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FIXED POINTS OF MODIFIED F-CONTRACTIONS IN S-METRIC SPACES

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Abstract. In this paper, we introduce a modified F-contraction in S-metric space. This modified form of F-contraction is via α-admissible mapping and we use it to examine the existence of fixed points in S-metric spaces. Sufficient examples are also given to examine the validity of the results obtained.

Keywords: fixed points; F-contractions; α-admissible; S-metric space.

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1. Introduction

In the year 2012, Wardowski [1] defined the notation of F-contraction to generalize Banach fixed point theorem. Samet et al. [2] also introduced the notation of α-admissible mappings. On the other hand Sedghi et al. [3] introduced the notion of S-metric space by generalizing metric space.

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The concept of α-admissible was extended in different directions. Bubul et al. [4] extended α-admissible mappings to (α,β)-admissible in S-metric like space. Priyobarta et al. [5] extended various forms of α-admissible in S-metric space. Bulbul et al. [6] also introduced $S-\beta-\psi$ contractive type mappings by extending $\alpha-\psi$-contractive mappings in S-metric space. There are various generalizations of α-admissible as well as F-contractions. These can be found in the literatures [7, 8, 9, 10, 11].

In this paper, we introduce a modified F-contraction by using α-admissible mappings and used it to examine the existence of fixed points in S-metric spaces.

2. Preliminaries

In 2012, Wardowski [1] defined a new concept of F- contraction as follows.

Definition 1. [1] *Let (X,d) be a metric space. A self- mapping $T: X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \le F(d(x,y)), \forall x,y \in X$$

where $F: \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

- **(F_1):** *F is increasing, i.e, for all $\alpha,\beta \in \mathbb{R}^+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;*
- **(F_2):** *For any sequence $\{a_n\}_{n=1}^{m}$ of positive real numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*
- **(F_3):** *There exists $k \in (0,1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.*

Let $\mathfrak{F}$ be the collection of all functions F satisfying (F_1), (F_2).

Wardowsksi [1] generalized the Banach Contraction Mapping Principle as follows.

Theorem 1. [1] *Let (X,d) be a complete metric space and $T: X \to X$ be an F-contraction. Then T has a unique fixed point.*

Following is the definition of c-comparision function.

Let Ψ be the family of functions $\psi: [0,\infty) \to [0,\infty)$ satisfying the following conditions

- **(i):** ψ is nondecreasing;
- **(ii):** $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

If $\psi \in \Psi$, then it is called c-comparison function. It is easy to show that $\psi(t) < t$ for all $t > 0$ and ψ is continuous at 0.

Definition 2. [3] *Let X be a non empty set and the mapping $S : X \times X \times X \to [0, +\infty)$ satisfies:*

1.: $S(x,y,z) = 0$ *if and only if* $x = y = z$ *for all* $x,y,z \in X$*;*

2.: $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$ *for all* $x,y,z,t \in X$*;*

Then, the pair (X,S) is called an S-metric space.

In 2012, Samet et al. [2] introduced the class of α-admissible mappings.

Definition 3. [2] *Let $\alpha : X \times X \to [0, \infty)$ be given mapping where $X \neq \phi$. A selfmapping T is called α-admissible if for all $x,y \in X$, we have*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1.$$

Priobarta et al. [5] extended α-admissible in the context of S-metric space as follows.

Definition 4. [5] *Let $\alpha_s : X \times X \times X \to [0, +\infty)$ be a given mapping where $X \neq \phi$. A selfmapping T is called α_S-admissible mapping if for all $x,y,z \in X$, we have*

$$\alpha_s(x,y,z) \geq 1 \Rightarrow \alpha_s(Tx,Ty,Tz) \geq 1.$$

Aydi et al. [7] introduced the following concept.

Definition 5. [7] *Let (X,d) be a metric space. A self-mapping $T : X \to X$ is said to be a modified F-contraction via α-admissible mappings if there exists $\tau > 0$ such that*

$$\begin{aligned} & d(Tx,Ty) > 0 \\ \Rightarrow \quad & \tau + F(\alpha(x,y)d(Tx,Ty)) \leq F(\Psi(d(x,y))) \end{aligned} \tag{1}$$

for all $x,y \in X$, where the mapping $F \in \mathfrak{F}$ and $\psi \in \Psi$.

If we let $F(t) = In(t)$ for $t > 0$, the contraction form (1) becomes

$$\alpha(x,y)d(Tx,Ty) \leq e^{-\tau}\psi(d(x,y)) \leq \psi(d(x,y)) \tag{2}$$

for all $x, y \in X, Tx \neq Ty$

(2) is considered as an $\alpha - \psi$-contraction which was introduced by Samet et al. [2].

We extend the concept of Aydi et al. [7] in S-metric space and introduce the following concept.

Definition 6. *Let (X,S) be an S-metric space. A self mapping $T : X \to X$ is said to be a modified F-contraction via α_s-admissible mappings if there exists $\tau > 0$ such that*

$$S(Tx,Ty,Tz) > 0$$

$$\Rightarrow \tau + F(\alpha_s(x,y,z))S(Tx,Ty,Tz) \leq F((S(x,y,z)) \tag{3}$$

for all $x, y, z \in X$ where the mapping $F \in \mathfrak{F}$ and $\psi \in \Psi$.

If we let $F(t) = ln(t)$ for $t > 0$, the contraction from (3) becomes

$$\alpha_s(x,y,z)S(Tx,Ty,Tz) \leq e^{-\tau}\psi(S(x,y,z)) \leq \psi(S(x,y,z)) \tag{4}$$

for all $x, y, z \in X$, $Tx \neq Ty \neq Tz$.

(4) is considered as an α_s- ψ-contraction.

In this paper, we introduce a modified F-contraction in S-metric space. This modified form of F-contraction is via α-admissible mapping and we use it to examine the existence of fixed points in S-metric spaces.

3. Main Results

We prove the following theorem.

Theorem 2. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a modified F-contraction via α_S- admissible mappings. Suppose that*

- **(i):** *T is α_s- admissible;*
- **(ii):** *there exists $x_0 \in X$ such that $\alpha_s(x_0, x_0, Tx_0) \geq 1$;*
- **(iii):** *T is continuous.*

Then T has a fixed point.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha_s(x_0, x_0, Tx_{0)}) \geq 1$. we define a sequence x_n in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. Suppose that $x_{n_0} = x_{n_0+1}$ for some n_0. So the proof is completed. Now, we assume that

$$x_n \neq x_{n+1} \text{ for all } n. \tag{5}$$

Since $\alpha_s(x_0, x_0, x_1) = \alpha_s(x_0, x_0, Tx_0) \geq 1$ and T is α_s-admissible, we get

$$\alpha_s(x_n, x_n, x_{n+1}) \geq 1, \text{ for all } n = 0, 1, \ldots \tag{6}$$

From (3) and (5), we have

$$\tau + F(\alpha_s(x_{n-1}, x_{n-1}, x_n)S(Tx_{n-1}, Tx_{n-1}, Tx_0) \leq F(\psi(S(x_{n-1}, x_{n-1}, x_n))$$

on account of (F_1) and (6), we find

$$\tau + F(S(x_n, x_n, x_{n+1})) \leq F(S(x_{n-1}, x_{n-1}, x_n)), \text{ for all } n \geq 1.$$

By letting $S_n = S(x_n, x_n, x_{n+1})$, the inequality above infer that

$$F(S_n) \leq F(S_{n-1}) - \tau \leq f(s_0) - n\tau \text{ for all } n \geq 1.$$

Conseqently, we obtain

$$\lim_{n\to\infty} F(S_n) = -\infty$$

By the property (F_2), we have

$$\lim_{n\to\infty} S_n = 0. \tag{7}$$

Now, due to (F_3), we have

$$\lim_{n\to\infty} S_n^k(F(S_n) = 0,$$

where $k \in (0, 1)$. By (7), the following holds for all $n \geq 0$.

$$\begin{aligned} 0 &\leq S_n^k F(S_n) - S_n^k(S_0) \leq S_n^k(F(S_0 - n\tau)) - S_n^k F(S_0) \\ &= -n\tau S_n^k \leq 0 \end{aligned} \tag{8}$$

letting $n \to \infty$ in (8), we find that

$$\lim_{n\to\infty} nS_n^k = 0.$$

So there exists $n_1 \in \mathbb{N}$ such that $S_n \leq 1/n^{1/k}$ for all $n \geq n_1$. For $m,n \in \mathbb{N}$ with $m > n \geq n_1$, we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S_n + 2S_{n+1} + \ldots + S_{n-1} \\ &\leq 2\sum_{i=1}^{\infty} 1/i^{1/k} \end{aligned}$$

Since $\sum_{i\leq 1} 1/i^{1/k}$ converges, the sequence $\{x_n\}$ is Cauchy in (X,S). From the completeness of X, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Finally, the continuity of T yields $Tu = u$, which completes the proof. □

Theorem 2 remains true if we replace the continuity hypothesis by the following property:

(H) If $\{x_n\}$ is a sequence in X such that $\alpha_n(x_n, x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha_s(x_{n(k)}, x_{n(k)}, x) \geq 1$ for all k.

Theorem 3. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a modified F-contraction via α_s-admissible mappings. Suppose that*

- **(i):** *T is α_s- admissible;*
- **(ii):** *there exists $x_0 \in X$ such that $\alpha_s(x_0, x_0, Tx_0) \geq 1$;*
- **(iii):** **(H)** *holds.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Following the lines in the proof of Theorem 2, we construct a sequence $\{x_n\}$ in (X,S) which is Cauchy and converges to some $u \in X$.

Suppose that there exists an increasing sequence $\{n(k)\} \subset N$ such that $x_{n(k)} = Tu$ for all $k \in N$. Letting $k \to \infty$, by the uniqueness of the limit, we find $Tu = u$. Hence, the proof is completed. As a result, we shall assume that there exists $k_0 \in N$ such that $x_{n(k)} \neq Tu$ for all $k \in N$ with $k \geq k_0$. Consequently, we have $Tx_{n(k)-1} \neq Tu$ for all $k \geq k_0$. Therefore, by (3), we have

$$\begin{aligned} &\tau + F(\alpha_s(x_{n(k)-1}, x_{n(k)-1}, u))S(Tx_{n(k)-1}, Tx_{n(k)-1}, Tu)) \\ &\leq F(\psi(S(x_{n(k)-1}, x_{n(k)-1}, u))). \end{aligned}$$

Regarding $\alpha(x_{n(k)-1}, x_{n(k)-1}, x) \geq 1$ and (F_1)

$$S(x_{n(k)}, x_{n(k)}, Tu) = S(x_{n(k)-1}, x_{n(k)-1}, Tu))$$
$$\leq \psi(S(x_{n(k)-1}, x_{n(k)-1}, u).$$

Since ψ is continuous at 0 and $S(x_{n(k)-1}, x_{n(k)-1}, u)) \to 0$,

$$\lim_{n\to\infty} \psi(S(x_{n(k)-1}, x_{n(k)-1}, u)) = 0.$$

Thus,

$$\lim_{n\to\infty} S(x_{n(k)+1}, x_{n(k)+1}, Tu)) = 0.$$

By the uniqueness of limit, $Tu = u$. □

We provide the following example.

Example 1. *Take $X = \{0,1,2\}$ and $T : X \to X$ such that $T0 = 0$ and $T1 = T2 = 1$. Consider $\alpha_s(1,1,2) = \alpha_s(2,2,1) = \alpha_s(1,1,1) = 1$.*

Let $x,y,z \in X$ such that $Ty \neq Tz$, so (x,y,z) is equal to (0,0,1), (0,0,2), (1,1,0) or (2,2,0). For these four cases, $\alpha_s(x,y,z) = 0$, so (4) holds. In other words, (3) holds for $F(t) = ln(t)$ and for any $\psi \in \Psi$ and any S-metric S. It is also obvious that the hypothesis (**H**) *is satisfied. Thus , applying Theorem 3, the mapping T has a fixed point. Here, we have two fixed points which are $u = 0$ and $u = 1$.*

Here, we underline the fact that the mapping considered in above examples has two fixed points, 0 and 1. Notice also that $\alpha_s(0,0,1) = 0 < 1$. For the uniqueness, we need an additional condition:

(**U**) *For all $x,y,z \in Fix(T)$, we have $\alpha_s(x,y,z) \geq 1$, where Fix (T) denotes the set of fixed points of T.*

Theorem 4. *Adding condition* (**U**) *to the hypothesis of Theorem 2 (resp. Theorem 3), we obtain that u is the unique fixed point of T.*

Proof. Suppose, on the contrary, that there exists $u, v \in X$ such that $u = Tu$ and $v = Tv$ with $u \neq v$. Then $Tu \neq Tv$, so by (3), we get

$$\tau + F(\alpha_s(u,u,v)S(Tu,Tu,Tv)) \leq F(\psi(S(u,u,v)))$$

that is,

$$\begin{aligned}\tau + F(\alpha_s(u,u,v)S(u,u,v)) &\leq F(\psi(S(u,u,v))) \\ &< F(S(u,u,v))\end{aligned}$$

which is a contradiction. Thus, $u = v$ which completes the proof. The following corollaries are immediate. □

Corollary 1. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a given mapping. Suppose there exists $\tau > 0$ such that*

$$\begin{aligned} & S(Tx,Ty,Tz) > 0 \\ (9) \qquad \Rightarrow \quad & \tau + F(S(Tx,Ty,Tz)) \leq (\psi(S(x,y,z))) \end{aligned}$$

for all $x,y,z \in X$ where F satisfies $(F_1)-(F_2)$.
Then T has a unique fixed point.

Proof. It is sufficient to take $\alpha_s(x,y,z) = 1$ in Theorem 4 □

Corollary 2. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a given mapping. Suppose there exists $\tau > 0$ such that*

$$\begin{aligned} & S(Tx,Ty,Tz) > 0 \\ (10) \qquad \Rightarrow \quad & \tau + F(S(Tx,Ty,Tz)) \geq F(cS(x,y,z)), \end{aligned}$$

for all $x,y,z \in X$ where F saties $(F_1)-(F_3)$ and $c \in (0,1)$.
Then T has a unique fixed point.

Proof. It follows from Corollary 1 with $\psi(t) = ct$
The investigation of existence of fixed points on metric spaces endowed with a partial order was intiated by Turinici [12]. □

Definition 7. *Let $(X,\leq)$ be a partially ordered set and $T : X \to X$ be a given mapping. It is said that T is nondecreasing with respect to $\leq$ if*

$$x,y \in X,\ x \leq y \Rightarrow Tx \leq Ty$$

Furthermore, a sequence $x_n \subset X$ *is said to be nondecreasing with respect to* $\leq$ *if*

$$x_{n(k)} \leq x \textit{ for all } k.$$

Definition 8. *Let* $(X,\leq)$ *be a partially ordered set and* S *be an* S*-metric on* X*. We say* $(X,\leq,S)$ *is regular if for every nondecerasing* $\{x_n\} \subset X$ *such that* $x_n \to x \in X$ *as* $n \to \infty$*, there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $x_{n(k)} \leq x$ *for all* k*.*

Under the set-up of partially ordered S-metric spaces, we have the following result.

Corollary 3. *Let* $(X,\leq)$ *be a partially ordered set and* S *be an* S*-metric on* X *such that* (X,S) *is complete. Let* $T : X \to X$ *be a nondecreasing mapping with respect to* $\leq$*. Suppose that there exist* $\tau \geq 0$*, such that* $\psi \in \Psi$ *and* $F \in \mathfrak{F}$ *such that*

$$\tau + F(S(Tx,Tx,Ty)) \leq F(\psi(S(x,x,y))),$$

for $x,y \in X$ *with* $x \geq y$ *and* $Tx \neq Ty$*. Suppose also that the following conditions hold:*

(i): *there exists* $x_0 \in X$ *such that* $x_0 \leq Tx_0$*;*

(ii): *either* T *is continuous;*

(iii): $r(X,\leq,S)$ *is regular.*

Then T *has a fixed point.*

Example 2. *Let* $X = [0,\infty)$ *and* $S(x,y,z) = |x-y| + |y-z|$ *for all* $x,y,z \in X$*. Take* $\tau > 0$*. Consider the mapping* $T : X \to X$ *given by*

$$Tx = \begin{cases} e^{\tau}(\frac{3x}{4}, \textit{ if } x \in [0,1] \\ e^{-tau}(\frac{3}{4}, \textit{ if } x > 1 \end{cases}$$

T *is continuous in* (X,S)*. Define the mapping* $\alpha_s : X \times X \times X \to [0,\infty)$ *by*

$$\alpha_s(x,y,z) = \begin{cases} 1, & \textit{if } x \in [0,1] \\ 0, & \textit{otherwise.} \end{cases}$$

Consider the function $\psi : [0,\infty) \to [0,\infty)$ *by*

$$\psi(t) = \begin{cases} \frac{3t}{4}, & \textit{if } t \in [0,1], \\ \frac{2t}{5} & \textit{otherwise} \end{cases}$$

Let $x,y,z \in X$ such that $\alpha_s(x,y,z) \geq 1$, so $x,y,z \in [0,1]$. Then $Tx,Ty,Tz \in [0,1]$, that is, $\alpha_s(Tx,Ty,Tz) = 1$. Hence, T is α_s-admissible. Mention that $\psi \in \Psi$ and $\alpha(0,0,T0) = 1$. In this case where $x,y,z \in [0,1]$ such that $Ty \neq Tz$, we have

$$\begin{aligned} \alpha(x,y,z)S(Tx,Ty,Tz) &= S(Tx,Ty,Tz) \\ &= e^{\tau}\frac{3}{4}(|x-y|+|y-z|) \\ &\leq e^{\tau}\psi S(x,y,z) \end{aligned}$$

In the other case where x or y or z is not in $[0,1]$, $\alpha(x,y,z) = 0$, so the above inequality is satisfied for all $x,y,z \in X$ with $Ty \neq Tz$. Thus, (3) *is satisfied with $F(t) = \ln(t)$ for $t > 0$. Moreover, t is easy to satisfy the hypothesis* **(U)** *is true. Thus, applying Theorem 3, the mapping T has a unique fixed point, which is $u = 0$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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