problem (1.1) produces

$$-(q(x)-\lambda^2) = \left(\frac{\lambda^2\theta'(x)}{\sin^2(\lambda\theta(x)} - \lambda^2\cot^2(\lambda\theta(x))\right)$$

and

$$\left(q(x)-\lambda^2-\lambda^2\frac{\cos^2(\lambda\theta(x))}{\sin^2(\lambda\theta(x))}\right)\sin^2(\lambda\theta(x)) = -\theta'(x)\lambda^2$$

therefore,

$$\theta'(x) = \left(\frac{-q(x)}{\lambda^2} + 1 + \frac{\cos^2(\lambda\theta(x))}{\sin^2(\lambda\theta(x)}\right)\sin^2(\lambda\theta(x)). \tag{1.10}$$

By using some trigonometric identity to the equation (1.10), the corresponding Prüfer equation that we obtain

$$\theta'(x) = 1 - \frac{q(x)}{2\lambda^2} + \frac{q(x)}{2\lambda^2}\cos(2\lambda\theta(x)). \tag{1.11}$$

3. Main Results

Now,we are ready to construct the asymptotics for eigenvalues.

Theorem 3.1. *Eigenvalues λ_k of the problem (1.1)-(1.3) satisfy*

$$\lambda_k = (k-\frac{1}{2}) + \frac{a_2}{a_1\pi(k-\frac{1}{2})} + O\left(\frac{1}{k}\right)$$

as $k\to\infty$

Proof. Integrating both sides of (1.11) from 0 to π with respect to x

$$\int_0^\pi \theta'(x)dx = \int_0^\pi \left(1 - \frac{q(x)}{2\lambda^2} + \frac{q(x)}{2\lambda^2}\cos(2\lambda\theta(x))\right)dx$$

by using the properties and definition of big O [47], we have seen that

$$\int_0^\pi q(x)\cos(2\lambda\theta(x))dx = O\left(\frac{1}{\lambda}\right)$$

then,

$$\theta(\pi) - \theta(0) = \pi - \frac{1}{2\lambda^2}\int_0^\pi q(x)dx + O\left(\frac{1}{\lambda^3}\right). \tag{1.12}$$

Now to find $\theta(\pi)$ and $\theta(0)$ we use (1.4), then from the boundary condition (1.2), we have the value of $\theta(0) = 0$ and to find the value of $\theta(\pi)$ from the boundary condition (1.3), we have

$$\lambda\cot(\lambda\theta(\pi)) = \frac{-(a_2\lambda + b_2)}{a_1\lambda + b_1}$$

after some calculation ,we obtain

$$\lambda\cot(\lambda\theta(\pi)) = \frac{-a_2}{a_1} + O\left(\frac{1}{\lambda}\right)$$

and

$$\lambda\theta(\pi) = arccot\left(\frac{-a_2}{a_1\lambda} + O\left(\frac{1}{\lambda^2}\right)\right). \tag{1.13}$$

By using Taylor expansion of *arccot* function for the right side of (1.13) we obtain

$$\theta(\pi) = \frac{(k-\frac{1}{2})\pi}{\lambda} - \frac{a_2}{a_1\lambda^2} + O\left(\frac{1}{\lambda^3}\right). \tag{1.14}$$

Substituting (1.14) into (1.12) for $\lambda = \lambda_k$ we get

$$\frac{(k-\frac{1}{2})\pi}{\lambda_k} - \frac{a_2}{a_1\lambda_k^2} + O\left(\frac{1}{\lambda_k^3}\right) = \pi - \frac{1}{\lambda_k^2}\int_0^\pi q(x)dx + O\left(\frac{1}{\lambda_k^3}\right)$$

and

$$\lambda_k = \frac{(k-\frac{1}{2})\pi}{\pi + \frac{a_2}{a_1\lambda_k^2} - \frac{1}{2\lambda_k^2}\int_0^\pi q(x)dx + O\left(\frac{1}{\lambda_k^2}\right)}$$

thus,

$$\lambda_k = (k - \frac{1}{2}) + \frac{a_2(k-\frac{1}{2})}{a_1\lambda_k^2\pi} - \frac{(k-\frac{1}{2})}{2\lambda_k^2\pi}\int_0^\pi q(x)dx + O\left(\frac{1}{(k)^2}\right)$$

As $\lambda \to \infty \Rightarrow \lambda_k \cong (k-\frac{1}{2})$,then

$$\lambda_k = (k-\frac{1}{2}) + \frac{a_2}{a_1\pi(k-\frac{1}{2})} - \frac{1}{2\pi(k-\frac{1}{2})}\int_0^{\pi} q(x)dx + O\left(\frac{1}{k^2}\right)$$

therefore,

$$\lambda_k = (k-\frac{1}{2}) + \frac{a_2}{a_1\pi(k-\frac{1}{2})} + O\left(\frac{1}{k}\right) \tag{1.15}$$

□

Theorem 3.2. *The nodal points of the problem (1.1)-(1.3) are*

$$x_j^k = \frac{j\pi}{k-\frac{1}{2}} + \frac{ja_2}{a_1(k-\frac{1}{2})^3} + \frac{1}{2k^2}\int_0^{x_j^k} q(t)dt + O\left(\frac{1}{k^3}\right)$$

Proof. Integrating (1.11) from 0 to x_j^k with respect to x

$$\int_0^{x_j^k} \theta'(x)dx = \int_0^{x_j^k}\left(1 - \frac{q(x)}{2\lambda^2} + \frac{q(x)}{2\lambda^2}\cos(2\lambda\theta(x))\right)dx$$

after some calculation, we get

$$\theta(x_j^k) - \theta(0) = x_j^k - \frac{1}{2\lambda^2}\int_0^{x_j^k} q(t)dt + O\left(\frac{1}{\lambda^3}\right) \tag{1.16}$$

By using Prüfer substitution (1.5), from boundary condition (1.2), we have

$$y(0) = s(0)\sin(\lambda\theta(0)) \Rightarrow \sin(\lambda\theta(0)) = 0, \theta(0) = 0.$$

Now we are going to find $\theta(x_j^k)$, since x_j^k are nodal points so $y(x_j^k) = 0$, therefore

$$y(x_j^k) = s(x_j^k)\sin(\lambda\theta(x_j^k)) = 0$$

while $s(x_j^k) \neq 0$,then

$$\sin(\lambda\theta(x_j^k)) = 0,$$

so

$$\lambda\theta(x_j^k) = j\pi \Rightarrow \theta(x_j^k) = \frac{j\pi}{\lambda}.$$

Now, we obtain that $\theta(0)=0$ and $\theta(x_j^k)=\frac{j\pi}{\lambda}$ plugging them into (1.16), for $\lambda=\lambda_k$ we determine

$$x_j^k=\frac{j\pi}{\lambda_k}+\frac{1}{2\lambda_k^2}\int_0^{x_j^k}q(t)dt+O\left(\frac{1}{\lambda_k^3}\right) \tag{1.17}$$

By inserting (1.15) into (1.17), we get

$$x_j^k=\frac{j\pi}{\left(k-\frac{1}{2}\right)\left(1+\frac{a_2}{a_1\pi(k-\frac{1}{2})^2}+O\left(\frac{1}{k^2}\right)\right)}+\frac{1}{2k^2}\int_0^{x_j^k}q(t)dt+O\left(\frac{1}{k^3}\right)$$

by using the properties and definition of big O [47], we obtain

$$x_j^k=\frac{j\pi}{k-\frac{1}{2}}+\frac{ja_2}{a_1(k-\frac{1}{2})^3}+\frac{1}{2k^2}\int_0^{x_j^k}q(t)dt+O\left(\frac{1}{k^3}\right)$$

This completes the proof. □

Theorem 3.3. *The nodal lengths of problem (1.1)-(1.3) are*

$$\ell_j^k=\frac{\pi}{k-\frac{1}{2}}+\frac{a_2}{a_1(k-\frac{1}{2})^3}+\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(t)dt+O\left(\frac{1}{k^3}\right)$$

Proof. By integrating (1.11) from x_j^k to x_{j+1}^k with respect to x

$$\int_{x_j^k}^{x_{j+1}^k}\theta'(x)dx=\int_{x_j^k}^{x_{j+1}^k}\left(1-\frac{q(x)}{2\lambda_k^2}+\frac{q(x)}{2\lambda_k^2}\cos(2\lambda\theta(x))\right)dx$$

$$\begin{aligned}\theta(x_{j+1}^k)-\theta(x_j^k)=&\int_{x_j^k}^{x_{j+1}^k}dx-\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(x)\cos(2\lambda\theta(x))dx\\&-\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(t)dt\end{aligned} \tag{1.18}$$

Since $\ell_j^k=x_{j+1}^k-x_j^k$, and in the same way in Theorem(2.2), we can find that $\theta(x_{j+1}^k)=\frac{(j+1)\pi}{\lambda_k}$, $\theta(x_j^k)=\frac{j\pi}{\lambda_k}$. Then insert them in (1.18), we obtain

$$\frac{(j+1)\pi}{\lambda_k}-\frac{j\pi}{\lambda_k}=x_{j+1}^k-x_j^k-\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(t)dt+O\left(\frac{1}{\lambda_k^3}\right),$$

$$\frac{\pi}{\lambda_k}=\ell_j^k-\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(x)dx+O\left(\frac{1}{\lambda_k^3}\right). \tag{1.19}$$

Now we use λ_k from (1.15) and plug it into (1.19)

$$\frac{\pi}{(k-\frac{1}{2})+\frac{a_2}{a_1\pi(k-\frac{1}{2})^2}+O\left(\frac{1}{k}\right)}=\ell_j^k-\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(x)dx+O\left(\frac{1}{\lambda_k^3}\right). \tag{1.20}$$

By using the property of big O , we get

$$\frac{\pi}{k-\frac{1}{2}}+\frac{\pi a_2}{a_1(k-\frac{1}{2})^3}+O\left(\frac{1}{k^3}\right)=\ell_j^k-\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(x)dx+O\left(\frac{1}{k^3}\right)$$

therefore,

$$\ell_j^k=\frac{\pi}{k-\frac{1}{2}}+\frac{a_2}{a_1(k-\frac{1}{2})^3}+\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(x)dx+O\left(\frac{1}{k^3}\right)$$

This completes the proof. □

Theorem 3.4. *For the problem (1.1)-(1.3) the potential function satisfies*

$$q(z)=\lim_{k\to\infty}2k\pi\left(\frac{(k-\frac{1}{2})}{\pi}-\frac{a_2}{a_1(k-\frac{1}{2})^2\pi\ell_j^k}-\frac{1}{\ell_j^k}\right)$$

for almost every $x\in(0,\pi)$ with $j=j_n(x)$.

Proof. By using nodal lengths

$$\ell_j^k=\frac{\pi}{k-\frac{1}{2}}+\frac{a_2}{a_1(k-\frac{1}{2})^3}+\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(t)dt+O\left(\frac{1}{k^3}\right)$$

$$\ell_j^k-\frac{\pi}{k-\frac{1}{2}}-\frac{a_2}{a_1(k-\frac{1}{2})^3}+O\left(\frac{1}{k^3}\right)=\frac{1}{2\lambda_k^2}\int_{x_j^k}^{x_{j+1}^k}q(t)dt$$

$$\frac{(k-\frac{1}{2})}{\pi}\,\ell_j^k-\frac{a_2}{a_1(k-\frac{1}{2})^2\pi}-1+O\left(\frac{1}{k^2}\right)=\frac{1}{2k\pi}\int_{x_j^k}^{x_{j+1}^k}q(t)dt. \tag{1.21}$$

Applying mean value theorem for integrals to (1.21), we obtain
where there exists $z\in[x_j^k,x_{j+1}^k]$ then $\Rightarrow\int_{x_j^k}^{x_{j+1}^k}q(t)dt=q(z)\ell_j^k$
we have

$$q(z)\ell_j^k=2k\pi\left(\ell_j^k-\frac{\pi}{k-\frac{1}{2}}-\frac{\pi a_2}{a_1(k-\frac{1}{2})^3}\right)+O\left(\frac{1}{k^3}\right).$$

Hence

$$q(z)=\lim_{k\to\infty}2k\pi\left(\frac{(k-\frac{1}{2})}{\pi}-\frac{a_2}{a_1(k-\frac{1}{2})^2\pi\ell_j^k}-\frac{1}{\ell_j^k}\right).$$

This completes the proof. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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