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ON A SOLVABLE $p-$DIMENSIONAL SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

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Abstract. In this paper, we investigate the solutions of the following system of $p-$nonlinear difference equations

$$x_{n+1}^{(i)} = \frac{a^{(i)} x_n^{(i+1)\,\mathrm{mod}(p)} x_{n-2}^{(i+1)\,\mathrm{mod}(p)}}{b^{(i)} x_{n-1}^{(i)} + c^{(i)} x_{n-2}^{(i+1)\,\mathrm{mod}(p)}}, n \in \mathbb{N}_0, p \in \mathbb{N}, i \in \{1, ..., p\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the sequences $\left(a^{(i)}\right)$, $\left(b^{(i)}\right)$, $\left(c^{(i)}\right)$, are non-zero real numbers and initial values $x_{-j}^{(i)}$, $j \in \{0, 1, 2\}$, $i \in \{1, ..., p\}$. Finally, we give some applications concerning aforementioned system of difference equations.

Keywords: Riccati difference equation; periodicity; general solution; system of difference equations; Pell sequence.

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1. Introduction

In the recent years, there has been a lot of interest in studying nonlinear difference equations and systems. Not surprisingly therefore, several studies have been published on this topic (see, e.g., [1]-[28], and the related references therein). Besides their theoretical value, most of the recent applications have appeared in many scientific areas such as biology (population dynamics in

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particular), ecology, physics, engineering and economics (see, e.g. [8],[9], [12], [22]). It is very worthy to find systems belonging to solvable nonlinear difference equations systems in closed-form.

Since the paper by Brand [5], the following one-dimensional nonlinear difference equation of Riccati type,

$$x_{n+1}=\frac{ax_n+d}{cx_n+b},n\in\mathbb{N}_0, \tag{1.1}$$

where the initial value x_0 is a real number or complex number and the parameters a,b,c and d are the real numbers with the restrictions $c\neq 0$ and $ab\neq cd$, have the most diverse and interesting properties, especially as regards the distribution of their cluster points. This finding, led Stević [24] to study the solutions of the Eq. (1.1).

In Abo-Zeid et al. [3] the authors presented the solutions of the one-dimensional system of non-linear difference equations which reduced to the Riccati difference equation under appropriate transformations,

$$x_{n+1}=\frac{x_nx_{n-2}}{\pm x_{n-1}\mp x_{n-2}},n\in\mathbb{N}_0. \tag{1.2}$$

Then, in [10] and [11], equations in (1.3) were generalized to the following equations

$$x_{n+1}=\frac{ax_{n-l}x_{n-k}}{bx_{n-p}\pm cx_{n-q}},n\in\mathbb{N}_0, \tag{1.3}$$

where $\max\{l,k,p,q\}$ is nonnegative integer and a,b,c are positive constants. Moreover, in [4] the authors presented the solutions of the following two-dimensional four systems of nonlinear difference equation generalization of Eq. (1.2):

$$x_{n+1}=\frac{y_ny_{n-2}}{x_{n-1}+y_{n-2}},y_{n+1}=\frac{x_nx_{n-2}}{\pm y_{n-1}\pm x_{n-2}},n\in\mathbb{N}_0. \tag{1.4}$$

But, two (resp. three)$-$dimensional system of difference equations in (1.4) was extended to the following two (resp. three)$-$dimensional system of difference equations with constant coefficients

$$x_{n+1}=\frac{y_ny_{n-2}}{bx_{n-1}+ay_{n-2}},y_{n+1}=\frac{x_nx_{n-2}}{dy_{n-1}+cx_{n-2}},n\in\mathbb{N}_0, \tag{1.5}$$

(resp.

$$x_{n+1}=\frac{y_n y_{n-2}}{bx_{n-1}+ay_{n-2}}, y_{n+1}=\frac{z_n z_{n-2}}{dy_{n-1}+cz_{n-2}}, z_{n+1}=\frac{x_n x_{n-2}}{fz_{n-1}+ex_{n-2}}, n\in\mathbb{N}_0), \tag{1.6}$$

and system (1.5) (resp. (1.6)) was solved using convenient transformations in [26] (resp. [17]). Its extension with constant coefficients and $p-$dimensional is a system of a huge interest. For this reason, another extension of system (1.6) is the following system of $p-$dimensional nonlinear difference equations

$$x_{n+1}^{(i)}=\frac{a^{(i)}x_n^{(i+1)\,\mathrm{mod}(p)}x_{n-2}^{(i+1)\,\mathrm{mod}(p)}}{b^{(i)}x_{n-1}^{(i)}+c^{(i)}x_{n-2}^{(i+1)\,\mathrm{mod}(p)}}, n\in\mathbb{N}_0, p\in\mathbb{N}, i\in\{1,...,p\}. \tag{1.7}$$

Now, we consider system (1.7) in the case when a $a^{(i)}\neq 0$ for all $i\in\{1,...,p\}$. Noticing that in this case, system (1.7) can be written in the form

$$x_{n+1}^{(i)}=\frac{x_n^{(i+1)\,\mathrm{mod}(p)}x_{n-2}^{(i+1)\,\mathrm{mod}(p)}}{\widetilde{b}^{(i)}x_{n-1}^{(i)}+\widetilde{c}^{(i)}x_{n-2}^{(i+1)\,\mathrm{mod}(p)}}, n\in\mathbb{N}_0, p\in\mathbb{N}, i\in\{1,...,p\}.$$

where $\widetilde{b}^{(i)}=\dfrac{b^{(i)}}{a^{(i)}}$ and $\widetilde{c}^{(i)}=\dfrac{c^{(i)}}{a^{(i)}}$, for all $i\in\{1,...,p\}$, we see that we may assume that $a^{(i)}=1$, for all $i\in\{1,...,p\}$. Hence we consider, without loss of generality, the system

$$x_{n+1}^{(i)}=\frac{x_n^{(i+1)\,\mathrm{mod}(p)}x_{n-2}^{(i+1)\,\mathrm{mod}(p)}}{b^{(i)}x_{n-1}^{(i)}+c^{(i)}x_{n-2}^{(i+1)\,\mathrm{mod}(p)}}, n\in\mathbb{N}_0, p\in\mathbb{N}, i\in\{1,...,p\}. \tag{1.8}$$

using the same notation for coefficients as in (1.7) except for the coefficients $a^{(i)}$, assuming that $a^{(i)}=1$, for all $i\in\{1,...,p\}$.

The remainder of the paper is organized as follows: In section 2, we study the solutions of the given system of the $p-$dimensional nonlinear rational difference equations by using convenient transformation. In the next section, we obtain well-known Fibonacci numbers and Pell numbers in the solutions of aforementioned system for some cases. Section 4 concludes.

2. EXPLICIT FORMULAS FOR THE SOLUTIONS OF SYSTEM (1.8)

Let $\left\{x_n^{(1)},x_n^{(2)},...,x_n^{(p)}\right\}_{n\geq -2}$ be a solution of system (1.8). If at least one of the initial values $x_{-j}^{(i)}$, $j\in\{0,1,2\}$, $i\in\{1,...,p\}$, is equal to zero, then the solutions of system (1.8) is not defined. For example, if $x_{n_0}^{(i_0)}=0$ for some $n_0\geq -2$, $i_0\in\{1,...,p\}$. Then from the system (1.8) it follows that $x_{n_0+1}^{(i_0)}=0$, and consequently $b^{(i_0)}x_{n_0+1}^{(i_0)}+c^{(i_0)}x_{n_0}^{(i_0+1)\,\mathrm{mod}(p)}=0$, from which it follows that $x_{n_0+3}^{(i_0)}$

is not defined. Thus, for every well-defined solution of system (1.8), we get that $\prod_{i=1}^{p} x_n^{(i)} \neq 0$, $n \geq -2$, if and only if $\prod_{i=1}^{p} x_{-j}^{(i)} \neq 0$, $j \in \{0,1,2\}$. Note that the system (1.8) can be written in the form

$$b^{(i)} \frac{x_{n-1}^{(i)}}{x_{n-2}^{(i+1)\,\mathrm{mod}(p)}} + c^{(i)} = \frac{x_n^{(i+1)\,\mathrm{mod}(p)}}{x_{n+1}^{(i)}},\ n \in \mathbb{N}_0,\ p \in \mathbb{N}, i \in \{1,...,p\}. \tag{2.1}$$

Next, by employing the change of variables

$$y_n^{(i)} = \frac{x_{n-1}^{(i+1)\,\mathrm{mod}(p)}}{x_n^{(i)}},\ n \geq -2,\ p \in \mathbb{N}, i \in \{1,...,p\}.$$

Then system (2.1) can be written as

$$y_{n+1}^{(i)} = \frac{b^{(i)}}{y_{n-1}^{(i)}} + c^{(i)},\ n \in \mathbb{N}_0,\ p \in \mathbb{N}, i \in \{1,...,p\}. \tag{2.2}$$

Let $z_{m,k}^{(i)} = y_{2m+k}^{(i)}$ for $m \geq -1$, $k \in \{1,2\}$, $i \in \{1,...,p\}$, $p \in \mathbb{N}$. Then, from (2.2) we see that $\left(z_{m,k}^{(i)}\right)_{m \geq -1}$, $k = 1,2$, $i \in \{1,...,p\}$, $p \in \mathbb{N}$, are $2p-$solutions to the following system of difference equations

$$t_m^{(i)} = \frac{b^{(i)}}{t_{m-1}^{(i)}} + c^{(i)},\ m \in \mathbb{N}_0, i \in \{1,...,p\},\ p \in \mathbb{N}. \tag{2.3}$$

System (2.3) is solvable. Let

$$t_m^{(i)} = u_{m+1}^{(i)} \left(u_m^{(i)}\right)^{-1},\ m \geq -1,\ i \in \{1,...,p\},\ p \in \mathbb{N}, \tag{2.4}$$

where $u_{-1}^{(i)} = 1$, $u_0^{(i)} = t_{-1}^{(i)}$, $i \in \{1,...,p\}$, $p \in \mathbb{N}$. From now on, we assume that the sequence $\left(t_m^{(i)}\right)_{m \geq -1, i \in \{1,...,p\}}$ is well defined. Then system (2.3) becomes

$$u_{m+1}^{(i)} = c^{(i)} u_m^{(i)} + b^{(i)} u_{m-1}^{(i)},\ m \in \mathbb{N}_0, i \in \{1,...,p\},\ p \in \mathbb{N}. \tag{2.5}$$

To solve system (2.5) we need to use the following lemma.

Lemma 2.1. *For $a,b \in \mathbb{R}$, consider the homogeneous linear second-order difference equation with constant coefficients*

$$x_{n+1} = ax_n + bx_{n-1}, n \in \mathbb{N}_0, \tag{2.6}$$

where $b \neq 0$ and $a^2+4b \neq 0$, the general solution for equation (2.6) as follows:

$$x_n = bx_{-1}s_{n-1} + x_0 s_n,\ n \geq -1,$$

s_{-2} is calculated by using the following relations $bs_{n-1} = s_{n+1} - as_n$ for $n=-1$.

Proof. The proof follows essentially the same arguments as in Stević [24]. □

Remark 2.1. *By taking $a=1$ (resp. $a=2$) and $b=1$ in equation (2.6), with $s_{-1}=0$, $s_0=1$, then the sequence $(s_n)_{n\geq -1}$ reduce to the well known Fibonacci (resp. Pell) sequence.*

Let $(s_m^{(i)})_{m\geq -1, i\in\{1,\dots,p\}}$ be the solution to system (2.5) such that $s_{-1}^{(i)}=0$ and $s_0^{(i)}=1$, for $i \in \{1,\dots,p\}$. Then, from Lemma 2.1, the general solutions to system (2.5) can be written in the following form

$$u_m^{(i)} = b^{(i)}u_{-1}^{(i)}s_{m-1}^{(i)} + u_0^{(i)}s_m^{(i)},\ m \geq -1, i \in \{1,\dots,p\}, \tag{2.7}$$

$s_{-2}^{(i)}$, $i \in \{1,\dots,p\}$ are calculated by using the following relations $b^{(i)}s_{m-1}^{(i)} = s_{m+1}^{(i)} - c^{(i)}s_m^{(i)}$, $i \in \{1,\dots,p\}$ for $m=-1$. From the system in (2.4) and the system (2.7), it follows that

$$t_m^{(i)} = \frac{b^{(i)}u_{-1}^{(i)}s_m^{(i)} + u_0^{(i)}s_{m+1}^{(i)}}{b^{(i)}u_{-1}^{(i)}s_{m-1}^{(i)} + u_0^{(i)}s_m^{(i)}} = \frac{b^{(i)}s_m^{(i)} + t_{-1}^{(i)}s_{m+1}^{(i)}}{b^{(i)}s_{m-1}^{(i)} + t_{-1}^{(i)}s_m^{(i)}},\ m \geq -1,\ i \in \{1,\dots,p\},\ p \in \mathbb{N}.$$

Hence,

$$z_{m,k}^{(i)} = \frac{b^{(i)}s_m^{(i)} + z_{-1,k}^{(i)}s_{m+1}^{(i)}}{b^{(i)}s_{m-1}^{(i)} + z_{-1,k}^{(i)}s_m^{(i)}},\ m \geq -1,\ i \in \{1,\dots,p\},\ p \in \mathbb{N},$$

for $k \in \{1,2\}$, that is,

$$y_{2m+k}^{(i)} = \frac{b^{(i)}s_m^{(i)} + y_{k-2}^{(i)}s_{m+1}^{(i)}}{b^{(i)}s_{m-1}^{(i)} + y_{k-2}^{(i)}s_m^{(i)}},\ m \geq -1,\ i \in \{1,\dots,p\},\ p \in \mathbb{N},\ \text{for } k \in \{1,2\}.$$

Using the change of variables

$$y_n^{(i)} = \frac{x_{n-1}^{(i+1)\,\mathrm{mod}(p)}}{x_n^{(i)}}, n \geq -2, p \in \mathbb{N}, i \in \{1,\dots,p\}, \tag{2.8}$$

thus we have

$$x_n^{(i)} = \frac{x_{n-1}^{(i+1)\,\mathrm{mod}(p)}}{y_n^{(i)}} = \frac{x_{n-2}^{(i+2)\,\mathrm{mod}(p)}}{y_n^{(i)}y_{n-1}^{(i+1)\,\mathrm{mod}(p)}} = \dots = \frac{x_{n-p+1}^{(i-1)}}{\displaystyle\prod_{j=0}^{p-2} y_{n-j}^{(i+j)\,\mathrm{mod}(p)}} = \frac{x_{n-p}^{(i)}}{\displaystyle\prod_{j=0}^{p-1} y_{n-j}^{(i+j)\,\mathrm{mod}(p)}}, n \geq p, p \in \mathbb{N}, i \in \{1,\dots,p\},$$

where $\prod_{j=i}^{l} y_j = 1$ if $l < i$. From all above mentioned we see that the following corollary holds.

Corollary 2.1. *Let* $\left\{x_n^{(1)}, x_n^{(2)}, ..., x_n^{(p)}\right\}_{n\geq -2}$ *be a solution of system* (1.8)*. Then for* $n \geq p$*,*

$$x_{pn+k}^{(i)} = \frac{x_k^{(i)}}{\prod_{l=0}^{n-1}\prod_{j=0}^{p-1} y_{p(n-l)+k-j}^{(i+j)\,\mathrm{mod}(p)}}, k \in \{0,1,...,p-1\}, i \in \{1,...,p\}, p \in \mathbb{N}.$$

Theorem 2.1. *Let* $\left\{x_n^{(1)}, x_n^{(2)}, ..., x_n^{(p)}\right\}_{n\geq -2}$ *be a solution of system* (1.8)*. Then for* $n \geq p$*, if* p *is even,*

$$x_{pn+k}^{(i)} = \frac{\left\{\prod_{j=1}^{\left[\frac{k}{2}\right]} y_{2j}^{(i+k-2j)\,\mathrm{mod}(p)}\right\}^{-1}\left\{\prod_{j=\left[\frac{k}{2}\right]+1}^{k} y_{2(k-j)+1}^{(i-k+2j-1)\,\mathrm{mod}(p)}\right\}^{-1} x_0^{(i+k)\,\mathrm{mod}(p)}}{\prod_{l=0}^{n-1}\left\{\prod_{j=0}^{\left[\frac{p-2}{2}\right]} y_{2(h_1(l,k,p,n)-j)}^{(i+2j+t)\,\mathrm{mod}(p)}\right\}\left\{\prod_{j=0}^{\left[\frac{p-2}{2}\right]} y_{2(h_1(l,k,p,n)+t-j)-1}^{(i+2j+1-t)\,\mathrm{mod}(p)}\right\}},$$

if p *is odd,*

$$x_{pn+k}^{(i)} = \frac{\left\{\prod_{j=1}^{\left[\frac{k}{2}\right]} y_{2j}^{(i+k-2j)\,\mathrm{mod}(p)}\right\}^{-1}\left\{\prod_{j=\left[\frac{k}{2}\right]+1}^{k} y_{2(k-j)+1}^{(i-k+2j-1)\,\mathrm{mod}(p)}\right\}^{-1} x_0^{(i+k)\,\mathrm{mod}(p)}}{\prod_{l=0}^{n-\left[\frac{n-2}{2}\right]}\left\{\prod_{j=0}^{\left[\frac{p-2h}{2}\right]} y_{2(h_2(l,k,p,n)+t\wedge t_2-j)}^{(i+2j+h)\,\mathrm{mod}(p)}\right\}\left\{\prod_{j=0}^{\left[\frac{p-2+2h}{2}\right]} y_{2(h_2(l,k,p,n)+h-j)-1}^{(i+2j+1-h)\,\mathrm{mod}(p)}\right\}}$$

$$\times \frac{\prod_{l=1}^{\left[\frac{n}{2}\right]}\left\{\prod_{j=0}^{\left[\frac{p-2+2h}{2}\right]} y_{2(h_3(l,k,p,n)-j-1+h)}^{(i+2j+1-h)\,\mathrm{mod}(p)}\right\}^{-1}}{\prod_{l=1}^{\left[\frac{n}{2}\right]}\left\{\prod_{j=0}^{\left[\frac{p+2t-2t_2}{2}\right]} y_{2(h_3(l,k,p,n)-j+t\wedge t_2)-1}^{(i+2j+h)\,\mathrm{mod}(p)}\right\}},$$

$k \in \{0,1,...,p-1\}, i \in \{1,...,p\}, p \in \mathbb{N}$, *where* $t = k - 2\left[\frac{k}{2}\right] \in \{0,1\}$, $t_2 = n - 2\left[\frac{n}{2}\right] \in \{0,1\}$, $h = t \vee t_2 - t \wedge t_2$, $h_1(l,k,p,n) = \left[\frac{p}{2}\right](n-l) + \left[\frac{k}{2}\right]$, $h_2(l,k,p,n) = h_1(2l,k,p,n) - l + \left[\frac{n}{2}\right]$, $h_3(l,k,p,n) = \left[\frac{p}{2}\right](2l+t_2-1) + \left[\frac{k}{2}\right] + l$, $[x]$ *is integral part of* x *and*

$$y_{2m+1}^{(i)} = \frac{b^{(i)}s_m^{(i)} + y_{-1}^{(i)}s_{m+1}^{(i)}}{b^{(i)}s_{m-1}^{(i)} + y_{-1}^{(i)}s_m^{(i)}}, y_{2(m+1)}^{(i)} = \frac{b^{(i)}s_m^{(i)} + y_0^{(i)}s_{m+1}^{(i)}}{b^{(i)}s_{m-1}^{(i)} + y_0^{(i)}s_m^{(i)}},\ m \geq -1,\ i \in \{1,...,p\},\ p \in \mathbb{N},$$

with $(s_m^{(i)})_{m\geq -1, i\in\{1,...,p\}}$ *be the solution to system* (2.5) *such that* $s_{-1}^{(i)}=0$ *and* $s_0^{(i)}=1$, *for* $i\in\{1,...,p\}$.

Proof. By Corollary 2.1, we obtain

$$x_{pn+k}^{(i)}=\frac{x_k^{(i)}}{\prod_{l=0}^{n-1}\prod_{j=0}^{p-1}y_{p(n-l)+k-j}^{(i+j)\bmod(p)}}, k\in\{0,1,...,p-1\}, i\in\{1,...,p\}, p\in\mathbb{N}.$$

Using (2.8), we get

$$x_{pn+k}^{(i)}=\frac{x_0^{(i+k)\bmod(p)}}{\left\{\prod_{l=0}^{n-1}\prod_{j=0}^{p-1}y_{p(n-l)+k-j}^{(i+j)\bmod(p)}\right\}\left\{\prod_{j=0}^{k-1}y_{k-j}^{(i+j)\bmod(p)}\right\}}, k\in\{0,1,...,p-1\}, i\in\{1,...,p\}, p\in\mathbb{N},$$

The rest of assertions are immediate. □

3. Some Applications

In this section, we will give some applications for some special cases of the coefficients of the system (1.7).

Corollary 3.1. *Let* $\left\{x_n^{(1)},...,x_n^{(2p)}\right\}_{n\geq -2}$ *be a well-defined solution to the following system,*

$$x_{n+1}^{(i)}=\frac{x_n^{(i+1)\bmod(2p)}x_{n-2}^{(i+1)\bmod(2p)}}{x_{n-1}^{(i)}+x_{n-2}^{(i+1)\bmod(2p)}}, n\in\mathbb{N}_0, i\in\{1,...,2p\},\ p\in\mathbb{N}. \tag{3.1}$$

Then

$$\begin{aligned}
x_{2pn+k}^{(i)}=x_0^{(i+k)\bmod(2p)}\prod_{l=1}^{n}&\left\{\prod_{j=\left[\frac{k-2p+2}{2}\right]}^{\left[\frac{k}{2}\right]}\frac{x_0^{(i+k-2j)\bmod(2p)}F_{pl+j-2}+x_{-1}^{(i+k-2j+1)\bmod(2p)}F_{pl+j-1}}{x_0^{(i+k-2j)\bmod(2p)}F_{pl+j-1}+x_{-1}^{(i+k-2j+1)\bmod(2p)}F_{pl+j}}\right.\\
&\left.\times\prod_{j=\left[\frac{k}{2}\right]}^{\left[\frac{k+2p-2}{2}\right]}\frac{x_{-1}^{(i-k+2j+1)\bmod(2p)}F_{pl+k-j-2}+x_{-2}^{(i-k+2j+2)\bmod(2p)}F_{pl+k-j-1}}{x_{-1}^{(i-k+2j+1)\bmod(2p)}F_{pl+k-j-1}+x_{-2}^{(i-k+2j+2)\bmod(2p)}F_{pl+k-j}}\right\}\\
&\times\prod_{j=1}^{\left[\frac{k}{2}\right]}\frac{x_0^{(i+k-2j)\bmod(2p)}F_{j-2}+x_{-1}^{(i+k-2j+1)\bmod(2p)}F_{j-1}}{x_0^{(i+k-2j)\bmod(2p)}F_{j-1}+x_{-1}^{(i+k-2j+1)\bmod(2p)}F_{j}}\\
&\times\prod_{j=\left[\frac{k}{2}\right]+1}^{k}\frac{x_{-1}^{(i-k+2j-1)\bmod(2p)}F_{k-j-1}+x_{-2}^{(i-k+2j)\bmod(2p)}F_{k-j}}{x_{-1}^{(i-k+2j-1)\bmod(2p)}F_{k-j}+x_{-2}^{(i-k+2j)\bmod(2p)}F_{k-j+1}},
\end{aligned}$$

for $n\in\mathbb{N}_0$, $k\in\{0,1,\dots,2p-1\}$, $i\in\{1,\dots,2p\}$, $p\in\mathbb{N}$, *where* $(F_n)_{n\geq-1}$ *is the solution to the following difference equation*

$$F_{n+1}=F_n+F_{n-1},n\in\mathbb{N}_0,$$

satisfying the initial conditions $F_{-1}=0$, $F_0=1$. *The sequence* $(F_n)_{n\geq-1}$ *is called the well-known Fibonacci sequence in literature.*

Proof. System (3.1) is obtained from system (1.7) with $a^{(i)}=b^{(i)}=c^{(i)}=1$, $i\in\{1,\dots,2p\}$, $p\in\mathbb{N}$. Hence, the sequence $(s_n^{(i)})_{n\geq-1,i\in\{1,\dots,2p\}}$ satisfying conditions $s_{-1}^{(i)}=0$ and $s_0^{(i)}=1$, for $i\in\{1,\dots,2p\}$ are the same and so we have $s_n^{(i)}=F_n,n\geq-1$. The rest of the proof is straightforward and hence omitted. □

Corollary 3.2. *Let* $\left\{x_n^{(1)},\dots,x_n^{(2p+1)}\right\}_{n\geq-2}$ *be a well-defined solution to the following system,*

$$x_{n+1}^{(i)}=\frac{x_n^{(i+1)\bmod(2p+1)}x_{n-2}^{(i+1)\bmod(2p+1)}}{x_{n-1}^{(i)}+2x_{n-2}^{(i+1)\bmod(2p+1)}},n\in\mathbb{N}_0,i\in\{1,\dots,2p+1\},\ p\in\mathbb{N}.\tag{3.2}$$

Then

$$\begin{aligned}
x_{(2p+1)n+k}^{(i)}=x_0^{(i+k)\bmod(2p+1)}&\prod_{l=0}^{n-\left[\frac{n-2}{2}\right]}\left\{\prod_{j=h}^{p}\frac{x_0^{(i+2j-h)\bmod(2p+1)}P_{m-1}+x_{-1}^{(i+2j-h+1)\bmod(2p+1)}P_m}{x_0^{(i+2j-h)\bmod(2p+1)}P_m+x_{-1}^{(i+2j-h+1)\bmod(2p+1)}P_{m+1}}\right.\\
&\left.\times\prod_{j=-h}^{p-1}\frac{x_{-1}^{(i+2j+1+h)\bmod(2p+1)}P_{m-t\vee t_2-1}+x_{-2}^{(i+2j+2+h)\bmod(2p+1)}P_{m-t\vee t_2}}{x_{-1}^{(i+2j+1+h)\bmod(2p+1)}P_{m-t\vee t_2}+x_{-2}^{(i+2j+2+h)\bmod(2p+1)}P_{m-t\vee t_2+1}}\right\}\\
&\times\prod_{l=0}^{\left[\frac{n-2}{2}\right]}\left\{\prod_{j=h}^{p+t-t_2+h}\frac{x_{-1}^{(i+2j-h)\bmod(2p+1)}P_{m-p-1}+x_{-2}^{(i+2j-h+1)\bmod(2p+1)}P_{m-p}}{x_{-1}^{(i+2j-h)\bmod(2p+1)}P_{m-p}+x_{-2}^{(i+2j-h+1)\bmod(2p+1)}P_{m-p+1}}\right.\\
&\left.\times\prod_{j=-h}^{p-1}\frac{x_0^{(i+2j+1+h)\bmod(2p+1)}P_{m-p-t\vee t_2-2}+x_{-1}^{(i+2j+2+h)\bmod(2p+1)}P_{m-p-t\vee t_2-1}}{x_0^{(i+2j+1+h)\bmod(2p+1)}P_{m-p-t\vee t_2-1}+x_{-1}^{(i+2j+2+h)\bmod(2p+1)}P_{m-p-t\vee t_2}}\right\}\\
&\times\prod_{j=1}^{\left[\frac{k}{2}\right]}\frac{x_0^{(i+k-2j)\bmod(2p+1)}P_{j-2}+x_{-1}^{(i+k-2j+1)\bmod(2p+1)}P_{j-1}}{x_0^{(i+k-2j)\bmod(2p+1)}P_{j-1}+x_{-1}^{(i+k-2j+1)\bmod(2p+1)}P_j}\\
&\times\prod_{j=\left[\frac{k}{2}\right]+1}^{k}\frac{x_{-1}^{(i-k+2j-1)\bmod(2p+1)}P_{k-j-1}+x_{-2}^{(i-k+2j)\bmod(2p+1)}P_{k-j}}{x_{-1}^{(i-k+2j-1)\bmod(2p+1)}P_{k-j}+x_{-2}^{(i-k+2j)\bmod(2p+1)}P_{k-j+1}}
\end{aligned}$$

for $n \in \mathbb{N}_0$, $k \in \{0,...,2p\}$, $i \in \{1,...,2p+1\}$, $p \in \mathbb{N}$, *where* $m = p(n-2l) - l + \left[\frac{k}{2}\right] + \left[\frac{n}{2}\right] + t \vee t_2 - j - 1$ *and* $(P_n)_{n \geq -1}$ *is the solution to the following difference equation*

$$P_{n+1} = 2P_n + P_{n-1}, n \in \mathbb{N}_0,$$

satisfying the initial conditions $P_{-1} = 0$, $P_0 = 1$. *The sequence* $(P_n)_{n \geq -1}$ *is called the Pell sequence in literature.*

Proof. System (3.2) is obtained from system (1.7) with $a^{(i)} = c^{(i)} = 1$ and $b^{(i)} = 2$, $i \in \{1,...,2p+1\}$, $p \in \mathbb{N}$. Hence, the sequence $(s_n^{(i)})_{n \geq -1, i \in \{1,...,2p+1\}}$ satisfying conditions $s_{-1}^{(i)} = 0$ and $s_0^{(i)} = 1$, for $i \in \{1,...,2p+1\}$ are the same and so we have $s_n^{(i)} = P_n, n \geq -1$. The rest of the proof is straightforward and hence omitted. □

4. CONCLUSION

In this paper, we represented the general solutions of $p-$dimensional systems of nonlinear rational difference equations with constant coefficients using suitable transformation reducing to the equations in Riccati type. Secondly, the solutions of this system are related to both Fibanacci numbers and Pell numbers for some special cases. Finally, we will give the following important open problem for system of difference equations theory to researchers. The system (1.7) can extend to equations more general than that in (1.7). For example, the $p-$dimensional system of nonlinear rational difference equations of $(\max\{m,k,l,s\}+1)-$order,

$$x_{n+1}^{(i)} = \frac{a^{(i)} x_{n-m}^{(i+1) \operatorname{mod}(p)} x_{n-l}^{(i+1) \operatorname{mod}(p)}}{b^{(i)} x_{n-k}^{(i)} + c^{(i)} x_{n-s}^{(i+1) \operatorname{mod}(p)}}, n \in \mathbb{N}_0, p \in \mathbb{N}, i \in \{1,...,p\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the sequences $\left(a^{(i)}\right)$, $\left(b^{(i)}\right)$, $\left(c^{(i)}\right)$, are non-zero real numbers and initial values $x_{-j}^{(i)}$, $j \in \{0,...,\max\{m,k,l,s\}\}$, $i \in \{1,...,p\}$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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