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BAYESIAN QUEUE MODELLING WITH LIKELIHOOD BINOMIAL NEGATIVE AND UNIFORM DISCRETE

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Abstract. Queues often occur in everyday life, including the queue for booking train tickets. Queue analysis has the benefit of analyzing whether a service is optimal or not. Long queues will certainly cause disappointment for prospective passengers booking train tickets. Meanwhile, many unemployed services cause losses in the provision of facilities. Queue analysis tests the distribution of the number of arrivals and the number of services following the Poisson or exponential distribution. Often the assumption test does not follow this distribution and is therefore considered a General distribution. In this study, the queue at the Tawang railway station in Semarang, Indonesia, service counter was modeled by looking for a general distribution using a Bayesian approach. The results are the likelihood distribution of the number of arrivals with a Uniform and Binomial Negative distribution. Besides, the likelihood distribution for the services has a uniform distribution. The prior used is the Poisson distribution with Jeffrey's approach. The resulting Posterior Distributions are Beta Distribution and Uniform Distribution. This shows that even though using a Bayesian approach, the resulting distribution is still General. The queue model for arrivals and services number is (G/G/2):(GD/∞/∞).
Keywords: queue theory; Bayesian; negative binomial; uniform discrete; beta
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1. INTRODUCTION

A queue is a group of customers in waiting line who want service from one or more in the service facilities in the system. The queuing process is associated with the arrival of a customer at

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a service facility, waiting in a queue if it cannot be served, being served, and finally leaving the facility after being served [1]. The definition of queue is a system that the number of customers wait in the line to get services [2]. The process of queue is a process which relates in customer's arrival in the system who wait in the line and get served, after that leave the line. Meanwhile, the queuing system is tools that consist of customers, servants, and a procedure that regulates customer service [3].

Queuing occurs because the customers at a system exceeds the available capacity to perform the service. It is because almost all economic, business, and social aspects operate with relatively limited resources. If there are too many servants, it will cost a lot. Conversely, if the service capacity is lacking, there will be queues, which can also cause losses in the form of losing customers. Thus, this queuing theory's main objective is to balance service costs and loss costs caused by waiting [4]. Queues occur in several daily activities, including queues for services at banks, hospitals, service vehicles, and so on [5]. One way to reduce queuing activities is by implementing online applications, such as queuing for train ticket purchases at the reservation counter. Prospective passengers can use the online train ticket application without queueing to buy tickets directly. However, not all potential passengers use the online application. There were still many prospective passengers who also queued for train ticket purchases at the reservation counters. Several possibilities cause prospective passengers to choose to queue rather than to buy online, including not having an online application and ordering tickets that exceed the limit set by the application (online ordering at least one day before departure). The queue that occurs at the ticket reservation counter requires an analysis of whether it is effective or not. Long lines will cause prospective passengers to feel disappointed with the service system. Meanwhile, if many servers are idle, it will cause losses for the company [3].

The queuing model was developed by many researchers, one of which is the Bayesian queue [6]. The bayesian principle combines the likelihood function with a prior distribution. It will produce a posterior distribution, which is used in queuing modeling. Meanwhile, the classical method only uses the variables' distribution to determine the queuing model. The Bayes method has the advantage of using prior information to determine the distribution. A bayesian queue is a queue that uses the posterior distribution in determining the distribution in the queue model [7]. The analysis of queue model is to identify the distribution of arrivals and time of services. In the Bayesian queuing model, the distribution determination is based on the previous distribution (prior) to form the posterior distribution [8]. This paper uses the Bayesian queue model to describe the queue modeling for train ticket reservations. The case study is a ticket booking service at Tawang railway station, Semarang city, Indonesia. The novelty of this paper is that other researchers have never done the modeling of the Bayesian queue at the train ticket booth.

2. MATERIALS AND METHODS

2.1. Steady State Concept

A Steady-state is a condition when the properties of the system do not change with time (constant). It is a requirement for analysis in the queuing model. To find out whether a system is steady state or not, we can use the formula as follows [9]:

$$\rho = \frac{\lambda}{c\mu}$$

ρ is the mean of customer arrivals to the line in point per unit time; λ is the average of customer arrivals; c is the number of servers; μ is the mean of customers who served in unit time. If the $\rho < 1$, then it is a steady state, so we can continue our analysis in the queue model. If the steady-state is unfulfilled, then we have to transform the data or add the observations.

2.2. Arrival Distribution

According [10], the arrival distribution of customers is usually characterized by inter-arrival time, namely the time between the arrival of two consecutive customers at a service facility. This pattern is deterministic or in the form of a random variable whose probability distribution is assumed to be known. This pattern can depend on the number of customers and whether they are in the system or not. It depends on this queuing system. The distribution of arrivals is an important factor that has a major influence on the smooth running of services. The arrival distribution is divided into individual arrivals (single arrivals) and group arrivals (bulk arrivals). These two components must receive adequate attention when designing service systems (Kakiay, 2004).

According to [5] the assumption about Poisson shows that customer arrivals are random and have an average arrival rate of . The length of the time interval between two customer arrivals of $1/\lambda$ is called the inter-arrival time. If the arrivals has Poisson distribution, then the time between arrivals has random and the distribution is exponential.

2.3. The Time services distribution

The form of service is determined by service time, namely the time required to serve customers at service facilities. The length of service is the time it takes to serve a customer. Therefore, the service time may be constant over time for all customers or it may be a random variable. For analysis purposes, the service time is considered a random variable that is scattered independently and equally and does not depend on arrival time. If the service time is randomly distributed, it is necessary to know the appropriate probability distribution to describe its behavior. Usually, if the service time is random, then the appropriate probability distribution is the exponential distribution [11].

2.4. Bayesian estimation

Let $\mathbf{y} = (y_1, \ldots, y_n)$ is observation with parameter θ, then the conditional distribution is $p(y|\theta)$, Based on Bayes theorem it can be written that [12]:

$$p(y|\theta)\, p(\theta) = p(\theta|y)\, p(y) \tag{1}$$

From the Eq. (1), it can be written as:

$$p(\theta|y) = \frac{p(y|\theta)\, p(\theta)}{p(y)} \tag{2}$$

In Eq. (2), $p(\theta)$ is distribution probability of prior; $p(y|\theta)$ is probability of likelihood distribution, and $p(\theta|y)$ is posterior distribution. The main objective is to obtain the posterior distribution which is used for estimation. There are several types of prior distributions based on the identification of data patterns, namely conjugate and non-conjugate [8]. Conjugate relates to the formla of the likelihood function, while the non-conjugate does not include the likelihood function. Prior determination is also obtained based on the identification the parameter, which is informative prior and non-informative prior. Informative prior refers to the provision of parameters from the prior distribution that has been selected whether the conjugate prior distribution or not. While the non-informative prior, the prior distribution is not consider as existing data or the prior distribution does not involve the information about parameters. One of the non-informative approaches is Jeffrey's method. It obtains that prior distribution $p(\theta)$ is square root from Fisher information, which has formula as follows:

$$p(\theta) = \sqrt{I(\theta)} \tag{3}$$

where $I(\theta)$ is expectation of Fisher information.

$$I(\theta) = -E_0\left[\frac{\partial^2 \log f(x;\theta)}{\partial^2\theta}\right] \tag{4}$$

2.5. Methodology of Research.

Field data collection was carried out for seven days, from January 1 to January 7, 2022, at Tawang railway station, Semarang, Central Java, Indonesia. The research variables are the data of the number of arrivals and served of customers at the reservation counter. The steps of analysis were as follows:

1. The testing of steady state in observation data
2. The distribution testing of the number of customers arrival and the number of served.
3. Determine the prior distribution based on the result of step (2).
4. Find the posterior distribution
5. Determine the queue model based on result in steps (5)
6. Determine the system performance measure

The software that was used in the analysis was R software.

3. RESULTS AND DISCUSSIONS

Table 1 provides data on the arrival frequency and service frequency of customers or potential passengers at the reservation counter.

Table 1. Arrival Data and Ordering Counter Service in Tawang railway station.

Observation	The Number of arrival	The Number of Served
1st day	111	111
2nd day	128	128
3rd day	114	114
4th day	122	122
5th day	94	94
6th day	121	121
7th day	111	111
Total	801	801

From the Table 1, the average the number of arrival and served were 114 people per a day. The next step was testing the steady-state process. The steady-state conditions must be met in the queuing model. The steady-state condition follows that the value of $\rho < 1$. It means that the mean of number of customers who come is smaller than the mean of customers served [13]. To get the value of ρ, necessary the average value of the number of customers who came and the average number of customers served. The time interval used is 60 minutes. Based on the data, it is known that

1. The mean of customers who come (λ) is 16.347 subscribers per 60 minutes.
2. The mean of customers served (μ) is 16.347 subscribers per 60 minutes.
3. The number of counters in the ordering section (c) is two counters.

So that the steady state value is $\rho = 0.5$ the value of the utility level of the ordering counter service facility is less than one, it can be concluded that the queue system is steady state.

After it is known that the steady state test is fulfilled, the next step is testing the compatibility of the distribution. In this paper, the distribution used is a discrete uniform distribution and a negative binomial distribution. The variables used in queuing modeling are the number of arrivals and served. For testing the Discrete Uniform distribution as follows [14] :

Hypothesis

H_0 : the number of arrivals (number of services) at the reservation counter distributed Uniform Discrete

H_1 : the number of arrivals (number of services) at the reservation counter do not have a uniform discrete distribution

By using the Kolmogorov Smirnov test, the test statistics used are [9]:

$$D = \max\left(\left|S(x_i) - F_0(x_i)\right|\right)$$

Where $S(x_i)$ is the cumulative distribution of the sample from the population, the number of arrivals or the number of served. $F_0(x_i)$ is the cumulative distribution of the Discrete Uniform distribution. The results obtained are as follows:

Table 2. The test of fit discrete uniform distribution of number arrival and served

Variables	D_{value}	**Decision**	**Conclusion**
The Number of arrival	0.14886	H_0 accepted	It has discrete uniform distribution
The number of served	0.18475	H_0 accepted	It has discrete uniform distribution

Criteria of testing is rejected H_0 for $D_{value} > D_{table}$; where the $D_{table} = \frac{1.35810}{\sqrt{49}} = 0.19401$. Based on table 2, the number of arrivals and the number of served are discrete uniform distribution. Then we would like to test the number of arrival and served with Negative Binomial distribution.

Hypothesis

H_0: the number of arrivals (and or the number of served) at the order counter has a negative Binomial distribution

H_1: the number of arrivals (and or the number of served) at the order counter is not negative binomial distribution

The statistics testing uses equation (1), the results can be obtained as follows:

Table 3. The test of fit the negative binomial distribution of number arrival and served

Variables	D_{value}	**Decision**	**Conclusion**
The Number of arrival	0.14987	H_0 Accepted	It has binomial negative distribution
The number of served	0.22045	H_0 Rejected	It hasn’t binomial negative distribution

Based on table 3, it can be concluded that the number of services is not negative binomial distribution. The next step is to determine the prior distribution. In this paper, the prior determination used non-informative priors using Jeffrey's. The prior distribution used is the non-informative prior of the Poisson distribution using Jeffrey's.

$$p(x;\lambda) = \frac{\exp(-\lambda)\lambda^x}{x!}$$

$$\log p(x;\lambda) = \log\left(\frac{\exp(-\lambda)\lambda^x}{x!}\right)$$

$$= x\log\lambda - \log x! - \lambda$$

$$\frac{\partial^2 \log p(x;\lambda)}{\partial\lambda^2} = -\frac{x}{\lambda^2}$$

$$I(\lambda) = -E_\lambda\left[-\frac{x}{\lambda^2}\right] = \frac{1}{\lambda^2}E_\lambda[x] = \frac{1}{\lambda}$$

So that, the prior distribution is as follows:

$$p(\lambda) = \frac{1}{\sqrt{\lambda}}$$

After getting the prior and likelihood distributions for the number of arrivals and the number of served, we have to find the Posterior distribution. The posterior distribution is the product of the likelihoods with the prior distribution.

Table 4. Posterior Distribution

Prior Distribution	**Likelihood**	**Posterior Distribution**
Poisson distribution	Binomial Negative Distribution	Beta Distribution
Poisson distribution	Uniform discrete distribution	Uniform discrete distribution

Based on table 4, it is known that if the Jaffrey's prior-distribution is from the Poisson distribution, with the likelihood used is the negative binomial distribution, it will produce a posterior distribution in the form of a beta distribution. Meanwhile, if the likelihood is a discrete uniform distribution, the posterior distribution has a uniform discrete distribution. So that the queuing model obtained from the posterior distribution is as follows:

Table 5. The queue model

The number of arrival distribution	**The number of served distribution**	**The queue model**
discrete uniform distribution	discrete uniform distribution	(G/G/2):(GD/∞/∞)
Beta distribution	discrete uniform distribution	(G/G/2):(GD/∞/∞)

Table 5 shows that the total distribution of arrivals has a discrete uniform distribution, while the distribution of the number of services is also a discrete uniform distribution so that the two distributions are classified as general distributions. The queue model obtained is (G / G / 2) :(GD / ∞ / ∞). Likewise, the number of arrivals distribution has a beta distribution and the number of services has a discrete uniform distribution indicating the queue model obtained (G / G / 2) :(GD / ∞ / ∞). It shows that with a different distribution, namely the beta distribution and the discrete uniform distribution in queuing modeling produces the same model, namely General. General distribution, namely the number of arrivals with a discrete uniform distribution and beta distribution while the number of services has a discrete uniform distribution. With the General distribution model, the queue system performance measures are obtained as follows:

Table 6. Performance system Measurement

Counter	L_q	L_s	W_q	W_s
Reservation	0.00125	1.00125	7.63078×10^{-5}	0.06125

Based on Table 6 it can be inferred that the estimated number of subscribers in the queue system every 60 minutes is L_s = 1.00125 subscribers. It means that in 60 minutes, the total number of customers in line and the customers being served are 1.00125 customers (one customer in line). Besides, the estimated number of subscribers in line every 60 minutes is L_q = 0.00125 subscribers. It means that the average customer waiting in line for service in 60 minutes is 0.00125 customers or we can say that the customer probably most served directly. The waiting time in the system is estimated at W_s = 0.06125 of 60 minutes or about 3.67500 minutes. Meanwhile, the estimated waiting time in the queue is W_q = 7.63078 $\times 10^{-5}$ of 60 minutes or about 0.00458 minutes. The probability that the service officer is unemployed is P_o = 0.3333, which means that the queue system has a busy chance of 66.67% and the remaining 33.33% is the chance that the queue system is not busy.

4. CONCLUSION

Based on the research on the queue booking counter at Tawang railway station, the number of arrivals and the number of served have a uniform distribution with a queuing model (G/G/2):(GD/∞/∞). If the number of arrivals has a Beta distribution and the number of services has a Uniform distribution, then the queuing model is also (G/G/2):(GD/∞/∞). In addition, when viewed from the performance measurement of the queuing system, it can be concluded that the performance at the reservation counter is already very good. Customers don't have to wait long to be served.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Chydzinski, Queues with the dropping function and non-Poisson arrivals, IEEE Access. 8 (2020), 39819–39829.

[2] T.J. Kakiay, Dasar Teori Antrian Untuk Kehidupan Nyata, Andi Ofset, Yogyakarta, 2004.

[3] P. Bedell, J.M. Smith, Topological arrangements of M/G/c/K, M/G/c/c queues in transportation and material handling systems, Computers Oper. Res. 39 (2012), 2800–2819.

[4] S.A. Bishop, H.I. Okagbue, P.E. Oguntunde, et al. Survey dataset on analysis of queues in some selected banks in Ogun State, Nigeria, Data Brief. 19 (2018), 835–841.

[5] A.M. Law, Simulation modeling and analysis, McGraw-Hill Education, Arizona,USA, 2015.

[6] Y. Mei, W. Gu, E.C.S. Chung, et al. A Bayesian approach for estimating vehicle queue lengths at signalized intersections using probe vehicle data, Transport. Res. Part C: Emerging Technol. 109 (2019), 233–249.

[7] B. Bigdeli, P. Pahlavani, High resolution multisensor fusion of SAR, optical and LiDAR data based on crisp vs. fuzzy and feature vs. decision ensemble systems, Int. J. Appl. Earth Obs. Geoinf. 52 (2016), 126–136.

[8] G. Casale, Approximating passage time distributions in queueing models by Bayesian expansion, Perform. Eval. 67 (2010), 1076–1091.

[9] T. Altiok, B. Melamed, Simulation modeling and analysis with ARENA, Academic Press, 2007.

[10] E. Sihotang, Sugito, Mustafid, et al. Analysis of queue and performance of automatic toll booths with a normal distribution (case study: Automatic booths toll gate muktiharjo), J. Phys.: Conf. Ser. 1524 (2020), 012093.

[11] J.F. Shortle, J.M. Thompson, D. Gross, et al. Fundamentals of queueing theory, John Willey and Sons, Inc., New York, 2018.

[12] J.J. Vargas-Sánchez, E. Causado, H. Mercado, Estimadores Bayesianos de distribuciones Weibull aplicados a un modelo de líneas de espera G/G/s, Rev. Métodos Cuantitativos para la Econ. y la Empres. 30 (2020), 142–162.

[13] A. Ahdika, Mathematical model and numerical simulation of dual-channel multiphase bulk arrival queuing system with heterogeneous service time, J. Phys. Conf. Ser. 1127 (2019), 012070.

[14] H.H. Taha, Operations research an introduction, Pearson Prentice Hall, England, 2007.