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BIFURCATION OF POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. This paper studies the bifurcation of positive solutions for a boundary-value problem of nonlinear fractional differential equations with integral boundary conditions. Using the topological degree theory and the bifurcation technique, the existence of positive solutions is investigated and some sufficient conditions are obtained.

Keywords: positive solution; fractional differential equation; integral boundary condition; bifurcation technique.

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1. Introduction

Since fractional-order models are found to be more adequate than integer-order models in some real-world problems. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. For details, see [16], [17], [18]. Consequently, in the last two decades, the study of fractional differential equations has attracted many scientists' interest due to their applications in chemical process, physics, biology, and so on. It have been given considerable attention by many authors, see [20], [21], [22] and the references therein.

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Owing to various applications of integral boundary value problems in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, undergroundwater flow, and population dynamics, the existence of solutions for fractional differential equations with integral boundary conditions has been extensively studied in recent years (see [3], [4], [5], [6] and the referencres therein).

In this paper, we aim to study the following boundary value problem of fractional differential equations with integral boundary conditions:

$$\begin{cases} {}^cD^{\alpha}u(t)+\eta f(t,u(t))=0 & 0<t<1, \\ u(0)=u''(0)=0, & u(1)=\beta\int_0^1 u(s)ds, \end{cases} \tag{1}$$

where $2<\alpha<3$, $0<\beta<2$, ${}^cD^{\alpha}$ is the Caputo fractional derivative, $\eta>0$ is a given constant, and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous function. We make the following assumptions:

(H1) $\exists(\overline{r},\overline{R})\in\mathbb{R}^2$ satisfying $0<\overline{r}<\overline{R}$, and there exist the functions $a_0,a^0,b_\infty,b^\infty\in C(J,\mathbb{R}^+)$ with $a_0(.),a^0(.),b_\infty(.),b^\infty(.)\not\equiv 0$ in any subinterval of $J:=[0,1]$ and functions $\zeta_1,\xi_1,\zeta_2,\xi_2\in C(J,\mathbb{R}^+)$ such that

$$a_0(t)(v-\xi_1(t,v))\le f(t,v)\le a^0(t)(v+\xi_2(t,v)) \quad \forall(t,v)\in J\times[0,\overline{r}],$$

and

$$b_\infty(t)(v-\zeta_1(t,v))\le f(t,v)\le b^\infty(t)(v+\zeta_2(t,v)) \quad \forall(t,v)\in J\times[\overline{R},+\infty),$$

where for $i\in\{1,2\}$ and uniformly with respect t we have

$$\xi_i(t,v)=o(v) \text{ as } v\to 0 \text{ and } \zeta_i(t,v)=o(v) \text{ as } v\to+\infty.$$

We use a bifurcation techniques and topology degree theory under the condition (H1) and other conditions to investigate the problem (1) and to obtain the existence of positive solutions.
Some special cases of (1) have been investigated. For example, Cabada and Wang [3] studied the existence of positive solutions of (1) with $\eta=1$, by using Guo-Krasnoselskii's fixed point theorem, and they proved the following:

Theorem 1. *Assume that one of the two following conditions is fulfilled:*

(i) *(Sublinear case)* $f_0=\infty$ *and* $f_\infty=0$.

(ii) *(Superlinear case)* $f_0 = 0$, $f_\infty = \infty$ *and there exist* $\mu > 0$ *and* $\theta > 0$ *for which* $f(t, \kappa x) \geq \mu \kappa^\theta f(t,x)$ *for all* $\kappa \in (0,1]$.

Then, the problem (1) *with* $\eta = 1$ *has at least one solution that belongs to the cone*

$$P = \{u \in C([0,1]) : u(t) \geq \frac{t\beta(\alpha-2)}{2\alpha}\|u\|, \quad \forall t,s \in [0,1]\}.$$

Where $f_0 := \lim_{u\to 0^+}\{\min_{t\in[0,1]} \frac{f(t,u)}{u}\}$ *and* $f_\infty := \lim_{u\to\infty}\{\max_{t\in[0,1]} \frac{f(t,u)}{u}\}$ *both uniformly with respect to* $t \in [0,1]$.

Remark 1. *Obviously, the condition (H1) means that the nonlinearity* $f(.,.)$ *is asymptotically linear at 0 and* ∞*, not necessarily linearizable or super-linear or sub-linear, then the conditions of Theorem* (1) *are differents then the condition (H1) and the method used in* [3] *is not helpful any more in this case.*

Remark 2. *Unfortunately, there have been a few papers studying such fractional differential equations using bifurcation ideas and to the best of our knowledge, there is no paper studying such fractional differential equations with integral boundary conditions using bifurcation techniques. The purpose of present paper is to fill this gap and the main method used here is bifurcation techniques and topological degree, not fixed point theorem on cone, upper and lower solutions technique, which is different from the references.*

The bifurcation technique was firstly proposed by Rabinowitz and then was extensively applied to the study of positive solutions for BVPs of integer order differential equations [11],[12]. For fractional differential equations, Liu and Yu [9] applied the bifurcation technique to the existence of positive solutions for a class of BVPs of fractional differential inclusions.

The rest of this paper is organized as follows. Section 2 contains some preliminary results. Section 3 presents the main results of this paper. In Section 4, two illustrative examples are worked out to support our obtained new results. In some sense, the proves given in this work follow similar steps to the ones obtained in [1], [8], [19].

2. Background Materials and Preliminaries

Firstly, we recall some well known results about fractional calculus. For details, please refer to [2], [10], [17] and references therein.

Definition 1. *For a function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as*

$$ {}^cD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n = [\alpha]+1, $$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2. *The Riemann-Liouville fractional integral of order α for a function f is defined as*

$$ I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, $$

provided that such integral exists.

Definition 3. *The Riemann-Liouville fractional derivative of order α for a function f is defined by*

$$ D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \quad n = [\alpha]+1, $$

provided that the right-hand side of the previous equation is pointwise defined on $(0,\infty)$.

Lemma 1. *Let $\alpha > 0$, then the fractional differential equation*

$$ {}^cD^{\alpha} u(t) = 0 $$

has a unique solution given by the expression

$$ u(t) = \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{j!} t^j. $$

Lemma 2. *Let $\alpha > 0$, then*

$$ I^{\alpha c}D^{\alpha} u(t) = u(t) - \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{j!} t^j. $$

Secondly, we recall some useful properties of Green's functions for BVP (1), which were proved in [3]. Throughout this paper, the Banach space is $C([0,1])$ with the norm $\|v\| = \max_{t\in[0,1]} |v(t)|, \quad \forall v \in C([0,1])$.

To solve BVP (1), we first consider the following linear boundary problem of fractional differential equation:

$$\begin{cases} {}^{c}D^{\alpha}u(t)+g(t)=0 \quad 0<t<1, \\ u(0)=u''(0)=0, \quad u(1)=\beta\int_0^1 u(s)ds, \end{cases} \tag{2}$$

where $g \in C([0,1])$ we cite the following two lemmas from reference [3].

Lemma 3. *(see*[3]*). Given $g \in C([0,1])$, then $u(t)=\int_0^1 G(t,s)g(s)ds$ is a solution of* (2)*, where*

$$G(t,s)=\begin{cases} \frac{2t(1-s)^{\alpha-1}(\alpha-\beta+\beta s)-(2-\beta)\alpha(t-s)^{\alpha-1}}{(2-\beta)\Gamma(\alpha+1)}, & 0\le s\le t\le 1, \\ \frac{2t(1-s)^{\alpha-1}(\alpha-\beta+\beta s)}{(2-\beta)\Gamma(\alpha+1)}, & 0\le t\le s\le 1. \end{cases} \tag{3}$$

Lemma 4. *(see*[3]*). Let $2<\alpha<3$ and $0<\beta<2$. For all $t,s\in[0,1]$, the function $G(t,s)$ defined by* (3) *has the following properties:*

$$\underbrace{\frac{t\beta(1-s)^{\alpha-1}(\alpha-2+2s)}{(2-\beta)\Gamma(\alpha+1)}}_{=tG(1,s)} \le G(t,s) \le \underbrace{\frac{2\alpha(1-s)^{\alpha-1}(\alpha-2+2s)}{(\alpha-2)(2-\beta)\Gamma(\alpha+1)}}_{=\frac{2\alpha}{\beta(\alpha-2)}G(1,s)}, \tag{4}$$

and

$$G(t,s)>0 \textit{ for all } t,s\in(0,1) \textit{ if and only if } \beta\in[0,2). \tag{5}$$

Thirdly, we list the following results on the topological degree of completely continuous operators.

Theorem 2. *(Schmitt and Thompson* [14]*. Let E be a real reflexive Banach space, $G: \mathbb{R}\times E\to E$ be completely continuous such that $G(\lambda,0)=0, \quad \forall\lambda\in\mathbb{R}$, and $a,b\in\mathbb{R}$ $(a<b)$ be such that $u=0$ is an isolated solution of the equation*

$$u-G(\lambda,u)=0, \quad u\in E, \tag{6}$$

for $\lambda=a$ and $\lambda=b$, where $(a,0),(b,0)$ are not bifurcation points of (6)*. Furthermore, assume that*

$$deg(I-G(a,.),B_r(0),0)\neq deg(I-G(b,.),B_r(0),0),$$

where $B_r(0)$ is an isolating neighborhood of the trivial solution.

Let

$$\Sigma = \overline{\{(\lambda,u) : (\lambda,u) \text{ is a solution of } (6) \text{ with } u \neq 0\}} \cup ([a,b]\times\{0\}).$$

Then there exists a connected component $\mathscr{C}$ (i.e. maximal closed connected subset) of Σ containing $[a,b]\times\{0\}$ in $\mathbb{R}\times E$ and either

- (i) *$\mathscr{C}$ is unbounded in $\mathbb{R}\times E$ or*
- (ii) *$\mathscr{C}\cap[(\mathbb{R}\setminus[a,b])\times\{0\}]\neq\emptyset$.*

Theorem 3. *(Schmitt [13]). Let E be a real reflexive Banach space. Let $G:\mathbb{R}\times E\to E$ be completely continuous, and let $a,b\in\mathbb{R}$ $(a<b)$ be such that the solution of (6) is, a priori, bounded in E for $\lambda=a$ and $\lambda=b$; that is, there exists an $R>0$ such that*

$$G(a,u)\neq u\neq G(b,u) \tag{7}$$

for all u with $\|u\|\geq R$. Furthermore, assume that

$$deg(I-G(a,.),B_R(0),0)\neq deg(I-G(b,.),B_R(0),0),$$

for sufficiently large $R>0$. Then there exists a closed connected set $\mathscr{C}$ of solutions to (6) that is unbounded in $[a,b]\times E$, and either

- (i) *$\mathscr{C}$ is unbounded in λ direction or*
- (ii) *there exists an interval $[c,d]$ such that $(a,b)\cap(c,d)=\emptyset$ and $\mathscr{C}$ bifurcates from infinity in $[c,d])\times E$.*

Lemma 5. *(Guo [15]). Let Ω be a bounded open set of real Banach space E, let θ denotes the zero element of E and let $G:\overline{\Omega}\to E$ be completely continuous. If there exists $y_0\in E$, $y_0\neq\theta$ such that*

$$x\in\partial\Omega,\quad \tau\geq 0\Rightarrow x-Gx\neq\tau y_0.$$

Then

$$deg(I-G,\Omega,\theta)=0.$$

Finally, in order to prove the main results, we choose the basic space

$$E := \{v \in C(J,\mathbb{R}^+) : v(0) = v''(0) = 0, \quad v(1) = \beta \int_0^1 v(s)ds\}.$$

Clearly, E is a Banach space with norm $\|v\| = \max_{t\in J} |v(t)|$ (for all $v \in E$).

Let

$$Q := \{v \in E : v(t) \geq \frac{t\beta}{2\alpha}(\alpha - 2)v(s) \geq 0, \quad \forall t, s \in J\}. \tag{8}$$

One can easily see that Q is a cone of E. In addition, from (8) we have

$$v(t) \geq \frac{t\beta}{2\alpha}(\alpha - 2)||v||, \quad \forall v \in Q, \quad \forall t \in J. \tag{9}$$

In order to use the bifurcation technique to investigate BVP (1), we deal with the following fractional boundary value problem with the parameter λ:

$$\begin{cases} {}^cD^{\alpha}u(t) + \lambda f(t,u(t)) = 0 & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \beta \int_0^1 u(s)ds. \end{cases} \tag{10}$$

We note that (λ,u) is said to be a solution of fractional BVP (10) if it satisfies (10). In addition, if $\lambda > 0$ and $u(t) > 0$ for $t \in (0,1)$, then (λ,u) is said to be a positive solution of BVP (10). It becomes apparent that if $\lambda > 0$ and $u \in Q\setminus\{\theta\}$ such that (λ,u) is a solution of fractional BVP (10), then by (9) we know that (λ,u) is a positive solution of BVP(10), where θ denotes the zero element of E. Define

$$\overline{f}(t,v) = \begin{cases} f(t,v), & (t,v) \in J \times \mathbb{R}^+, \\ f(t,0), & (t,v) \in J \times (-\infty,0). \end{cases}$$

Then $\overline{f}(t,v) \geq 0$ on $J \times \mathbb{R}$.

Lemma 6. *Set $A_\lambda : Q \to C([0,1])$ such that*

$$A_\lambda v(t) := \lambda \int_0^1 G(t,s)\overline{f}(s,v(s))ds. \tag{11}$$

Assume that (H1) is satisfied, then $A_\lambda : Q \to Q$ is completely continuous.

Proof. By assumption (H1) and using a similar process of the proof of Lemma 4.1 in [7] , we know $A_\lambda : Q \to Q$ is completely continuous. □

Let

$$\Sigma := \overline{\{(\lambda, v) \in \mathbb{R}^+ \times C([0,1]) : v = A_\lambda v, v \neq \theta\}}, \tag{12}$$

where θ is the zero element of $C(J, \mathbb{R}^+)$. From Lemma (6) and the definitions of $\overline{f}$ and the cone Q, one can see that $(\lambda, u) \in \Sigma \Rightarrow u \in Q$. Moreover, we have the following conclusion.

Lemma 7. *From Lemma* (3)*, if $u \in E$ is a fixed point of the operator A_λ, then (λ, u) is a solution of*

$$\begin{cases} {}^cD^\alpha v(t) + \lambda \overline{f}(t, v(t)) = 0 & t \in (0,1), \\ v(0) = v''(0) = 0, & v(1) = \beta \int_0^1 v(s)ds. \end{cases} \tag{13}$$

Furthermore, (λ, u) is a positive solution of BVP (13) *if and only if (λ, u) is a positive solution of BVP* (10).

For $a \in C(J, \mathbb{R}^+)$ with $a(t) \not\equiv 0$ in any subinterval of J, define the linear operator $L_a : C(J) \to C(J)$ by

$$L_a v(t) = \int_0^1 G(t,s)a(s)v(s)ds. \tag{14}$$

where $G(t,s)$ is defined by Lemma (3). From Lemma (3) and (4) and the well-known Krein-Rutman Theorem, one can obtain the following lemma.

Lemma 8. *The operator L_a defined by* (14) *is completely continuous and has a unique characteristic value $\lambda_1(a)$, which is positive, real and simple, and the corresponding eigenfunction $\phi(t)$ is of one sign in $(0,1)$, that is,*

$$\phi(t) = \lambda_1(a) L_a \phi(t), \textit{ for all } t \in J.$$

One can show that the operator L_a can be regarded as $L_a : L^2[0,1] \to L^2[0,1]$. This together with Lemma (8) guarantees that $\lambda_1(a)$ is also the characteristic value of L_a^* , where L_a^* is the conjugate operator of L_a. Denote φ^* by the nonnegative eigenfunction of L_a^* corresponding to $\lambda_1(a)$. Then, we have

$$\varphi^*(t) = \lambda_1(a) L_a^* \varphi^*(t), \quad \forall t \in J.$$

3. MAIN RESULTS

The main results of present paper are the following two theorems.

Theorem 4. *Suppose that (H1) holds and suppose either*

(i) $\lambda_1(a_0) < \eta < \lambda_1(b^\infty)$ *or*

(ii) $\lambda_1(b_\infty) < \eta < \lambda_1(a^0)$

holds. Then BVP (1) *has at least one positive solution.*

Theorem 5. *Suppose the following.*

(H2) *There exist* $R > 0$ *and* $h \in L(J, \mathbb{R}^+)$ *such that*

$$\forall t \in J, \forall v \in [0,R],\ f(t,v) \leq h(t)v \text{ and } \max\{\lambda_1(a_0), \lambda_1(b_\infty)\} < \eta < \frac{(\alpha-2)(2-\beta)\Gamma(\alpha+1)}{6\int_0^1 (2s+1)(1-s)^{\alpha-1} h(s)ds}.$$

Then BVP (1) *has at least two positive solutions.*

To prove Theorems (4) and (5), we first prove the following lemmas.

Lemma 9. *Suppose that (H1) holds. Let* $\lambda_1(a_0)$ *and* $\lambda_1(a^0)$ *be the unique characteristic value of* L_{a_0}, L_{a^0}*, respectively. Then we have*

(i) *Let* $[c,d] \subset \mathbb{R}^+$ *be a compact interval with* $[\lambda_1(a^0), \lambda_1(a_0)] \cap [c,d] = \emptyset$*. Then there exists* $\delta_1 \in (0, \bar{r})$ *such that*

$$v \neq A_\lambda v, \quad \forall \lambda \in [c,d], \quad \forall v \in E \text{ with } 0 < \|v\| \leq \delta_1;$$

(ii) *For* $\mu \in (0, \lambda_1(a^0))$*, there exists* $\delta_1 \in (0, \bar{r})$ *such that*

$$deg(I - A_\mu, B_\delta, 0) = 1, \quad \forall \delta \in (0, \delta_1];$$

(iii) *For* $\lambda > \lambda_1(a_0)$*, there exists* $\delta_2 \in (0, \bar{r})$ *such that*

$$deg(I - A_\lambda, B_\delta, 0) = 0, \quad \forall \delta \in (0, \delta_2].$$

Proof. **•:** Firstly, we prove conclusion (i). If this is not true, then

(15) $\exists \{(\mu_n, v_n)\} \subset [c,d] \times C([0,1]) : \quad \|v_n\| \to 0^+ (n \to +\infty)$ and $v_n = A_{\mu_n} v_n$.

Without loss of generality, assume that $\mu_n \to \mu \in [c,d]$ and $\|v_n\| < \overline{r}$ for all n. Notice that $v_n \in Q$. From (8), we know that $v_n(t) > 0$ in $(0,1)$. Set $w_n = \frac{v_n}{\|v_n\|}$. Then $w_n = \frac{A_{\mu_n} v_n}{\|v_n\|}$. From the definition of $\overline{f}(t,u)$, it is easy to obtain that $\{w_n\}$ is relatively compact in $C([0,1])$. Taking a subsequence and relabeling if necessary, then $w_n \to w$ in $C([0,1])$, $\|w\| = 1$ and $w \in Q$.

On the other hand, by (15) and the first part of (H1) we have

$$a_0(s)(v_n(s) - \xi_1(s, v_n(s))) \le \overline{f}(s, v_n(s)) \le a^0(s)(v_n(s) + \xi_2(s, v_n(s))), \quad \forall s \in J. \tag{16}$$

Therefore, from (5) and (11) we know

$$w_n(t) \le \mu_n \int_0^1 G(t,s) a^0(s)[w_n(s) + \frac{\xi_2(s, v_n(s))}{\|v_n\|}] ds \tag{17}$$

and

$$w_n(t) \ge \mu_n \int_0^1 G(t,s) a_0(s)[w_n(s) - \frac{\xi_1(s, v_n(s))}{\|v_n\|}] ds. \tag{18}$$

Let ψ^* and ψ_* be the positive eigenfunctions of $L^*_{a^0}, L^*_{a_0}$ corresponding to $\lambda_1(a^0)$ and $\lambda_1(a_0)$, respectively. Then from (17), it follows that

$$\langle w_n, \psi^* \rangle \le \mu_n \langle L_{a^0} w_n, \psi^* \rangle + \mu_n \int_0^1 \psi^*(t) [\int_0^1 G(t,s) a^0(s) \frac{\xi_2(s, v_n(s))}{\|v_n\|} ds] dt.$$

Letting $n \to +\infty$ and using condition (H1) again, we have

$$\langle w, \psi^* \rangle \le \mu \langle L_{a^0} w, \psi^* \rangle = \mu \langle w, L^*_{a^0} \psi^* \rangle = \mu \langle w, \frac{\psi^*}{\lambda_1(a^0)} \rangle,$$

which implies $\mu \ge \lambda_1(a^0)$. Similarly, one can obtain from (18) that $\mu \le \lambda_1(a_0)$. Summarizing the above, $\lambda_1(a^0) \le \mu \le \lambda_1(a_0)$, which is a contradiction with $\mu \in [c,d]$. Therefore, there exists $\delta_1 \in (0, \overline{r})$ such that

$$v \ne A_\lambda v, \quad \forall \lambda \in [c,d], \quad \forall v \in E \text{ with } 0 < \|v\| \le \delta_1.$$

- **:** Secondly, we prove conclusion (ii). Notice that $[0,\mu] \cap [\lambda_1(a^0), \lambda_1(a_0)] = \emptyset$. From (i), there exists $\delta_1 \in (0, \overline{r})$ such that

$$v \ne A_\lambda v, \quad \forall \lambda \in [0,\mu], \quad \forall v \in C([0,1]), \text{ with } 0 < \|v\| \le \delta_1.$$

Let $\tau = \frac{\lambda}{\mu}$, which means

$$v \neq \tau A_{\mu} v, \quad \forall \tau \in [0,1], \quad \forall v \in C([0,1]), \text{ with } 0 < \|v\| \leq \delta_1.$$

One can obtain from the homotopy invariance of topological degree that

$$\deg(I - A_{\mu}, B_{\delta}, 0) = \deg(I, B_{\delta}, 0) = 1, \quad \forall \delta \in (0, \delta_1].$$

- **•:** Finally, we prove conclusion (iii). From the Lemma (5), one needs to prove that for $\lambda > \lambda_1(a_0)$, there exists $\delta_2 \in (0, \bar{r})$ such that

$$v - A_{\lambda} v \neq \tau \varphi_0, \quad \forall \tau \geq 0, \quad \forall v \in C([0,1]), \text{ with } \|v\| = \delta_2, \tag{19}$$

where φ_0 is the positive eigenfunction of L_{a_0} corresponding to $\lambda_1(a_0)$. Suppose on the contrary that there exist $v_n \in C([0,1])$ with $\|v_n\| \to 0 \quad (n \to +\infty)$ and $\tau_n \geq 0$ such that $v_n - A_{\lambda} v_n = \tau_n \varphi_0$.

Set $w_n = \frac{v_n}{\|v_n\|}$. Then

$$w_n = \frac{A_{\lambda} v_n}{\|v_n\|} + \frac{\tau_n}{\|v_n\|} \varphi_0. \tag{20}$$

By virtue of $A_{\lambda} v_n \in Q$, we know $w_n \geq \frac{\tau_n}{\|v_n\|} \varphi_0$. As a result, $\frac{\tau_n}{\|v_n\|}$ is bounded. On the other hand, from (11), condition (H1), and Ascoli-Arzela theorem, it is easy to see that $\{\frac{A_{\lambda} v_n}{\|v_n\|}\}$ is relatively compact. This together with (20) guarantees that $\{w_n\}$ is also relatively compact. No loss of generality, suppose $w_n \to w$ as $n \to +\infty$.

Consequently, it follows from (11) and (20) that

$$w_n(t) \geq \lambda \int_0^1 G(t,s) a_0(s) [w_n(s) - \frac{\xi_1(s, v_n(s))}{\|v_n\|}] ds. \tag{21}$$

Also let ψ_* be the positive eigenfunction of $L_{a_0}^*$ corresponding to $\lambda_1(a_0)$. Then by (21), we know

$$\begin{aligned} \langle w_n, \psi_* \rangle &\geq \lambda \langle L_{a_0} w_n, \psi_* \rangle - \lambda \int_0^1 \psi_*(t) [\int_0^1 G(t,s) a_0(s) \frac{\xi_1(s, v_n(s))}{\|v_n\|} ds] dt \\ &= \lambda \langle w_n, L_{a_0}^* \psi_* \rangle - \lambda \int_0^1 \psi_*(t) [\int_0^1 G(t,s) a_0(s) \frac{\xi_1(s, v_n(s))}{\|v_n\|} ds] dt. \end{aligned}$$

Similar as in the proof of conclusion (i), we have $\frac{\xi_1(s,v_n(s))}{\|v_n\|} \to 0$ as $n \to +\infty$ uniformly with respect to $s \in (0,1)$, so we obtain that

$$\langle w, \psi_* \rangle \geq \lambda \langle w, L_{a_0}^* \psi_* \rangle = \lambda \langle w, \frac{\psi_*}{\lambda(a_0)} \rangle.$$

This means $\lambda \leq \lambda_1(a_0)$, which is a contradiction. Consequently, (19) holds. By virtue of Lemma (5), for each $\lambda > \lambda_1(a_0)$, there exists $\delta_2 > 0$ such that

$$\deg(I - A_\lambda, B_\delta, 0) = 0, \quad \forall \delta \in (0, \delta_2].$$

□

Theorem 6. $[\lambda_1(a^0), \lambda_1(a_0)]$ *is a bifurcation interval of positive solutions from the trivial solution for BVP* (10)*; that is, there exists an unbounded component* $\mathscr{C}_0$ *of positive solutions of BVP* (10)*, which meets* $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$*. Moreover, there exists no bifurcation interval of positive solutions from the trivial solution which is disjointed with* $[\lambda_1(a^0), \lambda_1(a_0)]$.

Proof. By virtue of (12) and Lemma (7), we need only to prove that there exists an unbounded component $\mathscr{C}_0$ of Σ, which meets $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$, and there exists no bifurcation interval of Σ from the trivial solution which is disjointed with $[\lambda_1(a^0), \lambda_1(a_0)]$. For fixed $n \in \mathbb{N}$ with $\lambda_1(a^0) - \frac{1}{n} > 0$, by the assertions (ii) and (iii) of Lemma (9) and their proof, there exists $r > 0$ such that such that all of the conditions of Theorem (2) are satisfied with

$$G(\lambda, u) = A_\lambda u, \quad a = \lambda_1(a^0) - \frac{1}{n}, \text{ and } b = \lambda_1(a_0) + \frac{1}{n}.$$

This together with Lemma (7) guarantees that there exists a closed connected set $\mathscr{C}_n$ of Σ containing $[\lambda_1(a^0) - \frac{1}{n}, \lambda_1(a_0) + \frac{1}{n}] \times \{0\}$ in $\mathbb{R}^+ \times C([0,1])$.

From the conclusion (i) of Lemma (9), the case (ii) of Theorem (2) cannot occur. Thus, $\mathscr{C}_n$ bifurcates from $[\lambda_1(a^0) - \frac{1}{n}, \lambda_1(a_0) + \frac{1}{n}] \times \{0\}$ and is unbounded in $\mathbb{R}^+ \times C([0,1])$. In addition, for any closed interval $[c,d] \subset [\lambda_1(a^0) - \frac{1}{n}, \lambda_1(a_0) + \frac{1}{n}] \backslash [\lambda_1(a^0), \lambda_1(a_0)]$, by the conclusion (i) of Lemma (9), there exists $\delta_1 > 0$ such that

$$\{v \in C([0,1]) : (\lambda, v) \in \Sigma, 0 < \|v\| < \delta_1, \lambda \in [c,d]\} = \emptyset.$$

Therefore, $\mathscr{C}_n$ must be bifurcated from $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$, which implies that $\mathscr{C}_n$ can be regarded as $\mathscr{C}_0$. In addition, using the conclusion (i) of Lemma (9) again, there exists no bifurcation interval of positive solutions from the trivial solution which is disjointed with $[\lambda_1(a^0), \lambda_1(a_0)]$. □

By a process similar to the above, one can obtain the following conclusions.

Lemma 10. *Let $\lambda_1(b_\infty)$ and $\lambda_1(b^\infty)$ be the unique characteristic value of $L_{b_\infty}, L_{b^\infty}$, respectively. Suppose that (H1) holds. Then we have*

(i) Let $[c,d] \subset \mathbb{R}^+$ be a compact interval with $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \cap [c,d] = \emptyset$. Then there exists $R_1 > \overline{R}$ such that

$$u \neq A_\lambda u, \quad \forall \lambda \in [c,d], \quad \forall u \in C([0,1]) \text{ with } \|u\| \geq R_1;$$

(ii) For $\mu \in (0, \lambda_1(b^\infty))$, there exists $R_1 > \overline{R}$ such that

$$deg(I - A_\mu, B_R, 0) = 1, \quad \forall R \geq R_1;$$

(iii) For $\lambda > \lambda_1(b_\infty)$, there exists $R_2 > \overline{R}$ such that

$$deg(I - A_\lambda, B_R, 0) = 0, \quad \forall R \geq R_2.$$

Theorem 7. $[\lambda_1(b^\infty), \lambda_1(b_\infty)]$ *is a bifurcation interval of positive solutions from infinity for BVP* (10)*, and there exists no bifurcation interval of positive solutions from infinity which is disjointed with* $[\lambda_1(b^\infty), \lambda_1(b_\infty)]$*. More precisely, there exists an unbounded component $\mathscr{C}_\infty$ of solutions to BVP* (10)*, which meets $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \infty$ and is unbounded in λ direction.*

Now we are in position to prove Theorems (4) and (5).

3.1. Proof of Theorem (4).

Certainly the solution of the form $(\eta, u) \quad (u \neq \theta)$ of (10) is a positive solution of BVP (1). So by Lemma (7), it is sufficient to show that there is a component $\mathscr{C}$ of Σ that crosses the hyperplane $\{\eta\} \times C([0,1])$, where $\Sigma \subset \mathbb{R}^+ \times C([0,1])$ is defined by (12).

Case (i). $\lambda_1(a_0) < \eta < \lambda_1(b^\infty)$

From Theorem (6) there exists an unbounded component $\mathscr{C}_0$ of positive solutions to BVP (10), which bifurcates from $[\lambda_1(a^0), \lambda_1(a_0)] \times \{\theta\}$. Therefore, there exists $(\mu_n, v_n) \in \mathscr{C}_0$ such that

$$\mu_n + \|v_n\| \to +\infty \qquad (n \to +\infty). \tag{22}$$

If there exists $n \in \mathbb{N}$ such that $\mu_n \geq \eta$, the result follows. If this is false, we have $\mu_n < \eta$ for all $n \in \mathbb{N}$. Notice that $(0,\theta)$ is the only solution of (10) with $\lambda = 0$. By the conclusion (i) of Lemma (9) and the conclusion (i) of Lemma (10), we have $\mathscr{C}_0 \cap (\{0\} \times C([0,1]) = \emptyset$. Therefore, $\mu_n \in (0,\eta)$ for all $n \in \mathbb{N}$.

Taking a subsequence and relabeling if necessary, suppose $\mu_n \to \mu^*$ as $(n \to +\infty)$. Then $\mu^* \in [0,\eta]$. This together with (22) guarantees that $\|v_n\| \to +\infty$. Choose $[c,d] = [0,\lambda_1(b^\infty) - \frac{1}{m}]$ for $m \in \mathbb{N}$. By the conclusion (i) of Lemma (10), it follows that $\mu^* > \lambda_1(b^\infty) - \frac{1}{m}$ for each $m \in \mathbb{N}$, which means $\eta < \lambda_1(b^\infty) \leq \mu^*$.This is a contradiction. Then BVP (1) has a positive solution u with $(\eta,u) \in \mathscr{C}_0$.

Case (ii)**.** $\lambda_1(b_\infty) < \eta < \lambda_1(a^0)$

By Theorem (7), there exists an unbounded component $\mathscr{C}_\infty$ of solutions to (10) which bifurcates from $[\lambda_1(b^\infty),\lambda_1(b_\infty)] \times \infty$, and is unbounded in λ direction. If $\mathscr{C}_\infty \cap (\mathbb{R}^+ \times \{0\}) = \emptyset$, using the fact that $\mathscr{C}_\infty \cap (\{0\} \times C([0,1]) = \emptyset$ and $\mathscr{C}_\infty$ is unbounded in λ direction, we know that $\mathscr{C}_\infty$ must cross the hyperplane $\{\eta\} \times C([0,1])$. If $\mathscr{C}_\infty \cap (\mathbb{R}^+ \times \{0\}) \neq \phi$, from Theorem (6) and $\mathscr{C}_\infty \cap (\{0\} \times C([0,1]) = \emptyset$, we have $\mathscr{C}_\infty \cap (\mathbb{R}^+ \times \{0\}) \in [\lambda_1(a^0),\lambda_1(a_0)] \times \{0\}$. So $\mathscr{C}_\infty$ joins $[\lambda_1(a^0),\lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(b^\infty),\lambda_1(b_\infty)] \times \infty$. This together with $\lambda_1(b_\infty) < \eta < \lambda_1(a^0)$ guarantees that $\mathscr{C}_\infty$ crosses the hyperplane $\{\eta\} \times C([0,1])$. Then BVP (1) has a positive solution u with $(\eta,u) \in \mathscr{C}_\infty$.

3.2. **Proof of Theorem** (5)**.**

First we show that there exists $\varepsilon > 0$ such that

$$\Sigma \cap ([0,\eta+\varepsilon] \times \partial B_R) = \emptyset, \tag{23}$$

where $B_R = \{v \in C([0,1]) : \|v\| < R\}$, and $\Sigma \subset \mathbb{R}^+ \times C([0,1])$ is defined by (12). In fact, from assumption (H2), it follows that there exists $\varepsilon > 0$ such that

$$\frac{6(\eta+\varepsilon)}{(\alpha-2)(2-\beta)\Gamma(\alpha+1)} \int_0^1 (2s+1)(1-s)^{\alpha-1} h(s)ds < 1.$$

If there is a solution (λ,u) of $u = A_\lambda v$, such that $0 \leq \lambda \leq \eta+\varepsilon$ and $\|v\| = R$, then

$$0 \leq u(t) \leq \|v\| = R \text{ for } t \in J.$$

By virtue of (11) and Lemma (4), we have

$$\begin{aligned} R = \|v\| &= \max_{t\in J} \lambda \int_0^1 G(t,s)\overline{f}(s,u(s))ds \\ &\leq (\eta+\varepsilon)R \max_{t\in J} \int_0^1 G(t,s)h(s)ds \\ &\leq \frac{6(\eta+\varepsilon)R}{(\alpha-2)(2-\beta)\Gamma(\alpha+1)} \int_0^1 (2s+1)(1-s)^{\alpha-1}h(s)ds \\ &< R. \end{aligned}$$

which is a contradiction. Thus, $\Sigma \cap ([0,\eta+\varepsilon]\times \partial B_R) = \emptyset$.

Next, from Theorem (6), there exists an unbounded components $\mathscr{C}_0$ of solutions to (10), which meet $[\lambda_1(a^0),\lambda_1(a_0)]\times\{0\}$. By (23) we know $\mathscr{C}_0 \cap ([0,+\varepsilon]\times \partial B_R) = \emptyset$. This together with the fact that $\mathscr{C}_0$ is unbounded, $\lambda_1(a_0) < \eta$, and $\mathscr{C}_0 \cap (\{0\}\times C([0,1])) = \emptyset$ guarantees that $\mathscr{C}_0$ crosses the hyperplane $\{\eta\}\times C([0,1])$. Then BVP (1) has a positive solution u_1 with

$$(\eta,u_1)\in \mathscr{C}_0 \text{ and } \|u_1\| < R.$$

Similarly, by Theorem (7) and (23), the BVP (1) has a positive solution $u_2(t)$ with

$$(\eta,u_2)\in \mathscr{C}_\infty \text{ and } \|u_2\| > R.$$

Consequently, BVP (1) has at least two positive solutions.

Immediately, from the proof of Theorem (5), we have the following Corollary.

Corollary 1. *Suppose that assumption (H2) holds. In addition, suppose that one of the following two conditions holds:*

(i) $\lambda_1(a_0) < \eta$; *or*

(ii) $\lambda_1(b_\infty) < \eta$.

Then BVP (1) *has at least one positive solution.*

Remark 3. *Corollary* (1) *is different from Theorem* (5) *though their results are similar.*

4. Examples

Example 1. *Let ρ be the unique characteristic value of L corresponding to positive eigenfunction with $a(t) \equiv \frac{3}{2}$ in (2.11). From Lemma 2.10 it follows that ρ exists and we have*

$$\exists \phi : \phi(t) = \rho \int_0^1 G(t,s)\frac{3}{2}\phi(s)ds, \text{ for all } t \in J.$$

We consider the following BVP

$$\begin{cases} {}^cD^{\frac{5}{2}}u(t) + \eta f(t,u(t)) = 0 & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \beta \int_0^1 u(s)ds, \end{cases} \tag{24}$$

where $\beta \in (0,2)$ and

$$f(t,u) = \begin{cases} \rho(u + tu^2), & t \in J, u \in [0,1], \\ \rho(4u - 3 + t), & t \in J, u \in [1,3], \\ \rho(3u + t\sqrt{\frac{u}{3}}), & t \in J, u \in [3,+\infty). \end{cases} \tag{25}$$

Then BVP (24) *has at least one positive solution.*

Proof. There is no doubt that $f_0, f_\infty \notin \{0,\infty\}$. Consequently, from theorem (1) we cannot affirm that BVP (24) with $(\eta,\beta) = (1,1)$ has a positive solution. Otherwise, BVP (24) can be regarded as the form (1) and from (25), one can see that the condition (H1) of theorem (4) is satisfied with $a_0(t) = a^0(t) = \rho$, $b_\infty(t) = b^\infty(t) = 3\rho$ and $\xi_1(t,u) = \xi_2(t,u) = tu^2, \zeta_1(t,u) = \zeta_2(t,u) = \frac{t}{3}\sqrt{\frac{u}{3}}$. By the definition of ρ, it is easy to see that $\frac{1}{2} = \lambda_1(b_\infty) < 1 < \lambda_1(a^0) = \frac{3}{2}$.
Therefore, BVP (24) has at least one positive solution as $\eta \in (\frac{1}{2}, \frac{3}{2}), \beta \in (0,2)$. □

Example 2. *Let ρ be the unique characteristic value of L corresponding to positive eigenfunction with $a(t) \equiv \frac{1}{2}$ in (2.11). From Lemma 2.10 it follows that ρ exists and we have*

$$\exists \phi : \phi(t) = \rho \int_0^1 G(t,s)\frac{1}{2}\phi(s)ds, \text{ for all } t \in J.$$

We consider the following BVP

$$\begin{cases} {}^cD^{\frac{5}{2}}u(t) + \eta f(t,u(t)) = 0 & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \beta \int_0^1 u(s)ds, \end{cases} \tag{26}$$

where $\beta \in (0,2)$ *and*

$$f(t,u) = \rho u(3 + \sin(tu) + \sin(\frac{t}{u}))$$

Then BVP (26) *has at least two positive solutions.*

Proof. BVP (26) can be regarded as the form (1). Choose $a_0(t) = b_\infty(t) = 2\rho$, $a^0(t) = b^\infty(t) = 4\rho$ and $\xi_1(t,u) = -\frac{1}{2}u\sin(tu)$, $\xi_2(t,u) = \frac{1}{4}u\sin(tu)$, $\zeta_1(t,u) = -\frac{1}{2}u\sin(\frac{t}{u})$, $\zeta_2 = \frac{1}{4}u\sin(\frac{t}{u})$. It is easy to see $\xi_i(t,u) = o(|u|)$ as $|u| \to 0$, and $\xi_i(t,u) = o(|u|)$ as $|u| \to +\infty$ both uniformly with respect to $t \in J$ $(i = 1,2)$. Therefore, (H1) is satisfied.

Choose $h(t) = 5\rho$, thus $f(t,u) \leq h(t)u$.

By the definition of ρ, it is easy to see that

$$\underbrace{\max\{\lambda_1(a_0), \lambda_1(b_\infty)\}}_{=\frac{1}{4}} < \underbrace{\eta}_{=1} < \underbrace{\frac{(\alpha-2)(2-\beta)\Gamma(\alpha+1)}{6\int_0^1(2s+1)(1-s)^{\alpha-1}h(s)ds}}_{=\frac{3\sqrt{\pi}}{8\rho}}$$

As a result, by Theorem (5), BVP (26) has at least two positive solutions. Otherwise, from theorem (1) we cannot affirm that BVP (24) with $(\eta,\beta) = (1,1)$ has a positive solution because it is clear that $f_0, f_\infty \notin \{0,\infty\}$.

□

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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