Available online at http://scik.org
J. Math. Comput. Sci. 4 (2014), No. 2, 206-225
ISSN: 1927-5307

# ATTRACTIVITY AND EXISTENCE RESULTS FOR HYBRID DIFFERENTIAL EQUATIONS WITH ANTICIPATION AND RETARDATION

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**Abstract.** In this paper, we establish an existence and global attractivity results for a hybrid differential equation of quadratic type on unbounded intervals of real line with the mixed arguments of anticipations and retardation. A positivity result is also obtained under some usual natural conditions. Our hypotheses and claims have also been explained with the help of a natural realization.

**Keywords**: hybrid differential equation; quadratic functional differential equation; hybrid fixed point theorem; existence theorem; attractivity of solution and asymptotic stability.

**2010 AMS Subject Classification:** 34A12, 34D23, 47H10.

## 1. Statement of the Problem

Let $t_0 \in \mathbb{R}$ be a fixed real number and let $J_\infty = [t_0, \infty)$ be a closed but unbounded interval in $\mathbb{R}$. Let $\mathscr{CRB}(J_\infty)$ denote the class of functions $a : J_\infty \to \mathbb{R}_+ - \{0\}$ satisfying the following properties:

(i) $a$ is continuous,

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Received February 3, 2013

(ii) $\lim_{t\to\infty} a(t) = \infty$.

There do exist functions $a : J_\infty \to \mathbb{R}_+ - \{0\}$ satisfying the above two conditions. In fact, if $a_1(t) = |t|+1$, $a_2(t) = e^{|t|}$, then $a_1, a_2 \in \mathscr{CRB}(J_\infty)$. Again, the class of continuous and strictly monotone functions $a : J_\infty \to \mathbb{R}_+ - \{0\}$ satisfy the above criteria. Note that if $a \in \mathscr{CRB}(J_\infty)$, then the reciprocal function $\overline{a} : J_\infty \to \mathbb{R}_+$ defined by $\overline{a}(t) = \frac{1}{a(t)}$ is continuous and $\lim_{t\to\infty} \overline{a}(t) = 0$.

Given a function $a \in \mathscr{CRB}(J_\infty)$, we consider the following functional hybrid differential equation (in short HDE),

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)x(t)}{f(t,x(t),x(\alpha(t)))}\right] = g(t,x(t),x(\gamma(t))),\ t \in J_\infty, \\ x(t_0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where $f : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, and $\alpha, \gamma : J_\infty \to J_\infty$ are continuous functions and $g : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

(i) the map $(x,y) \mapsto \dfrac{a(t)x}{f(t,x,y)}$ is well defined for each $t \in J_\infty$; and
(ii) the functions $\alpha$ and $\gamma$ are respectively anticipatory and retardatory, that is $\alpha(t) \geq t$ and $\gamma(t) \leq t$ for all $t \in J_\infty$ with $\alpha(t_0) = t_0$.

**Definition 1.1.** By a *solution* for the functional differential equation (1.1) we mean a function $x \in BC(J_\infty, \mathbb{R})$ such that

(i) the function $t \mapsto \dfrac{a(t)x(t)}{f(t,x(t),x(\alpha(t)))}$ is continuous on $J_\infty$,
(ii) $x$ satisfies the equations in (1.1) on $J_\infty$,

where $BC(J_\infty, \mathbb{R})$ is the space of bounded and continuous real-valued functions defined on $J_\infty$.

The HDE (1.1) is a quadratic perturbation of first and second type for the initial value problem of linear first order ordinary differential equations. It is also new to the literature and includes a good number of known hybrid differential equations. The details of different types of perturbations of a linear differential equation appears in Dhage [9]. Let $k : J_\infty \to \mathbb{R}$ be a continuous function such that $\lim_{t\to\infty} e^{K(t)} = \infty$, where $K(t) = \int_{t_0}^{t} k(s)\,ds$. Then the hybrid differential equation,

$$\left.\begin{aligned} &\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t),x(\alpha(t)))}\right]+k(t)\left[\frac{x(t)}{f(t,x(t),x(\alpha(t)))}\right] \\ &\qquad\qquad = g(t,x(t),x(\gamma(t))),\ t\in J_\infty, \\ &x(t_0)=x_0\in\mathbb{R}, \end{aligned}\right\} \tag{1.2}$$

is of the type of HDE (1.1) on the interval $J_\infty$. The special case when $f(t,x,y)=f(t,x)$ and $g(t,x,y)=g(t,x)$, the HDE (1.1) reduces to quadratic HDE

$$\left.\begin{aligned} &\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right]+k(t)\left[\frac{x(t)}{f(t,x(t))}\right]=g(t,x(t)),\ t\in J_\infty, \\ &x(t_0)=x_0\in\mathbb{R}. \end{aligned}\right\} \tag{1.3}$$

The HDE (1.3) with $J_\infty=\mathbb{R}_+$ has been discussed in Dhage [5] for the local attractivity results under mixed Lipschitz and compactness type conditions. Finally, HDE (1.1) again includes as a special case the well-known Bernoulli's equation,

$$\left.\begin{aligned} &x'(t)+k(t)x(t)=q(t)x^n(t),\ t\in J_\infty, \\ &x(0)=x_0\in\mathbb{R}, \end{aligned}\right\} \tag{1.4}$$

where $q:J_\infty\to\mathbb{R}$ and $n$ is a nonnegative real number. A special case of (1.3) with $f(t,x)=1$ has been treated in Burton and Furumochi [2] for asymptotic stability of solutions. Thus, our HDE (1.1) is more general and therefore, the global attractivity and ultimate positivity results proved in this paper are of great interest and include the existence as well as attractivity results for the above mentioned HDEs (1.1)-(1.3) on $J_\infty$ as special cases.

## 2. Auxiliary Results

Let $X$ be a non-empty set and let $\mathscr{T}:X\to X$. An invariant point under $\mathscr{T}$ in $X$ is called a fixed point of $\mathscr{T}$, that is, the fixed points are the solutions of the functional equation $\mathscr{T}x=x$. Any statement asserting the existence of fixed point of the mapping $\mathscr{T}$ is called a fixed point theorem for the mapping $\mathscr{T}$ in $X$. The fixed point theorems are obtained by imposing the conditions on $T$ or on $X$ or on both $\mathscr{T}$ and $X$. By experience, better the mapping $\mathscr{T}$ or $X$, we have better fixed point principles. As we go on adding richer structure to the non-empty set $X$, we derive richer fixed point theorems useful for applications to different areas of mathematics and particularly to nonlinear differential and integral equations. Below we give some fixed point theorems useful in establishing the attractivity and ultimate positivity of the solutions for FHDE (1.1) on unbounded intervals. Before stating these results we give some preliminaries.

Let $X$ be an infinite dimensional Banach space with the norm $\|\cdot\|$. A mapping $\mathscr{T} : X \to X$ is called *$\mathscr{D}$-Lipschitz* if there is a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|\mathscr{T}x - \mathscr{T}y\| \leq \phi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$, where $\phi(0) = 0$. If $\phi(r) = kr$, $k > 0$, then $\mathscr{T}$ is called *Lipschitz* with the Lipschitz constant $k$. In particular, if $k < 1$, then $\mathscr{T}$ is called a *contraction* on $X$ with the contraction constant $k$. Further, if $\phi(r) < r$ for $r > 0$, then $\mathscr{T}$ is called *nonlinear $\mathscr{D}$-contraction* and the function $\phi$ is called $\mathscr{D}$-function of $\mathscr{T}$ on $X$. There do exist $\mathscr{D}$-functions and the commonly used $\mathscr{D}$-functions are $\phi(r) = kr$ and $\phi(r) = \frac{r}{1+r}$, etc. (see Banas and Dhage [1] and the references therein).

**Definition 2.1.** An operator $\mathscr{T}$ on a Banach space $X$ into itself is called totally bounded if for any bounded subset $S$ of $X$, $\mathscr{T}(S)$ is a relatively compact subset of $X$. If $\mathscr{T}$ is continuous and totally bounded, then it is called completely continuous on $X$.

Our essential tool used in the paper is the following fixed point theorem of Dhage [7] for a quadratic operator equation involving two operators in a Banach algebras $X$.

**Theorem 2.1.** (Dhage [7]) *Let $S$ be a non-empty, closed convex and bounded subset of the Banach algebra $X$ and let $\mathscr{A} : X \to X$ and $\mathscr{B} : S \to X$ be two operators such that*

(a) *$\mathscr{A}$ is $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function $\psi$,*

(b) *$\mathscr{B}$ is completely continuous,*

(c) *$x = \mathscr{A}x\mathscr{B}y$ for all $y \in S \Longrightarrow x \in S$, and*

(d) *$M\psi(r) < r$, where $M = \|\mathscr{B}(S)\| = \sup\{\|\mathscr{B}x\| : x \in S\}$.*

*Then the operator equation*

$$\mathscr{A}x\mathscr{B}x = x \tag{2.2}$$

*has a solution in $S$.*

A collection of a good number of applicable fixed point theorems may be found in the monographs of Granas and Dugundji [10], Deimling [3], Zeidler [13] and the references therein. In the following section we give different types of characterizations of the solutions for nonlinear functional differential equations on unbounded intervals of real line.

## 3. Characterizations of Solutions

We seek solutions of the HDE (1.1) in the space $BC(J_\infty,\mathbb{R})$ of continuous and bounded real-valued functions defined on $J_\infty$. Define a standard supremum norm $\|\cdot\|$ and a multiplication "$\cdot$" in $BC(J_\infty,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J_\infty} |x(t)|$$

and

$$(x\cdot y)(t) = (xy)(t) = x(t)y(t),\ t\in J_\infty.$$

Clearly, $BC(J_\infty,\mathbb{R})$ becomes a Banach algebra with respect to the above norm and the multiplication in it. Let $\mathscr{A},\mathscr{B} : BC(J_\infty,\mathbb{R}) \to BC(J_\infty,\mathbb{R})$ be two continuous operators and consider the following operator equation in the Banach algebra $BC(J_\infty,\mathbb{R})$,

$$\mathscr{A}x(t)\,\mathscr{B}x(t) = x(t) \tag{3.1}$$

for all $t\in J_\infty$. Below we give different characterizations of the solutions for the operator equation (3.1) in the space $BC(J_\infty,\mathbb{R})$.

**Definition 3.1.** We say that solutions of the operator equation (3.1) are *locally attractive* if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(J_\infty,\mathbb{R})$ for some $x_0\in BC(J_\infty,\mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.1) belonging to $\overline{B}_r(x_0)$ we have that

$$\lim_{t\to\infty}(x(t)-y(t)) = 0. \tag{3.2}$$

In the case when the limit (3.2) is uniform with respect to the set $\overline{B}_r(x_0)$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t)-y(t)| \le \varepsilon \tag{3.3}$$

for all $x,y\in \overline{B}_r(x_0)$ being solutions of (3.1) and for $t\ge T$, we will say that solutions of equation (3.1) are *uniformly locally attractive* on $J_\infty$.

**Definition 3.2.** A solution $x = x(t)$ of equation (3.1) is said to be *globally attractive* if (3.2) holds for each solution $y = y(t)$ of (3.1) in $BC(J_\infty, \mathbb{R})$. In other words, we may say that solutions of the equation (3.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (3.1) in $BC(J_\infty, \mathbb{R})$, the condition (3.2) s satisfied. In the case when the condition (3.2) is satisfied uniformly with respect to the space $BC(J_\infty, \mathbb{R})$, i.e., if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (3.2) is satisfied for all $x, y \in BC(J_\infty, \mathbb{R})$ being the solutions of (3.1) and for $t \geq T$, we will say that solutions of the equation (3.1) are *uniformly globally attractive* on $J_\infty$.

**Remark.** Let us mention that the details of the global attractivity of solutions may be found in a recent paper of Hu and Yan [12] while the concepts of uniform local and global attractivity (in the above sense) may be found in Banas and Dhage [1].

Now we introduce the new concept of local and global ultimate positivity of the solutions for the operator equation (3.1) in the space $BC(J_\infty, \mathbb{R})$.

**Definition 3.3.** (Dhage [8]) A solution $x$ of the equation (3.1) is called *locally ultimately positive* if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(J_\infty, \mathbb{R})$ for some $x_0 \in BC(J_\infty, \mathbb{R})$ such that $x \in \overline{B}_r(0)$ and

$$\lim_{t \to \infty} \big[ |x(t)| - x(t) \big] = 0. \tag{3.4}$$

In the case when the limit (3.4) is uniform with respect to the solution set of the operator equation (3.1) in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$\big| |x(t)| - x(t) \big| \leq \varepsilon \tag{3.5}$$

for all $x$ being solutions of (3.1) in $BC(J_\infty, \mathbb{R})$ and for $t \geq T$, we will say tithat solutions of equation (3.1) are *uniformly locally ultimately positive* on $J_\infty$.

**Definition 3.4.** (Dhage [8]) A solution $x \in BC(J_\infty, \mathbb{R})$ of the equation (3.1) is called *globally ultimately positive* if (3.4) is satisfied. In the case when the limit (3.5) is uniform with respect to the solution set of the operator equation (3.1) in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that (3.5) is satisfied for all $x$ being solutions of (3.1) in in $BC(J_\infty, \mathbb{R})$ and for $t \geq T$, we will say that solutions of equation (3.1) are *uniformly globally ultimately positive* on $J_\infty$.

**Remark.** We note that global attractivity implies the local attractivity and uniform global attractivity implies the uniform local attractivity of the solutions for the operator equation (3.1) on $J_\infty$. Similarly, global ultimate positivity implies local ultimate positivity of the solutions for the operator equation (3.1) on unbounded intervals. However, the converse of the above two statements may not be true.

## 4. Attractivity and Positivity Results

Now, in this section, we discuss the attractivity results for the first order ordinary differential equation (1.1) on $J_\infty$.

**Definition** A function $\beta : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if

(i) the map $t \mapsto \beta(t,x,y)$ is measurable for all $x, y \in \mathbb{R}$

(ii) the map $(x,y) \mapsto \beta(t,x,y)$ is jointly continuous for all $t \in J_\infty$.

The following theorem is often times used in the study of nonlinear discontinuous and specially Carathéodory theory of nonlinear differential equations.

**Theorem 4.1.** (Carathéodory) *Let $\beta : J_\infty \times \mathbb{R} \to \mathbb{R}$ be a mapping such that $g(\cdot,x)$ is measurable for all $x \in \mathbb{R}$ and $\beta(t,\cdot)$ is continuous for all $t \in J_\infty$. Then the map $(t,x) \mapsto \beta(t,x)$ is jointly measurable.*

We need the following hypotheses in the sequel.

($A_1$) The function $x \mapsto \dfrac{a(t_0)x}{f(t_0,x,x)}$ is injective in $\mathbb{R}$.

($A_2$) The function $f$ is continuous and there exists a function $\ell \in BC(J_\infty, \mathbb{R}_+)$ and a constant $K > 0$ such that

$$\big|f(t,x_1,x_2) - f(t,x_1,x_2)\big| \le \frac{\ell(t)\max\{|x_1 - x_2|, |x_2 - y_2|\}}{K + \max\{|x_1 - x_2|, |x_2 - y_2|\}}$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Moreover, $\sup\limits_{t \in J_\infty} \ell(t) = L$.

($B_1$) The function $g$ is Carathéodory.

($B_2$) There exists a function $b \in BC(J_\infty, \mathbb{R}_+)$ such that

$$|g(t,x,y)| \le b(t)$$

for all $t \in J_\infty$ and $x, y \in \mathbb{R}$. Moreover, we assume that

$$\lim_{t\to\infty} \overline{a}(t) \int_{t_0}^{t} b(s)\,ds = 0.$$

**Remark 4.1.** If the hypothesis ($B_2$) holds and $a \in \mathscr{CRB}(J_\infty)$, then $\overline{a} \in BC(J_\infty, \mathbb{R}_+)$. Again, the function $w : \mathbb{R}_+ \to \mathbb{R}$ defined by the expression $w(t) = \overline{a}(t) \int_{t_0}^{t} b(s)\,ds$ is continuous and the number $W = \sup_{t \geq t_0} w(t)$ exists.

The following lemma is useful in the sequel.

**Lemma 4.1.** *Assume that the hypothesis ($A_1$) holds. Then, for any function $h \in L^1(J_\infty, \mathbb{R}_+)$, the function $x \in BC(J_\infty, \mathbb{R}_+)$ is a solution of the HDE*

$$\frac{d}{dt}\left[\frac{a(t)x(t)}{f(t,x(t),x(\alpha(t)))}\right] = h(t),\ t \in J_\infty, \tag{4.1}$$

*and*

$$x(0) = x_0 \tag{4.2}$$

*if and only if x satisfies the hybrid integral equation (HIE)*

$$x(t) = \left[f(t,x(t),x(\alpha(t)))\right]\left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \overline{a}(t)\int_{t_0}^{t} h(s)\,ds\right),\ t \in J_\infty. \tag{4.3}$$

**Proof.** Let $h \in L^1(J_\infty, \mathbb{R}_+)$ be bounded with bound M. Assume first that $x$ is a solution of the HDE (4.1)-(4.2). By definition, the map $t \mapsto \dfrac{a(t)x(t)}{f(t,x(t),x(\alpha(t)))}$ is continuous, and so, whence $\dfrac{d}{dt}\left[\dfrac{a(t)x(t)}{f(t,x(t),x(\alpha(t)))}\right]$ is integrable on $J_\infty$. Applying integration to (4.1) from $t_0$ to $t$, we obtain the HDE (4.3) on $J_\infty$.

Conversely, assume that the function $x$ satisfies the HIE (4.3) on $J_\infty$. Since $h$ is bounded, it can be proved that the function $t \mapsto \dfrac{a(t)x(t)}{f(t,x(t),x(\alpha(t)))}$ is continuous for each $x \in BC(J_\infty, \mathbb{R}_+)$ and hence differential on $J_\infty$. By direct differentiation of the HIE (3.1), we obtain the HDE (4.1) -(4.2). Again, substituting $t = t_0$ in the HIE (4.3) yields

$$\frac{a(t_0)x(t_0)}{f(t_0,x(t_0),x(t_0))} = \frac{a(t_0)x_0}{f(t_0,x_0,x_0)}.$$

Since the mapping $x \mapsto \dfrac{a(t_0)x}{f(t,x,x)}$ is injective in $\mathbb{R}$, we obtain $x(t_0) = x_0$. Hence the proof of the lemma is complete.

Our main existence and attractivity result is

**Theorem 4.2.** *Assume that the hypotheses ($A_1$) - ($A_2$) and ($B_1$) - ($B_2$) hold. Further, assume that*

$$L\left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\|+W\right)\leq K. \tag{4.4}$$

*Then the HDE (1.1) has a solution and solutions are uniformly globally attractive defined on $J_\infty$.*

**Proof.** Now, using hypotheses ($H_7$) and ($H_8$) it can be shown that the HDE (1.1) is equivalent to the functional integral equation

$$x(t)=\big[f(t,x(t),x(\alpha(t)))\big]\left(\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)}+\overline{a}(t)\int_{t_0}^{t}g(s,x(s),x(\gamma(s)))\,ds\right), \tag{4.5}$$

for all $t\in J_\infty$. Set $X=BC(J_\infty,\mathbb{R})$ and define a closed ball $\overline{B}_r(0)$ in $X$ centered at origin of radius $r$ given by

$$r=\big(L+F_0\big)\left(Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\|+W\right).$$

Define the operators $\mathscr{A}$ on $X$ and $\mathscr{B}$ on $\overline{B}_r(0)$ by

$$\mathscr{A}x(t)=f(t,x(t),x(\alpha(t))),\ t\in J_\infty \tag{4.6}$$

and

$$\mathscr{B}x(t)=\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)}+\overline{a}(t)\int_{t_0}^{t}g(s,x(s),x(\gamma(s)))\,ds,\ \ t\in J_\infty. \tag{4.7}$$

Then the FIE (4.5) is transformed into the operator equation as

$$\mathscr{A}x(t)\,\mathscr{B}x(t)=x(t),\ t\in J_\infty. \tag{4.8}$$

We show that $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.1 on $BC(J_\infty,\mathbb{R})$. First we we show that the operators $\mathscr{A}$ and $\mathscr{B}$ define the mappings $\mathscr{A}\,:\,X\to X$ and $\mathscr{B}\,:\,\overline{B}_r(0)\to X$. Let $x\in X$ be arbitrary. Obviously, $\mathscr{A}x$ is a continuous function on $J_\infty$. We show that $\mathscr{A}x$ is bounded on $J_\infty$. Thus, if $t\in J_\infty$, then we obtain:

$$\begin{aligned}|\mathscr{A}x(t)|=|f(t,x(t),x(\alpha(t)))|&\leq|f(t,x(t),x(\alpha(t)))-f(t,0,0)|+|f(t,0,0)|\\&\leq\ell(t)\frac{\max\{|x(t)|,|x(\alpha(t))|\}}{K+\max\{|x(t)|,|x(\alpha(t))|\}}+F_0\\&\leq L+F_0.\end{aligned}$$

Therefore, taking the supremum over $t$, $\|\mathscr{A}x\| \le L + F_0 = N$. Thus $\mathscr{A}x$ is continuous and bounded on $J_\infty$. As a result $\mathscr{A}x \in X$. Similarly, it can be shown that $\mathscr{B}x \in X$ and in particular, $\mathscr{A}: X \to X$ and $\mathscr{B}: \overline{B}_r(0) \to X$. We show that $\mathscr{A}$ is a Lipschitz on $X$. Let $x, y \in X$ be arbitrary. Then, by hypothesis (H$_3$),

$$\begin{aligned}
\|\mathscr{A}x - \mathscr{A}y\| &= \sup_{t \in J_\infty} |\mathscr{A}x(t) - \mathscr{A}y(t)| \\
&\le \sup_{t \in J_\infty} \ell(t) \frac{\max\{|x(t)-y(t)|, |x(\alpha(t)) - y(\alpha(t))|\}}{K + \max\{|x(t)-y(t)|, |x(\alpha(t)) - y(\alpha(t))|\}} \\
&\le \frac{L\|x-y\|}{K + \|x-y\|} \\
&= \psi(\|x-y\|)
\end{aligned}$$

for all $x, y \in X$. This shows that $\mathscr{A}$ is a $\mathscr{D}$-Lipschitz on $X$ with $\mathscr{D}$-function $\psi(r) = \frac{Lr}{K+r}$.

Next we shows that $\mathscr{B}$ is a completely continu0ous operator on $\overline{B}_r(0)$. First, we show that $\mathscr{B}$ is continuous on $\overline{B}_r(0)$. To do this, let us fix arbitrarily $\varepsilon > 0$ and let $\{x_n\}$ be a sequence of points in $\overline{B}_r(0)$ converging to a point $x \in \overline{B}_r(0)$. Then we get:

$$\begin{aligned}
|(\mathscr{B}x_n)(t) - (\mathscr{B}x)(t)| &\le |\overline{a}(t)| \int_0^t |g(s, x_n(s), x_n(\gamma(s))) - g(s, x(s), x(\gamma(s)))| ds \\
&\le |\overline{a}(t)| \int_0^t [|g(s, x_n(s), x_n(\gamma(s)))| + |g(s, x(s), x(\gamma(s)))|] ds \\
&\le 2|\overline{a}(t)| \int_0^t h(s)\, ds \\
&= 2w(t).
\end{aligned} \tag{4.9}$$

Hence, by virtue of hypothesis (H$_2$), we infer that there exists a $T > 0$ such that $w(t) \le \varepsilon$ for $t \ge T$. Thus, for $t \ge T$, from the estimate (3.3) we derive that

$$|(\mathscr{B}x_n)(t) - (\mathscr{B}x)(t)| \le 2\varepsilon \quad \text{as} \quad n \to \infty.$$

Furthermore, let us assume that $t \in [t_0, T]$. Then, by dominated convergence theorem, we obtain the estimate:

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{B}x_n(t) &= \lim_{n\to\infty}\left[\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \overline{a}(t)\int_0^t g(s,x_n(s),x_n(\gamma(s)))\,ds\right] \\
&= \frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \overline{a}(t)\int_0^t \left[\lim_{n\to\infty} g(s,x(s),x_n(\gamma(s)))\right]ds \qquad (4.10) \\
&= \mathscr{B}x(t)
\end{aligned}$$

for all $t \in [t_0, T]$. Moreover, it can be shown as below that $\{\mathscr{B}x_n\}$ is an equicontinuous sequence of functions in $X$. Now, following the arguments similar to that given in Granas *et al.* [11], it is proved that $\mathscr{B}$ is a a continuous operator on $\overline{B}_r(0)$.

Next, we show that $\mathscr{B}$ is compact on $\overline{B}_r(0)$. To finish, it is enough to show that every sequence $\{\mathscr{B}x_n\}$ in $\mathscr{B}(\overline{B}_r(0))$ has a Cauchy subsequence. Now, by hypothesis ($H_2$),

$$\begin{aligned}
|\mathscr{B}x_n(t)| &\le \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| |\overline{a}(t)| + |\overline{a}(t)|\int_0^t |g(s,x_n(s),x_n(\gamma(s)))|\,ds \\
&\le \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + w(t) \qquad (4.11) \\
&\le \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + W
\end{aligned}$$

for all $t \in \mathbb{R}_+$. Taking the supremum over $t$, we obtain

$$\|\mathscr{B}x_n\| \le \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + W$$

for all $n \in \mathbb{N}$. This shows that $\{\mathscr{B}x_n\}$ is a uniformly bounded sequence in $\mathscr{B}(\overline{B}_r(0))$.

Next, we show that $\{\mathscr{B}x_n\}$ is also a equicontinuous sequence in $\mathscr{B}(\overline{B}_r(0))$. Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there is a constant $T_1 > 0$ such that $|w(t)| < \frac{\varepsilon}{8}$ for all $t \ge T_1$. Similarly, since $\lim\limits_{t\to\infty} \overline{a}(t) = 0$, for above $\varepsilon > 0$, there is a $T_2 > 0$ such that $|\overline{a}(t)| < \dfrac{\varepsilon}{8\left|\dfrac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right|}$ for all $t \ge T_2$. Thus, if $T = \max\{T_1, T_2\}$, then $|w(t)| < \dfrac{\varepsilon}{8}$ and $|\overline{a}(t)| < \dfrac{\varepsilon}{8\left|\dfrac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right|}$ for all

$t \geq T$. Let $t, \tau \in \mathbb{R}_+$ be arbitrary. If $t, \tau \in [t_0, T]$, then we have

$$\begin{aligned}
|\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| &\leq \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \left| \overline{a}(t) \int_0^t g(s,x_n(s),x_n(\gamma(s)))\,ds - \overline{a}(\tau) \int_0^\tau g(s,x_n(s),x_n(\gamma(s)))\,ds \right| \\
&\leq \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \left| \overline{a}(t) \int_0^t g(s,x_n(s),x_n(\gamma(s)))\,ds - \overline{a}(\tau) \int_0^t g(s,x_n(s),x_n(\gamma(s)))\,ds \right| \\
&\quad + \left| \overline{a}(\tau) \int_0^t g(s,x_n(s),x_n(\gamma(s)))\,ds - \overline{a}(\tau) \int_0^\tau g(s,x_n(s),x_n(\gamma(s)))\,ds \right| \\
&\leq \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)| \left| \int_0^t g(s,x_n(s),x_n(\gamma(s)))\,ds \right| \\
&\quad + |\overline{a}(\tau)| \left| \int_\tau^t g(s,x_n(s),x_n(\gamma(s)))\,ds \right| \\
&\leq \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)| \int_0^T h(s)\,ds + \|\overline{a}\| \left| \int_\tau^t h(s)\,ds \right| \\
&\leq \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)| \int_0^T h(s)\,ds + \|\overline{a}\|\,|p(t) - p(\tau)| \\
&\leq \left[ \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| + \|h\|_{L^1} \right] |\overline{a}(t) - \overline{a}(\tau)| + \|\overline{a}\|\,|p(t) - p(\tau)|
\end{aligned}$$

where, $p(t) = \int_{t_0}^t h(s)\,ds$ and $\|h\|_{L^1} = \int_{t_0}^\infty h(s)\,ds$.

By the uniform continuity of the function $\overline{a}$ and $p$ on $[t_0, T]$, for above $\varepsilon$ we have the numbers $\delta_1 > 0$ and $\delta_2 > 0$ depending only on $\varepsilon$ such that

$$|t - \tau| < \delta_1 \Longrightarrow |\overline{a}(t) - \overline{a}(\tau)| < \frac{\varepsilon}{8\left[ \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| + \|h\|_{L^1} \right]}$$

and

$$|t - \tau| < \delta_2 \Longrightarrow |p(t) - p(\tau)| < \frac{\varepsilon}{8\|\overline{a}\|}.$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Then

$$|t - \tau| < \delta_3 \Longrightarrow |\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| < \frac{\varepsilon}{4}$$

for all $n \in \mathbb{N}$. Again, if $t, \tau > T$, then we have a $\delta_4 > 0$ depending only on $\varepsilon$ such that

$$\begin{aligned}
|\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| &\leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| |a(t) - a(\tau)| \\
&\quad + \left|\overline{a}(t)\int_0^t g(s,x_n(s))\,ds - \overline{a}(t)\int_0^\tau g(s,x_n(s),x_n(\gamma(s)))\,ds\right| \\
&\leq \left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| |\overline{a}(t)| + |\overline{a}(\tau)| + w(t) + w(\tau) \\
&< \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$ whenever $|t - \tau| < \delta_4$. Similarly, if $t, \tau \in \mathbb{R}_+$ with $t < T < \tau$, then we have

$$|\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| \leq |\mathscr{B}x_n(t) - \mathscr{B}x_n(T)| + |\mathscr{B}x_n(T) - \mathscr{B}x_n(\tau)|.$$

Take $\delta = \min\{\delta_3, \delta_4\} > 0$ depending only on $\varepsilon$. Therefore, from the above obtained estimates, it follows that

$$|\mathscr{B}x_n(t) - \mathscr{B}x_n(T)| < \frac{\varepsilon}{2} \quad \text{and} \quad |\mathscr{B}x_n(T) - \mathscr{B}x_n(\tau)| < \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$ whenever $|t - \tau| < \delta$. As a result, $|\mathscr{B}x_n(t) - \mathscr{B}x_n(\tau)| < \varepsilon$ for all $t, \tau \in J_\infty$ and for all $n \in \mathbb{N}$ whenever $|t - \tau| < \delta$. This shows that $\{\mathscr{B}x_n\}$ is a equicontinuous sequence in $X$. Now an application of Arzelà-Ascoli theorem yields that $\{\mathscr{B}x_n\}$ has a uniformly convergent subsequence on the compact subset $[t_0, T]$ of $J_\infty$. Without loss of generality, call the subsequence to be the sequence itself. We show that $\{\mathscr{B}x_n\}$ is Cauchy in $X$. Now $|\mathscr{B}x_n(t) - \mathscr{B}x(t)| \to 0$ as $n \to \infty$ for all $t \in [t_0, T]$. Then for given $\varepsilon > 0$ there exits an $n_0 \in \mathbb{N}$ such that

$$\sup_{t_0 \leq p \leq T} \overline{a}(p) \int_{t_0}^p |g(s,x_m(s),x_m(\gamma(s))) - g(s,x_n(s),x_n(\gamma(s)))|\,ds < \frac{\varepsilon}{2}$$

for all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$, then we have

$$
\begin{aligned}
\|\mathscr{B}x_m - \mathscr{B}x_n\| &= \sup_{t_0 \le p < \infty} \left| \overline{a}(p) \int_{t_0}^{p} |g(s,x_m(s),x_m(\gamma(s))) - g(s,x_n(s),x_n(\gamma(s)))|\, ds \right| \\
&\le \sup_{t_0 \le p \le T} \left| \overline{a}(p) \int_{t_0}^{p} |g(s,x_m(s),x_m(\gamma(s))) - g(s,x_n(s),x_m(\gamma(s)))|\, ds \right| \\
&\quad + \sup_{p \ge T} \overline{a}(p) \int_{t_0}^{p} \big[ |g(s,x_m(s),x_m(\gamma(s)))| + |g(s,x_n(s),x_m(\gamma(s)))| \big]\, ds \\
&< \varepsilon.
\end{aligned}
$$

This shows that $\{\mathscr{B}x_n\} \subset \mathscr{B}(\overline{B}_r(0)) \subset X$ is Cauchy. Since $X$ is complete, $\{\mathscr{B}x_n\}$ converges to a point in $E$. As $\mathscr{B}(\overline{B}_r(0))$ is closed $\{\mathscr{B}x_n\}$ converges to a point in $\mathscr{B}(\overline{B}_r(0))$. Hence $\mathscr{B}(\overline{B}_r(0))$ is relatively compact and consequently $\mathscr{B}$ is a continuous and compact operator on $\overline{B}_r(0)$ into $X$.

Next, we estimate the value of the constant $M$. By definition of $M$, one has

$$
\begin{aligned}
\|\mathscr{B}(\overline{B}_r(0))\| &= \sup\{\|\mathscr{B}x\| \;:\; x \in \overline{B}_r(0)\} \\
&= \sup\left\{ \sup_{t \in J_\infty} |\mathscr{B}x(t)| \;:\; x \in \overline{B}_r(0) \right\} \\
&\le \sup_{x \in \overline{B}_r(0)} \left\{ \sup_{t \in J_\infty} \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t)| \right. \\
&\qquad \left. + \sup_{t \in J_\infty} |\overline{a}(t)| \int_{t_0}^{t} |g(s,x(s),x(\gamma(s)))|\, ds \right\} \\
&\le \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| \|\overline{a}\| + W.
\end{aligned}
$$

Thus, we have

$$
\|\mathscr{B}x\| \le \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| \|\overline{a}\| + W
$$

for all $x \in \overline{B}_r(0)$. Next, let $x, y \in X$ be arbitrary. Then,

$$\begin{aligned}
|x(t)| &\leq |\mathscr{A}x(t)|\,|\mathscr{B}y(t)| \\
&\leq \|\mathscr{A}x\|\,\|\mathscr{B}y\| \\
&\leq \|\mathscr{A}(X)\|\,\|\mathscr{B}(\overline{B}_r(0))\| \\
&\leq \big(L+F_0\big)\,M \\
&\leq \big(L+F_0\big)\left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\|+W\right)
\end{aligned}$$

for all $t \in J_\infty$. Therefore, we have

$$\|x\| \leq \big(L+F_0\big)\left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\|+W\right) = r.$$

This shows that $x \in \overline{B}_r(0)$ and hypothesis (c) of Theorem 2.1 is satisfied. Again,

$$M\phi(r) \leq \frac{L\left(\Big|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\Big|\,\|\overline{a}\|+W\right)r}{K+r} < r$$

for $r > 0$, because

$$L\left(\Big|\frac{a(t_0)x_0}{g(t_0,x_0,x_0)}\Big|\,\|\overline{a}\|+W\right) \leq K.$$

Therefore, hypothesis (d) of Theorem  is satisfied. Now we apply Theorem 2.1 to the operator equation $\mathscr{A}x\mathscr{B}x = x$ to yield that the HDE (1.1 ) has a solution on $J_\infty$. Moreover, the solutions of the HDE (1.1) are in $\overline{B}_r(0)$. Hence, solutions are global in nature.

Finally, let $x, y \in \overline{B}_r(0)$ be any two solutions of the HDE (1.1) on $J_\infty$. Then

$$
\begin{aligned}
&|x(t)-y(t)| \\
&\quad \leq \Bigg| [f(t,x(t),x(\alpha(t)))] \left( \frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \overline{a}(t) \int_{t_0}^{t} g(s,x(s),x(\gamma(s)))\,ds \right) \\
&\qquad - [f(t,y(t),y(\alpha(t)))] \left( \frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \overline{a}(t) \int_{t_0}^{t} g(s,y(s),y(\gamma(s)))\,ds \right) \Bigg| \\
&\quad \leq \Bigg| \big[ f(t,x(t),x(\alpha(t))) - f(t,y(t),y(\alpha(t))) \big] \\
&\qquad\qquad \times \left( \frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \overline{a}(t) \int_{t_0}^{t} g(s,x(s),x(\gamma(s)))\,ds \right) \Bigg| \\
&\qquad + \left| f(t,y(t),y(\alpha(t))) \Big( \overline{a}(t) \int_{t_0}^{t} g(s,x(s),x(\gamma(s)))\,ds - g(s,y(s),y(\gamma(s)))\,ds \Big) \right| \\
&\quad \leq |f(t,x(t),x(\alpha(t))) - f(t,y(t),y(\alpha(t)))| \left( \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| |\overline{a}(t)| + w(t) \right) \\
&\qquad\qquad + 2\big[ |f(t,x(t),x(\alpha(t))) - f(t,0,0)| + |f(t,0,0)| \big] w(t) \\
&\quad \leq \ell(t) \frac{|x(t)-y(t)|}{K+|x(t)-y(t)|} \left( \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| \|\overline{a}\| + W \right) \\
&\qquad + 2\left[ \frac{\ell(t)|y(t)|}{K+|y(t)|} + F_0 \right] w(t) \\
&\quad \leq \frac{L\left( \left| \frac{a(t_0)x_0}{f(t_0,x_0,x_0)} \right| \|\overline{a}\| + W \right) |x(t)-y(t)|}{K+|x(t)-y(t)|} + 2(L+F_0)w(t).
\end{aligned}
\tag{4.12}
$$

Taking the limit superior as $t \to \infty$ in the above inequality yields, $\lim_{t\to\infty} |x(t)-y(t)| = 0$. Therefore, there is a real number $T > 0$ such that $|x(t)-y(t)| < \varepsilon$ for all $t \geq T$. Consequently, the solutions of HDE (1.1) are uniformly globally attractive on $J_\infty$. This completes the proof. □

**Remark 4.2.** The conclusion of Theorem 4.1 also remains true under the following modified conditions:

(i) The hypothesis ($A_2$) is replaced with

($A_3$) The function $f$ is continuous and there exist a function $\ell \in BC(J_\infty, \mathbb{R}_+)$ such that $\big| f(t,x_1,x_2) - f(t,x_1,x_2) \big| \leq \ell(t)\,\psi(\max\{|x_1-x_2|, |x_2-y_2|\})$ for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in$

$\mathbb{R}$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing function and $L = \sup_{t\in J_\infty} \ell(t)$,

and

(ii) the inequality (4.4) is replaced by

$$L\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| \|\overline{a}\| + W\right)\psi(r) < r, \quad \forall r > 0.$$

**Theorem 4.3.** *Assume that the hypotheses ($A_1$), ($A_3$) and ($B_1$) - ($B_2$) hold. Then the HDE (1.1) has a solution and solutions are uniformly globally attractive and ultimately positive defined on $J_\infty$.*

**Proof.** By Theorem 4.1, the HDE (1.1) has a global solution in the closed ball $\overline{B}_r(0)$, where the radius $r$ is given as in the proof of Theorem 4.1, and the solutions are uniformly globally attractive on $J_\infty$. We know that for any $x, y \in \mathbb{R}$, one has the inequality, $|x|\,|y| = |xy| \geq xy$, and therefore,

$$\big||xy| - (xy)\big| \leq |x|\,\big||y| - y\big| + \big||x| - x\big|\,|y| \tag{4.13}$$

for all $x, y \in \mathbb{R}$. Now, for any solution $x \in \overline{B}_r(0)$, one has

$$\begin{aligned}
&\big||x(t)| - x(t)\big| \\
&\leq \left|\big[f(t,x(t),x(\alpha(t)))\big]\left(\left|\frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right| - \frac{a(t_0)x_0}{f(t_0,x_0,x_0)}\right)\overline{a}(t)\right| \\
&\quad + \big[f(t,x(t),x(\alpha(t)))\big] \\
&\qquad \times \left|\left|\overline{a}(t)\int_{t_0}^{t} g(s,x(s),x(\gamma(s)))\,ds\right| - \overline{a}(t)\int_{t_0}^{t} g(s,x(s),x(\gamma(s)))\,ds\right| \\
&\quad + \Big||f(t,x(t),x(\alpha(t)))| - f(t,x(t),x(\alpha(t)))\Big| \\
&\qquad \times \left|\frac{a(t_0)x_0\overline{a}(t)}{f(t_0,x_0,x_0)} + \overline{a}(t)\int_{t_0}^{t} g(s,x(s),x(\gamma(s)))\,ds\right| \\
&\leq 2[L + F_0]w(t).
\end{aligned} \tag{4.14}$$

Taking the limit superior as $t \to \infty$ in the above inequality (4.14), we obtain the estimate that $\lim_{t\to\infty} \big||x(t)| - x(t)\big| = 0$. Therefore, there is a real number $T > 0$ such that $\big||x(t)| - x(t)\big| \leq \varepsilon$

for all $t \geq T$. Hence, solutions of the HDE (1.1) are uniformly globally attractive as well as ultimately positive defined on $J_\infty$. This completes the proof. □

In the following we indicate an example to illustrate the abstract ideas contained in Theorem 4.2.

**Example 4.1** Let $J_\infty = \mathbb{R}_+ = [0,\infty) \subset \mathbb{R}$. Given a function $a(t) = e^t \in \mathscr{CRB}(\mathbb{R}_+)$, consider the following functional hybrid differential equation with the mixed arguments of anticipation and retardation,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{e^t x(t)}{1+\dfrac{t}{t^2+1}\left(\dfrac{\tan^{-1}(|x(t)|)+\tan^{-1}(|x(2t)|)}{2+\tan^{-1}(|x(t)|)+\tan^{-1}(|x(2t)|)}\right)}\right] &= \frac{e^{-t}\log(|x(t)|+|x(t/2)|)}{1+|x(t)|+|x(t/2)|}, \\ x(0) &= 0. \end{aligned}\right\} \tag{4.15}$$

for all $t \in \mathbb{R}_+$. Define the functions $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$ and $g : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,x,y) = 1 + \frac{t}{t^2+1}\left[\frac{\tan^{-1}(|x|)+\tan^{-1}(|y|)}{2+\tan^{-1}(|x|)+\tan^{-1}(|y|)}\right]$$

and

$$g(t,x,y) = \frac{e^{-t}\log(|x|+|y|)}{1+|x|+|y|}.$$

Clearly, the function $f$ satisfies the hypothesis ($A_1$) on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. Now, it can be shown as in Banas and Dhage [1] that the function $f$ satisfies the hypothesis ($A_3$) with $\ell(t) = \dfrac{t}{t^2+1}$ and $\phi(r) = \dfrac{\tan^{-1} r}{1+\tan^{-1} r}$ for $r > 0$. Clearly, the function $f$ satisfies the hypothesis ($A_3$). Also the function $g$ satisfies the hypotheses ($B_1$)-($B_2$) with $b(t) = e^{-t}$. Hence, $L = \dfrac{1}{2}$ and $W = 1$. Also, we have $\lim\limits_{t\to\infty} \ell(t) = \lim\limits_{t\to\infty} \dfrac{t}{t^2+1} = 0$ and $\lim\limits_{t\to\infty} w(t) = \lim\limits_{t\to\infty} e^{-t}\displaystyle\int_0^t e^{-s}\,ds = 0$. Hence by Theorem 4.2, the HDE (4.15) has a solution and solutions are globally attractive defined on $\mathbb{R}_+$.

**Remark 4.3.** Finally, we remark that the ideas of this paper may be extended to a more general functional differential equations,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)x(t)}{f(t,x(\alpha_1(t)),\cdots,x(\alpha_n(t)))}\right] &= g(t,x(\gamma_1(t)),\cdots,x(\gamma_n(t))),\ t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{4.16}$$

for proving the similar results on unbounded intervals $J_\infty$ of real line under appropriate modifications. The abstract results so obtained are useful to prove the existence and attractivity theorems for the HDE of the form

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{e^t x(t)}{1+\dfrac{t}{t^2+1}\cdot\dfrac{\sum_{i=1}^n \tan^{-1}|x(it)|}{n+\sum_{i=1}^n \tan^{-1}|x(it)|}}\right] &= \frac{e^{-t}\log(\sum_{i=1}^n |x(t/i)|)}{1+\sum_{i=1}^n |x(t/i)|},\ t\in J_\infty, \\ x(0)&=0. \end{aligned}\right\} \tag{4.17}$$

**Remark 4.4.** If $g$ is assumed to be continuous function on $J_\infty \times \mathbb{R}\times\mathbb{R}$, then the attractiity and existence results for the HDE (1.1) may be obtained via another approach of using measure of noncompactness. See the details of this procedure that appears in Banas and Dhage [1] and the reerences given therein.

## 5. The Conclusion

From foregoing discussion, it is clear that the fixed point theorems are useful for proving the existence theorems as well as for characterizing the solutions of different types of functional differential equations on unbounded intervals of real line. The choice of the fixed point theorems depends upon the situations and the circumstances of the nonlinearities involved in the problems. The clever selection of the fixed point theorems yields very powerful existence results as well as different characterizations of the nonlinear functional differential equations. In this article, we have been able to prove the existence as well as global attractivity and ultimate positivity of the solutions for three types of nonlinear functional differential equations unbounded intervals. However, other nonlinear functional differential equations can be treated in the similar way for these and some other characterizations such as monotonic global attractivity, monotonic asymptotic attractivity and monotonic ultimate positivity of the solutions for such equations on unbounded intervals of real line. In a forthcoming paper, it is proposed to discuss the global asymptotic and monotonic attractivity of solutions for nonlinear functional differential equations via classical and applicable hybrid fixed point theory.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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