COMPUTING PSEUDOSPECTRA OF POLYNOMIAL EIGENVALUE PROBLEMS: DIRECT APPROACH VS LINEARIZATION

ZHENGSHENG WANG*, YUANJUN LI, AND CHUNTAO LIU

Department of Mathematics, Nanjing University of Aeronautics and Astronautics,

Nanjing 210016, P.R.China

Abstract. Pseudospectra of polynomial eigenvalue problems (PEPs) have been investigated in recent years. In this paper, we firstly investigate the grid method, linearization method for pseudospectra of PEPs. Then, we explore a method for computing the pseudospectra of large PEPs, which is so called generalized projection method that projects to reduce the size of the problem directly using the generalized Arnoldi iteration. At last, numerical experiments and comparisons are given to illustrate the efficiency of generalized projection method and the different sensitivity of these methods.

Keywords: Pseudospectra; Grid method; Generalized Arnoldi iteration, Linearization; Polynomial eigenvalue problem.

2000 AMS Subject Classification: 65F15

1. Introduction

Pseudospectra are an established tool for gaining insight into the sensitivity of the eigenvalues of a matrix to perturbations. Their use is widespread with applications in areas such as fluid mechanics, Markov chains, and control theory. Especially, pseudospectra associated with the standard and generalized eigenvalue problems have been widely investigated in recent years. We refer to Trefethen[18, 20] for thorough surveys of pseudospectra and their computation for a single matrix; see also the Web site[4]. Most of the existing work is for the standard eigenproblem, although attention has also been given to matrix pencils[7, 9, 16].

*Corresponding author

E-mail addresses: wangzhengsheng@nuaa.edu.cn (Z.S. Wang), lyj861125@163.com (Y. Li), liuchuntao9@126.com (C. Liu)

In this paper, we investigate pseudospectra of polynomial eigenvalue problems(PEPs) which is to find the solutions (λ, x) of

$$P(\lambda)x = (\lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0)x = 0 \tag{1}$$

where $A_k \in \mathcal{C}^{n \times n}, k = 0, 1, \cdots, m$.

The If $x \neq 0$ then λ is called an eigenvalue and x the corresponding right eigenvector; y is a left eigenvector if $y^* P(\lambda) = 0$. The set of eigenvalues of P is denoted by $\Lambda(P)$. When A_m is nonsingular P has mn finite eigenvalues, while if A_m is singular P has infinite eigenvalues[2, 8, 15]. Throughout this paper we assume that PEPs have only finite eigenvalues (and pseudoeigenvalues); how to deal with infinite eigenvalues is described in [7].

For notational convenience, we introduce

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0$$

The definition of the ε-pseudospectrum of P in [16] as following

$$\Delta_\varepsilon P(\lambda) = \{\lambda \in C : (P(\lambda) + \Delta p(\lambda))x = 0, x \neq 0, \|A_k\| \leq \varepsilon \alpha_k, k = 0 : m\} \tag{2}$$

Here the α_k are nonnegative parameters that allow freedom in how perturbations are measured, for example, in an absolute sense $\alpha_k \equiv 1$ or a relative sense $\alpha_k = \|A_k\|$. By setting $\alpha_k = 0$ we can force $\Delta A_k = 0$ and thus keep A_k unperturbed. The norm, here and throughout, is any subordinate matrix norm.

Specifically, when $m = 1$ and $\alpha_1 = 1$, The definition (2) reduces to the standard definition of ε-pseudospectra of a single matrix. when $m = 2$, it is so called the pseudospectra of quadratic eigenvalue problems(QEPs). And when $m = 3$, it is so called the pseudospectra of cubic eigenvalue problems(CEPs).

There is a generalization of the equivalence for the ε-pseudospectrum of PEPs in [7] as following.

Proposition 1

$$\Delta_\varepsilon P(\lambda) = \{\lambda \in C : \|P(\lambda)^{-1}\| \geq (\varepsilon p(|\lambda|))^{-1}\}$$

where $p(x) = \sum_{k=0}^m \alpha_k x^k$.

In this paper, we firstly investigate the grid method, linearization method for pseudospectra of PEPs. Then, we propose a new projection method for pseudopectra of large PEPs, which constructs a generalized projection method, so called generalized Arnoldi iteration. In this method, we directly project the original PEP onto a properly chosen low dimensional subspace to reduce to a PEP with lower order by the generalized Arnoldi iteraion. Then compute the pseudospectra of the reduced PEP using the above small dense problem methods. Numerical experiments described in the later section show the efficiency of this methods. At last, numerical experiments and comparisons are given.

The outline of this paper is as follows. In Section 2, we investigate the grid method, linearization method for pseudospectra of PEPs. In Section 3, the generalized projection method for pseudospectra of PEPs is proposed. In Section 4, some numerical experiments and comparisons are given. Finally, we make some concluding remarks.

2. The direct approach and linearization

In recent years, researchers have been studying for numerical methods[3, 6, 7, 9, 13, 16, 20] which can be applied to compute the pesudospectra of PEPs. In general, for small dense problems, there are the following main methods.

The basic method is the grid method based on Golub-Reinsch SVD (Singular Value Decomposition) by evaluting the scaled resolvent norm on a grid of points z in the complex plane and sending the results to a contour plotter. Another approach named transfer function method based on generalized Schur decomposition using the QZ algorithm for computing pseudospectra of PEPs [5, 6, 12, 13, 16, 21]. The third kind of the methods are the solvent-type methods which includes the following three special methods: the Newton's method, the Bernoulli iteration for QEPs[16], and the Schur method based on generalized Schur decomposition by the QZ method[6].

All the methods described above are intended for small to medium-scale problems for which Schur and other reductions are possible. For large, possibly sparse, structured problems, different techniques are necessary. These techniques can be classified into two categories: (1) those that approximate the norm of the resolvent directly, and (2) those that project to reduce the size of the problem and then compute the pseudospectra of the reduced problem using the above methods for small problems.

The characterization of the ε-pseudospectrum in Proposition 1 is the basis of the methods for computing pseudospectra of PEP. The main idea is to compute $p(|z|)\|P(\lambda)^{-1}\|$ on a grid in $\mathcal{C}$ and send results to a contour plotter. Especially, for the 2-norm, $\|P(\lambda)^{-1}\|_2 = (\sigma_{min}(P(z)))^{-1}$. This is so called the direct approach which consists of approximating $\|P(z)^{-1}\|$ at each grid point z. Techniques analogous to those used for single matrices can be applied, such as the Lanczos method applied to $P(z)^*P(z)$ or its inverse. See [19, 20] for more details and further references.

Algorithm 2.1 The grid method for pseudospectra of PEPs

(1) Construct a mesh Ω over a region of the complex plane that includes $\Lambda_\varepsilon(A)$;

(2) Compute $p(|z|)\|P(\lambda)^{-1}\|$ for every node z of Ω;

(3) Use a visualisation tool to display the computed pseudospectra.

In another hand, the classical approach in solving PEPs is to turn it into a linear eigenvalue problem. In the case of (2) this leads to the linearized generalized eigenvalue problem (GEP)

$$\left(\lambda \begin{bmatrix} A_m & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I \end{bmatrix} - \begin{bmatrix} -A_{m-1} & -A_{m-2} & \cdots & -A_0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \right) \begin{bmatrix} \lambda^{m-1}x \\ \lambda^{m-2}x \\ \vdots \\ x \end{bmatrix} = 0 \qquad (3)$$

or an equivalent standard eigenvalue problem (SEP)

$$\begin{bmatrix} -A_m^{-1}A_{m-1} & -A_m^{-1}A_{m-2} & \cdots & -A_m^{-1}A_0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} \lambda^{m-1}x \\ \lambda^{m-2}x \\ \vdots \\ x \end{bmatrix} = \lambda \begin{bmatrix} \lambda^{m-1}x \\ \lambda^{m-2}x \\ \vdots \\ x \end{bmatrix} \qquad (4)$$

where we assume throughtout the paper that A_m is nonsingular.

If $(\lambda, \begin{bmatrix} \lambda^{m-1}x^T, \lambda^{m-2}x^T, \cdots, x^T \end{bmatrix}$ is an eigenpair of (3), then x is an eigenvector of PEPs (1) associated with the eigenvalue λ. Therefore, we may compute the pseudospectra of PEP using the methods for those of SEP or GEP by this linearization technique. However, it suffers some disadvantages, such as

solving the GEP (3) or SEP (4) of m times of the dimension of the original PEP, and more importantly, the loss of original structure of the PEP in the prosess of linearization.

The projection methods for single matrix, Toh and Trefethen[17] and Wright and Trefethen[22] approximate the resolvent norm by the Arnoldi method. Simoncini and Gallopoulos[13] show that a better but more costly version. However, these techniques are not applicable to the polynomial eigenvalue problem of degree larger than one because of the lack of a Schur form for the Arnoldi method to approximate.

Here, we apply the Arnoldi projection iteration to the SEP (4) and the use the grid method to compute the pseudospectra. We call it the linearization method.

Algorithm 2.2 The linearization method for pseudospectra of PEPs

(1) Do the Arnoldi iteration to the SEP (4) and get the small projection matrix ;
(2) Compute the pseudospectra of the small projection matrix using the grid method;
(3) Use a visualisation tool to display the computed pseudospectra of the small projection matrix which approximate the pseudospectra of original PEPs.

3. The generalized Arnoldi method

3.1 The generalized projection iteration

In this section, we discuss a generalized projection procedure. Firstly, we discuss a generalized Krylov subspace of a square matrix sequence $\{A_j\}$ and a vector sequence u_j. Then we study a generalized Arnolid procedure for generating an orthonormal basis of the generalized Krylov subspace[1].

Let $A_0, A_1, \cdots, A_{m-1}$ be a square matrix sequence of size n, and $u_0, u_1, \cdots, u_{m-1}$ be an n-dimensional vector sequence with $u_{m-1} \neq 0$. Then the sequences

$$r_0, r_1, r_2, \cdots, r_{p-1} \tag{5}$$

where

$r_0 = u_1$

$r_1 = u_2$

$\cdots$

$r_{m-2} = u_{m-1}$

$r_{m-1} = A_{m-1}u_{m-1} + A_{m-2}u_{m-2} + \cdots + A_0u_0$

$r_j = A_{m-1}r_{j-1} + A_{m-2}r_{j-2} + \cdots + A_0r_{j-m}$ for $j \geq m$

is called a generalized Krylov sequence based on $\{A_j\}$ and $\{u_j\}$.

$$\mathcal{K}_p(A; u) = span\{r_{m-2}, r_{m-1}, \cdots, r_{p+j-3}\} \tag{6}$$

is called an mth generalized Krylov subspace.

In fact, if one applies a Krylov subspace technique to (4), then an associated Krylov subspace would naturally be

$$\mathcal{K}_p(F; v) = span\{v, Fv, F^2v, \cdots, F^{p-1}v\} \tag{7}$$

where v is an initial vector of length $n \times m$, and matrix F as follow.

$$F = \begin{bmatrix} -A_m^{-1}A_{m-1} & -A_m^{-1}A_{m-2} & \cdots & -A_m^{-1}A_0 \\ I & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}$$

For notational convenience, we denote $A_{m-1} = -A_m^{-1}A_{m-1}, A_{m-2} = -A_m^{-1}A_{m-2}, \cdots, A_0 = -A_m^{-1}A_0$ and $v = [u_{m-1}^T, u_{m-2}^T, \cdots, u_0^T]^T$.

Then it immediately derives that the generalized Krylov vectors r_j of length n defined in (4) and the standard Krylov vectors $\{F^j v\}$ of length nm defined in (5) is related as the following form

$$\begin{bmatrix} r_j \\ r_{j-1} \\ \vdots \\ r_{j-m+1} \end{bmatrix} = F^{j-m+2}v, \text{ for } j \geq m-1 \tag{8}$$

Equation (8) indicates that the subspace $\mathcal{K}_p(A; u)$ of $\mathcal{C}^N$ should be able to provide sufficient information to let us directly work with the PEP, instead of using the subspace $\mathcal{K}_p(F; v)$ for the linearized eigenvalue problem (4).

The following procedure constructs the vectors $q_1^{(m-1)}, q_2^{(m-1)}, \cdots, q_p^{(m-1)}$ such that $\{q_1^{(m-1)}, q_2^{(m-1)}, \cdots, q_p^{(m-1)}\}$ is an orthonormal basis of the subspace $\mathcal{K}_p(A; u)$ which can generate through the following generalized Arnoldi procedure. The algorithm is described as follows.

Algorithm 3.1 The generalized Arnoldi iteration

(1) Start: Choose an initial unit vector $\beta = \|u_{m-1}\|_2$

(2) $\begin{bmatrix} q_1^{(m-1)} \\ q_1^{(m-2)} \\ \vdots \\ q_1^{(0)} \end{bmatrix} = \frac{1}{\beta} \begin{bmatrix} u_{m-1} \\ u_{m-2} \\ \vdots \\ u_0 \end{bmatrix}$

(3) For $j = 1, 2, 3, \cdots, p$ do the generalized full orthogonalization process

(4) $\begin{bmatrix} q_{j+1}^{(m-1)} \\ q_{j+1}^{(m-2)} \\ \vdots \\ q_{j+1}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} q_j^{(m-1)} \\ q_j^{(m-2)} \\ \vdots \\ q_j^{(0)} \end{bmatrix}$

(5) For $i = 1, 2, 3, \cdots, j$

(6) $h_{i,j} = (q_i^{(m-1)})^T q_{j+1}^{(m-1)}$

(7) $\begin{bmatrix} q_{j+1}^{(m-1)} \\ q_{j+1}^{(m-2)} \\ \vdots \\ q_{j+1}^{(0)} \end{bmatrix} = \begin{bmatrix} q_{j+1}^{(m-1)} \\ q_{j+1}^{(m-2)} \\ \vdots \\ q_{j+1}^{(0)} \end{bmatrix} - h_{i,j} \begin{bmatrix} q_i^{(m-1)} \\ q_i^{(m-2)} \\ \vdots \\ q_i^{(0)} \end{bmatrix}$

(8) End for

(9) $h_{j+1,j} = \|q_{j+1}^{(m-1)}\|_2$

(10) If $h_{j+1,j} = 0$, then stop

(11) $\begin{bmatrix} q_{j+1}^{(m-1)} \\ q_{j+1}^{(m-2)} \\ \vdots \\ q_{j+1}^{(0)} \end{bmatrix} = \frac{1}{h_{j+1,j}} \begin{bmatrix} q_{j+1}^{(m-1)} \\ q_{j+1}^{(m-2)} \\ \vdots \\ q_{j+1}^{(0)} \end{bmatrix}$

(12) End for

Let $\widetilde{H}_p$ denote the $(p+1) \times p$ upper Hessenberg matrix whose nonzero entries $h_{i,j}(i = 1, 2, \cdots, p+1, j = 1, 2, \cdots, p)$ are defined by Algorithm 2.1. H_m is the $m \times m$ matrix obtained from $\widetilde{H}_p$ by deleting the last row.

From the generalized Arnoldi iteration, we have the following relations:

$$\sum_{i=0}^{m-1} A_i Q_p^{(i)} = Q_p^{(m-1)} H_p + q_{p+1}^{(m-1)} e_p^T h_{p+1,p} \tag{9}$$

$$Q_p^{(j)} = Q_p^{(j-1)} H_p + q_{p+1}^{(j-1)} e_p^T h_{p+1,p}, 1 \le j \le n-1 \tag{10}$$

where $Q_p^{(i)} = [q_1^{(i)}, q_2^{(i)}, \cdots, q_p^{(i)}]$.

With $\widetilde{H}_p$, (9) and (10) can be written in the compact form:

$$\begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} Q_p^{(m-1)} \\ Q_p^{(m-2)} \\ \vdots \\ Q_p^{(0)} \end{bmatrix} = \begin{bmatrix} Q_{p+1}^{(m-1)} \\ Q_{p+1}^{(m-2)} \\ \vdots \\ Q_{p+1}^{(0)} \end{bmatrix} \widetilde{H}_p \tag{11}$$

This relation assembles the similarity between the generalized Arnoldi iteration and the classical well-known Arnoldi iteration[12, 21].

Bao[1] proved that the vector sequence $q_1^{(m-1)}, q_2^{(m-1)}, \cdots, q_p^{(m-1)}$ indeed is an orthonormal basis of the generalized Krylov subspace $\mathcal{K}_p(A; u)$.

3.2 The generalized projection method for pseudospectra of CEPs

For simplicity, for the rest of this paper, we are concerned only with $m = 3$. For taking $m = 3$, i.e., the pseudospectra of CEPs, we have the following simple algorithm with memory saving.

Algorithm 3.2 The generalized Arnoldi iteration ($m = 3$)

(1) Start: $q_1 = u_2/\|u_2\|_2; g_1 = u_1/\|u_2\|_2; f = u_0/\|u_2\|_2$

(2) For $j = 1, 2, \cdots, p$

(3) $q_{j+1} = A_2 q_j + A_1 g_j + A_0 f$

(4) $g_{j+1} = q_j$

(5) For $i = 1, 2, \cdots, j$
- $h_{i,j} = q_i^* q_{j+1}$
- $q_{j+1} = q_{j+1} - h_{i,j} q_i$
- $g_{j+1} = g_{j+1} - h_{i,j} g_i$

(6) End for

(7) $h_{j+1,j} = \|q_{j+1}\|_2$

(8) If $h_{j+1,j} = 0$, then stop

(9) $q_{j+1} = q_{j+1}/h_{j+1,j}; g_{j+1} = g_{j+1}/h_{j+1,j}$
$f = (P(:, 1 : j) - u_0 H(1, 1 : j)) H(2 : j+1, 1 : j)^{-1} e_j$

(10) End for

Finally, the orthogonality of the basis vectors $q_1^{(m-1)}, q_2^{(m-1)}, \cdots, q_j^{(m-1)}$ is directly obtained from the orthogonalization inner for-loop (step 5) and normalization is step 9 of Algorithm 3.2.

Furthermore, we have a new version of the generalized Arnoldi iteration by using the relations in Algorithm 3.1. In view of (10), for $Q_{p+1}^{(0)}$ partitioned as $Q_{p+1}^{(0)} = [q_1^{(0)}, \widehat{Q}^{(0)}]$, we can write

$$Q_p^{(1)} = Q_{p+1}^{(0)} \widetilde{H}_{p+1} = q_1^{(0)} \widetilde{H}_p(2 : p+1, :)$$

So we can compute the vector $q_{j+1}^{(0)}$ from $Q_j^{(1)}$ and $H(1 : j+1, 1 : j)$. The new procedure reduces memory requirement by almost $1/n$.

Then, we apply the generalized Krylov subspace and its orthonormal basis generated by the generalized Arnoldi procedure to develop a projection method for computing the pesudospectra of CEPs

$$P(\lambda)x = (\lambda^3 M_3 + \lambda^2 M_2 + \lambda M_1 + M_0)x = 0 \tag{12}$$

where $M_i \in \mathcal{C}^{n \times n}, (i = 0, 1, 2, 3)$, and $x \in \mathcal{C}$ is the eigenvector corresponding to the eigenvalue λ.

We follow the orthogonal Rayleigh-Ritz approximation procedure to derive a method for pseudospectra which approximates a large-scale CEP by a small-size CEP. Following the standard derivation, to apply Rayleigh-Ritz approximation technique based on the generalized subspace $\mathcal{K}_p(A; u)$ with $A_2 = -(M_3)^{-1}M_2, A_1 = -(M_3)^{-1}M_1, A_0 = -(M_3)^{-1}M_0$, we seek an approximate eigenpair (θ, z), where$\theta \in \mathcal{C}$ and $z \in \mathcal{K}_p(A; u)$, by imposing the following orthogonal condition, also called the Galerkin condition,

$$(\theta^3 M_3 + \theta^2 M_2 + \theta M_1 + M_0)z \perp \mathcal{K}_p(A; u) \tag{13}$$

or equivalently,

$$v^*(\theta^3 M_3 + \theta^2 M_2 + \theta M_1 + M_0)z = 0, v \in \mathcal{K}_p(A; u) \tag{14}$$

Since $z \in \mathcal{K}_p(A; u)$, it can be written as

$$z = Q_p h$$

where matrix Q_p of $n \times p$ is an orthonormal basis of $z \in \mathcal{K}_p(A; u)$ generated by GAR iteration (Algorithm 2.2), and h is an m-dimension vector. By (13) and (14), it yields that θ and h must satisfy the reduced CEP

$$\widehat{P}(\theta)h = (\theta^3 \widehat{M}_3 + \theta^2 \widehat{M}_2 + \theta \widehat{M}_1 + \widehat{M}_0)h = 0 \tag{15}$$

where

$$\widehat{M}_j = Q_p^* M_j Q_p, j = 0, 1, 2, 3 \tag{16}$$

The eigenpair (θ, h) of (15) defines the Ritz pair (θ, z). The Ritz pair is an approximate eigenpair of the CEP (12). We note that by explicitly formulating the matrices $\widehat{M}_j (j = 0, 1, 2, 3)$, essential structures of $\widehat{M}_j$ are preserved. For example, if M_1 is symmetric positive definite, so is $\widehat{M}_1$. As a result, essential spectral properties of the CEP will be preserved.

For sufficiently large p, the pseudospectra of original CEP (12) can be reasonably approximated by the corresponding pseudospectra of reduced CEP (15) as follows

$$\Lambda_\varepsilon(P) \approx \Lambda_\varepsilon(\widehat{P})$$

For $p \ll n$, the computation of pseudospectra of reduced CEP (15) will be $O((p/n)^3)$ times faster than that of original problem.

Therefore, we briefly describe the generalized Arnoldi method for computing pesudospectra of CEP (12) as follows.

Algorithm 3.2 The generalized Arnoldi method for pseudospectra of CEPs

(1) Compute the matrices $\widehat{M}_j = Q_p^* M_j Q_p, j = 0, 1, 2, 3$ in (16) by generalized Arnoldi iteration;

(2) Compute the pseudospectra of the reduced CEP (15) using the methods for pseudospectra of small dense CEP;

(3) Use a visualisation tool to display the computed pseudospectra.

4. The generalized Arnoldi method

In this section, we present some numerical experiments to illustrate the effectiveness of the method. For simplicity, here we take some not so large matrices as examples which are from real engineering problems and we hope to investigate more large problems in the same way in the future. Here, all the computations are finished with MATLAB 7.1 on PC (Intel(R) Pentium(R) 4, CPU Processor 3.4GHz 3.39GHz, Memory 0.99GB).

In the following numerical experiments, we give some comparison in terms of CUPtime (execution time) and pseudospectra plots of the different techniques. These techniques are as following:

- Grid-SVD: the gird SVD method directly on original PEPs (Algorithm 2.1);
- Lin-Grid: the grid SVD method on the linearization form(4) of the PEPs;
- Lin-Arn(m)-Grid: the grid SVD method on projection reduced problem by Arnoldi iteration of the linearization form (4) of the original PEP (Algorithm 2.2);
- G-Arn(m): the generalized Arnoldi method of the PEPs (Algorithm 3.2)

The starting vector u_2 of the G-Arn(m) method is chosen as a vector with all components equal to 1 and $u_0 = u_1 = 0$. According to the definition of pseudospectra of PEP, the so-called exact pseudospectra of the PEP are computed by the Grid-SVD method. And the small dense pseudospectra method of PEPs in Algorithm 3.1 may also be chosen as the Grid-SVD method here.

Example 1: The wing problem This is a very small problems. It can show the difference of direct method and lineariztion form methods for the pseudospectra of PEPs (here, $m = 2$, i.e., QEP). This example is based on a quadratic polynomial $Q(\lambda) = \lambda^2 M_2 + \lambda M_1 + M_0$ from [4]. The eigenproblem for $Q(\lambda)$ arose from the analysis of the oscillations of a wing in an airstream. The matrices are

$$M_2 = \begin{bmatrix} 17.6 & 1.280 & 2.890 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix}, M_1 = \begin{bmatrix} 7.66 & 2.450 & 2.100 \\ 0.23 & 1.040 & 0.223 \\ 0.60 & 0.756 & 0.658 \end{bmatrix},$$

$$M_0 = \begin{bmatrix} 121.0 & 18.9 & 15.9 \\ 0 & 2.70 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}$$

We Consider the 100×100 grid over the complex region of $min(real(e)) - h \le Re \le max(real(e)) + h, min(imag(e)) - h \le Im \le max(imag(e)) + h$, $h = 5$ and e is the set of the eigenvalues of the original QEPs, which is bigger than the complex region of the eigenvalues of the QEP.

Figure 1 shows the boundaries of ε-pseudospectra of the original QEP of wing problem by the Grid-SVD method with ε between 10^{-3} and $10^{-0.8}$. The eigenvalues are plotted as '×'. The solid curve marks the boundary of the ε-pseudospectra for $\varepsilon = 10^{-0.8}$. The picture shows that the pair of complex eigenvalues $\lambda = 0.88 \pm 8.4i$ are much more sensitive to perturbations than the other two complex pairs.

The eigenvalues of $Q(\lambda)$ are the same as those of the linearized problem $F - \lambda I$, where

$$F = \begin{bmatrix} -M_2^{-1}M_1 & -M_2^{-1}M_0 \\ I & 0 \end{bmatrix} \tag{19}$$

Figure 2 shows boundaries of ε-pseudospectrum for this matrix F, for the same ε as in Figure 1.
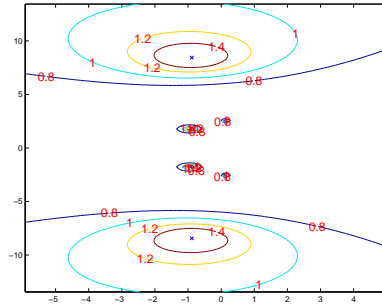

FIGURE 1.   pseudospectra of wing problem. Pseudospectra QEP with $\varepsilon \in [10^{-3}, 10^{-0.8}]$ by the Grid-SVD method .
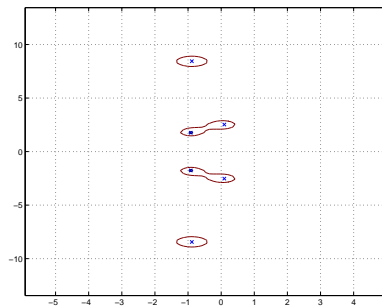

FIGURE 2.   pseudospectra of wing problem. For pseudospectrum boundaries of linerization form of QEP with $\varepsilon \in [10^{-3}, 10^{-0.8}]$.

Clearly, the ε-pseudospectra of the linearized form (19) do not give useful information about the behavior of the eigensystem of $Q(\lambda)$ under perturbations. This emphasizes the importance of defining and computing pseudospectra for the quadratic eigenvalue problem in its original form or generalized projection with its original form, instead of linearization directly or linearization-projection.

Example 2: The damped mass-spring system We now consider the connected damped mass-spring system illustrated in Figure 3. The ith mass of weight mi is connected to the $(i + 1)$st mass by a spring and a damper with constants k_i and d_i, respectively. The ith mass is also connected to the ground by a spring and a damper with constants κ_i and τ_i, respectively. The vibration of this system is governed by a second-order differential equation

$$M\frac{d^2}{dt^2}x + D\frac{d}{dt}x + Kx = 0$$

where the mass matrix $M = diag(m_1, \cdots, m_n)$ is diagonal, and the damping matrix C and stiffness matrix K are symmetric tridiagonal. The differential equation leads to the quadratic eigenvalue problem

$$(\lambda^2 M + \lambda C + K)x = 0$$

In our experiments, we took all the springs (respectively, dampers) to have the same constant $\kappa = 5$ (respectively, $\tau = 10$), except the first and last, for which the constant is 2κ (respectively, 2τ), and we took $m_i \equiv 1$. Then

$$C = \tau * tridiag(-1, 3, -1), K = \kappa * tridiag(-1, 3, -1)$$

and the quadratic eigenvalue problem is overdamped. We take an $n = 50$ degree of freedom mass-spring system over a 100×100 grids. The plots of the pseudospectra are given in Figure 4 and Figure 5.
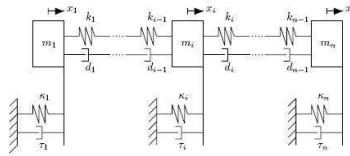

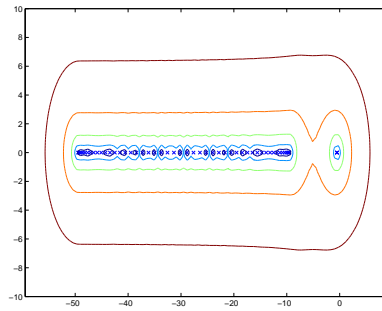FIGURE 3.   An n degree of freedom damped mass-spring system.

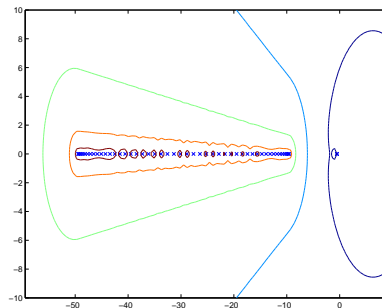FIGURE 4.   pseudospectra of damped mass-spring system by Lin-Grid method with 5 contour lines.

FIGURE 5.   pseudospectra of damped mass-spring system by Grid-SVD method with 5 contour lines.

In Table 1 we give the comparison in terms of CUPtime (in seconds) of the different methods. While the precise times are not important, the conclusion is clear: in this example, the Grid-SVD method is much faster than the Lin-Grid mehtod. (The high speed of the SVD method relative to its flop count is attributable to MATLABs very efficient svd function.)

TABLE I: *Comparison in terms of CUPtime of the different methods.*

Freedom n	Methods	CPUtime (in seconds)
50	Grid-SVD	19.095694
	Lin-Grid	79.889766
10	Grid-SVD	1.956311

Example 3: The dynamic analysis in structural engineering We use some structural engineering matrices from the Harwell-Boeing collection in MatrixMarket[10] to compare different methods for computing pseudospectra of cubic eigenvalue problem(CEP)(14). These matrices all represent dynamic analysis in structural engineering.

$$M_3 = 5I, M_2 = 3tridiag(-1, 3, -1), M_1 = bcsstm, M_0 = bcsstk$$

The data of matrices pair (M_1, M_0) are extracted from $(bcsstm01, bcsstk01)$, $(bcsstm04, bcsstk04)$, and $(bcsstm06, bcsstk06)$, respectively. These matrices are $42 \times 48, 132 \times 132$ and 420×420 ,respectively.

We consider the 50×50 and 100×100 respectively grid over the complex region of $min(real(eig)) - h \leq Re \leq max(real(eig)) + h$, $min(imag(eig)) - h \leq Im \leq max(imag(eig)) + h$, $h = 100$ and eig is the set of the eigenvalues of the original CEP. So the interesting pseudospectra region is bigger (more h) than the complex region of the eigenvalues of the original CEP.

In Table 2, we give the comparison in terms of CUPtime (in seconds) of the different methods. While the precise times are not important, the conclusion in this example is clear:

- The Grid-SVD method is much faster than the Lin-Grid.
- Lin-Arn(10)-Grid is much faster than the Grid-SVD.
- G-Arn(m) is much faster than the Grid-SVD.

TABLE II: *Comparison in terms of CUPtime.*

	Methods	*CPUtime (in seconds)*
50×50 *grid*	*Grid-SVD*	*5.426956*
bcsstm01	*Lin-Grid*	*60.582968*
bcsstk01	*Lin-Arn(10)-Grid*	*0.280336*
	G-Arn(10)	*0.439379*
	G-Arn(20)	*1.199913*
	G-Arn(48)	*7.723941*
100×100 *grid*	*Grid-SVD*	*23.332106*
bcsstm01	*G-Arn(10)*	*2.368047*
bcsstk01	*G-Arn(20)*	*5.626782*
50×50 *grid*	*Grid-SVD*	*58.878288*
bcsstm04	*Lin-Grid*	*1332.509283*
bcsstk04	*Lin-Arn(10)-Grid*	*1.595170*
50×50 *grid*	*Grid-SVD*	*1600.872610*
bcsstm06	*G-Arn(10)*	*0.830783*
bcsstk06	*G-Arn(50)*	*7.375079*
	G-Arn(200)	*209.351960*

The following plots give the comparisons of the pesudospectra of the CEP by different methods. Obviously, we can find the conclusion clearly as following:

- The pseudopsetrca of the lineariztion form loss much sensitivity than the pseudospectra of the original CEP. In another word, the pseudospectra of the linearized form do not give relative exact information of the pseudospectra of original problem. This emphasizes the importance of

defining and computing pseudospectra for the PEP in its original form or generalized projection with its original form, instead of linearization directly or linearization-projection.

- The generalized projection method, G-Arn(m), is a fast and efficient method to compute the pseudospectra of CEP.
- When $m = n$ for G-Arn(m), i.e., the dimension of the generalized Krylov subspace is n , in this case, the plots of pseudospectra by G-Arn(m) and Grid-SVD coincide completely.
- With the m of G-Arn(m) increasing, the pseudospectra by the G-Arn(m) is better and better close to the exact pseudospectra.
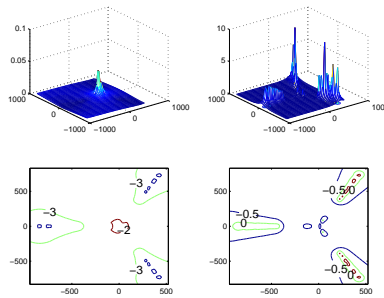
FIGURE 6. Comparison of plot of pseudospectra between Grid-SVD method and the Lin-Grid over 50×50 gird with bcsstm01 and bcsstk01. The left two plots are made by Lin-Grid, and right two plots are made by Grid-SVD
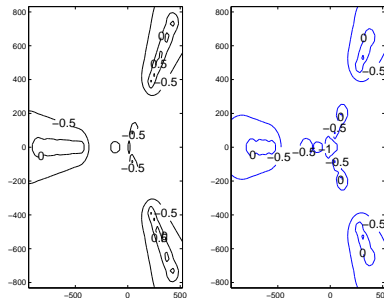

FIGURE 7. Comparison of plot of pseudospectra between Grid-SVD method and G-Arn(10) over 50×50 gird with bcsstm01 and bcsstk01. The left plot (black line) is Grid-SVD and the right plot (blue line) is G-Arn(10).

Conclusion and remarks

In this paper, we gave a comparison of the direct approaches and linearization for computing pseudospectra of PEPs. Numerical examples show the difference between the direct approaches and linearization and give some conclusions.

There remain many problems to be studied. We will continue to investigate the condition numbers of the PEPs and its linearizations. Also, we try to propose some structure-preserving algorithm applied to the linearization. Those need further research.
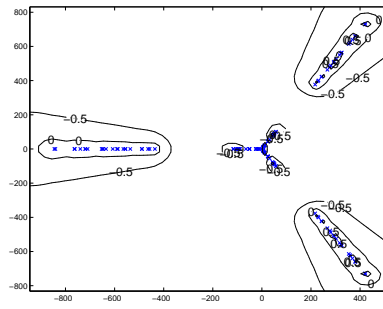
FIGURE 8.   Comparison of plot of pseudospectra between Grid-SVD method (solid line) and G-Arn(48) (dash line) over 50×50 gird with bcsstm01 and bcsstk01. The eigenvalues are plotted as '$\times$'. The plots coincide completely.
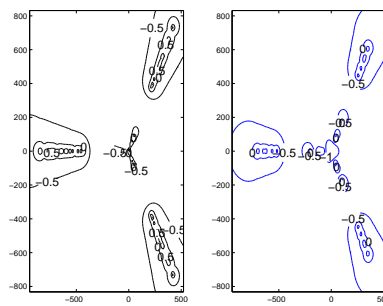

FIGURE 9.   Comparison of plot of pseudospectra between Grid-SVD method and G-Arn(10) over 100×100 gird with bcsstm01 and bcsstk01. The left plot (black line) is Grid-SVD and the right plot (blue line) is G-Arn(10).
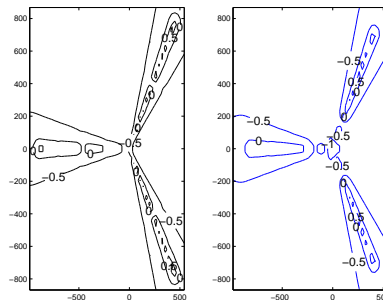

FIGURE 10.   Comparison of plot of pseudospectra between Grid-SVD method and G-Arn(50) over 50×50 gird with bcsstm06 and bcsstk06. The left plot (black line) is Grid-SVD and the right plot (blue line) is G-Arn(50).

REFERENCES

[1] L. Bao, Y.Q. Lin, Y.M. Wei, Restarted generalized Krylov subspace methods for solving large-scale polynomial eigenvalue problems, Numer. Algor. 50(2009)17-32.

[2] T. Betcke, Optimal scaling of generalized and polynomial eigenvalue problems, SIAM J. Matrix Anal. Appl. 30(4)(2008)1320-1338.

[3] T. Braconnier, N.J. Higham, Computing the field of values and pseudospectra using the Lanczos method with continuation, BIT 36(1996)422-440.
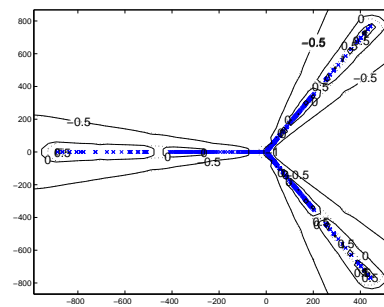
FIGURE 11.   Comparison of plot of pseudospectra between Grid-SVD method(solid line) and G-Arn(200) (dash line) over 50×50 gird with bcsstm06 and bcsstk06. The eigenvalues are plotted as '$\times$'.

[4] M. Embree, L.N. Trefethen, Pseudospectra gateway, http://www.comlab.ox.ac.uk/pseudospectra/.

[5] G.H. Golub, C.F. Loan, Matrix computations, 2nd ed., Johns Hopkins University Press, Baltimore, MD, 1989.

[6] N.J. Higham, H.M. Kim, Numerical Analysis of a Quadratic Matrix Equation, IMA J. Numer. Anal. 20(2000)499-519.

[7] N.J. Higham, F. Tisseur, More on pseudospectra for polynomial eigenvalue problems and applications in control theory, Linear Algebra Appl. 351-352(2002)435-453.

[8] M.E. Hochstenbach, G.L.G. Sleijpen. Harmonic and refined Rayleigh-Ritz for polynomial eigenvalue problem. Numer. Linear Algebra Appl. 15(2008)35-54.

[9] P. Lancaster, P. Psarrokos, On the pseudospectra of matrix polynomials, SIAM J. Matrix Anal. Appl., 27(2005)115-129.

[10] Matrix Market, http://math.nist.gov/MatrixMarket/.

[11] K. Meerbergen, Locking and restarting quadratic eigenvalue solvers, SIAM J. Sci. Comput. 22(2001)1814-1839.

[12] Y. Saad, Numerical methods for large eigenvalue problems, Manchester University Press, Manchester, UK,1992.

[13] V. Simoncini, E. Gallopoulos, Transfer functions and resolvent norm approximation of large matrices, Electron. Trans. Numer. Anal. 7(1998)190-201.

[14] G.L.G. Sleijpen, A.G.L. Booten, D.R. Fokkema, H.A.Van der Vorst, Jacobi- Davidson type methods for generalized eigenproblems and polynomial eigenproblems, BIT 36(1996)595-633.

[15] F. Tisseur, K. Meerbergen, The quadratic eigenvalue problem, SIMAV Review 43(2001)235-286.

[16] F. Tisseur, N.J. Highm, Structured pseudospectra for polynomial eigenvalue problems with applications, SIAM J. Matrix Anal. Appl. 23(2001)187-208.

[17] K.C. Toh, L.N. Trefethen, Calculation of pseudospectra by the Arnoldi iteration, SIAM J.Sci. Comput. 17(1996)1-15.

[18] L.N. Trefethen, Pseudospectra of linear operators, SIAM Review 39(1997)383-406.

[19] L.N. Trefethen, Computation of pseudospectra, Acta Numerica 8(1999)247-295.

[20] L.N. Trefethen, Spectra and pseudospectra: The behavior of nonnormal matrices and operator, Princeton University Press, Princeton, 2005.

[21] H.A. van Vorst, Computational methods for large eigenvalue problems, North-Holland(Elsevier), Amsterdam, 2002.

[22] T.G. Wright, L.N. Trefethen, Large-scale computation of pseudospectra using ARPACK and eigs, SIAM J.Sci. Comput. 23(2001)591-605.