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SINGLE CELL NUMEROV TYPE DISCRETIZATION FOR 2D BIHARMONIC AND TRIHARMONIC EQUATIONS ON UNEQUAL MESH

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Abstract: In this article using nine point single cell, we report difference methods of accuracy of $O(k^2+h^4)$ for the solution of two dimensional multi-harmonic elliptic equations on unequal mesh, where *k*>0 and *h*>0 are grid sizes in *y*- and *x*-coordinates respectively. In all cases, we use Numerov type discretization. For a fixed value of (k/h^2), the proposed methods behave like fourth order in nature. We do not require to discretize the boundary conditions and the values of $\left(\nabla^2\right)^n u$, *n*=1,2,… are obtained as by-product of the methods. The resulting matrix system is solved by using the block iterative methods. Comparative results are provided to demonstrate the fourth order behaviour of the proposed methods.

Keywords: Numerov type discretization; fourth order accuracy; Laplacian; biharmonic and triharmonic equations; normal derivatives; maximum absolute errors.

2000 AMS Subject Classification: 65N06

1. Introduction

We consider the two dimensional biharmonic and triharmonic elliptic partial differential equations of the form

$$\nabla^4 u(x,y) \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f(x,y), \qquad 0<x,y<1 \tag{1}$$

and

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$$\nabla^6 u(x,y) \equiv \frac{\partial^6 u}{\partial x^6} + 3\left(\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4}\right) + \frac{\partial^6 u}{\partial y^6} = f(x,y), \qquad 0<x,y<1 \tag{2}$$

where $(x,y) \in \Omega = \{(x,y) | 0 < x, y < 1\}$ with boundary $\partial\Omega$ and $\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ represents as the two dimensional Laplacian of the function $u(x,y)$.

Dirichlet boundary conditions of second kind for equation (1) are given by

$$u = g_{11}(x,y), \quad \frac{\partial^2 u}{\partial n^2} = g_{12}(x,y), \quad (x,y) \in \partial\Omega, \tag{3}$$

and for equation (2) are given by

$$u = g_{21}(x,y), \quad \frac{\partial^2 u}{\partial n^2} = g_{22}(x,y), \quad \frac{\partial^4 u}{\partial n^4} = g_{23}(x,y),, \quad (x,y) \in \partial\Omega. \tag{4}$$

where $(\partial u / \partial n)$ is the directional derivative acting along the outward normal $\hat{n}$ to the boundary curve of the solution region. We assume that the function $u(x,y)$ is sufficiently smooth and required higher order partial derivatives of $u(x,y)$ exist in the solution domain Ω.

The biharmonic and triharmonic equations are fourth order elliptic partial differential equations, which encountered in areas of continuum mechanics, including linear elasticity theory and viscous flow problems. Different techniques for the numerical solution of the 2D biharmonic and triharmonic equations have been considered in the literature. Smith [1] and Ehrlich [2, 3] have solved 2D biharmonic equations using coupled second order accurate finite difference equations. Bauer and Riess [4] have used block iterative method to solve the equation. Later, kwon *et al* [5], Stephenson [6], Mohanty and Pandey [7], Evans and Mohanty [8] have developed certain second- and fourth-order finite difference approximations for the second biharmonic problems using 9-point compact cell. Dehghan and Mohebbi [9] have discussed high order accuracy methods for multi-dimensional biharmonic equations of second kind. Recently, using uniform mesh with equal mesh length, Mohanty [10,11] and Mohanty *et al* [12] have discussed fourth order compact finite difference schemes for the solution of 2D biharmonic and triharmonic equations, in which they used five function evaluations. Using five function evaluations, Mohanty *et al* [13] have developed fourth order discretization for the solution of two dimensional elliptic equations on an unequal mesh. In this paper, we split the differential equations (1) and (2) into system of two and three elliptic differential equations and introduce new ideas to handle boundary conditions without discretizing them in the system of elliptic equations. We use only 9-point compact cell (see fig.1) and unequal mesh for the discretization of differential equations (1) and (2). In all case, we use Numerov type discretization and only three functions evaluation. The given Dirichlet boundary conditions are exactly satisfied and no approximations for derivatives need to be carried out at the boundaries. The main advantage of this work is that we require only three evaluations of function *f*, whereas in our previous work five evaluations of function *f* were required to obtain the numerical solution of $u(x,y)$. Thus the proposed method requires less algebraic operations as compared to our earlier methods. In next section, we give the completely mathematical details of the methods. In section 3, we discuss the appropriate block iterative methods for the resulting linear systems. In order to illustrate the method and its fourth order convergence, we have solved *two* problems in section 4. Concluding remarks are given in section 5.

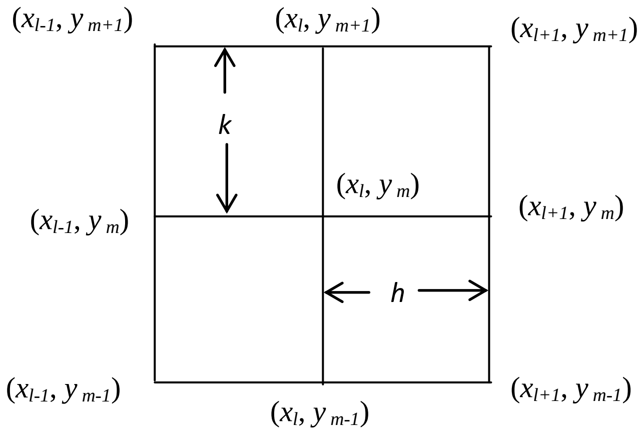

Fig.1: 9-point 2D single computational cell

2. Numerov type discretization

We consider our region of interest, a rectangular domain $\Omega = [0,1] \times [0,1]$. A grid with spacing $h > 0$ and $k > 0$ in the directions x- and y- respectively are first chosen, so that the mesh points (x_l, y_m) denoted by (l,m) are defined as $x_l = lh$ and $y_m = mk$, $l =$

$0,1,\ldots,N+1$, $m=0,1,\ldots,M+1$, where N, M are positive integers such that $(N+1)h=1$ and $(M+1)k=1$.

Let us denote the mesh ratio parameter by $p=(k/h)>0$. For convergence of the numerical scheme it is essential that our parameter remains in the range $0<\sqrt{6}p<1$.

We define: $$\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v(x,y), \quad (x,y)\in\Omega, \tag{5.1}$$

$$\nabla^4 u(x,y) \equiv \nabla^2 v \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = w(x,y),\ (x,y)\in\Omega. \tag{5.2}$$

Note that, the Dirichlet boundary conditions for the equation (1) are given by (3). Since the grid lines are parallel to coordinate axes and the values of u are exactly known on the boundary, this implies, the successive tangential partial derivatives of u are known exactly on the boundary. For example, on the line y=0, the values of $u(x,0)$ and $u_{yy}(x,0)$ are known, i.e., the values of $u_x(x,0)$, $u_{xx}(x,0)$,… etc are known on the line y=0. This implies the values of $u(x,0)$ and $\nabla^2 u(x,0) \equiv u_{xx}(x,0)+u_{yy}(x,0)$ are known on the line y=0. Similarly the values of u and $\nabla^2 u$ are known on all sides of the square region Ω.

Similarly for the differential equation (2), the values of $u, \nabla^2 u$ and $\nabla^4 u$ are known on all sides of the square region Ω.

Thus the Dirichlet boundary conditions (3) for the equation (1) may be replaced by

$$u = g_{11}(x,y), \quad v = \nabla^2 u = g_{13}(x,y), \quad (x,y)\in\partial\Omega, \tag{6}$$

and (4) for the equation (2) are replaced by

$$u = g_{21}(x,y), \quad v = \nabla^2 u = g_{24}(x,y), \quad w = \nabla^4 u = g_{25}(x,y),, \quad (x,y)\in\partial\Omega. \tag{7}$$

Then we re-write the boundary value problem (1) and (6) in coupled form

$$\frac{\partial^2 u}{\partial x^2} = v(x,y) - \frac{\partial^2 u}{\partial y^2}, \quad (x,y) \in \Omega, \tag{8.1}$$

$$\frac{\partial^2 v}{\partial x^2} = f(x,y) - \frac{\partial^2 v}{\partial y^2}, \quad (x,y) \in \Omega, \tag{8.2}$$

and (2) and (7) in a system of three Poisson equations of the form

$$\frac{\partial^2 u}{\partial x^2} = v(x,y) - \frac{\partial^2 u}{\partial y^2}, \quad (x,y) \in \Omega, \tag{9.1}$$

$$\frac{\partial^2 v}{\partial x^2} = w(x,y) - \frac{\partial^2 v}{\partial y^2}, \quad (x,y) \in \Omega, \tag{9.2}$$

$$\frac{\partial^2 w}{\partial x^2} = f(x,y) - \frac{\partial^2 w}{\partial y^2}, \quad (x,y) \in \Omega, \tag{9.3}$$

and the exact Dirichlet boundary conditions for all above equations are given by (6) and (7), respectively.

At the grid point (x_l, y_m), let $U_{l,m}, V_{l,m}, W_{l,m}$ and $u_{l,m}, v_{l,m}$, $w_{l,m}$ be the exact and approximate solution values of $u(x,y)$, $v(x,y)$ and $w(x,y)$, respectively. Let $f_{l,m}$ be the exact value of $f(x,y)$ at the grid point (x_l, y_m).

We need the following approximations:

$$\overline{U}_{yy\,l,m} = (U_{l,m+1} - 2U_{l,m} + U_{l,m-1})/(k^2), \tag{10.1}$$

$$\overline{U}_{yy\,l\pm1,m} = (U_{l\pm1,m+1} - 2U_{l\pm1,m} + U_{l\pm1,m-1})/(k^2), \tag{10.2}$$

$$\overline{V}_{yy\,l,m} = (V_{l,m+1} - 2V_{l,m} + V_{l,m-1})/(k^2), \tag{10.3}$$

$$\overline{V}_{yy\,l\pm1,m} = (V_{l\pm1,m+1} - 2V_{l\pm1,m} + V_{l\pm1,m-1})/(k^2), \tag{10.4}$$

$$\overline{W}_{yy\,l,m} = (W_{l,m+1} - 2W_{l,m} + W_{l,m-1})/(k^2), \tag{10.5}$$

$$\overline{W}_{yy\,l\pm1,m} = (W_{l\pm1,m+1} - 2W_{l\pm1,m} + W_{l\pm1,m-1})/(k^2). \tag{10.6}$$

Then at each grid point (x_l, y_m), l=1(1)N, m=1(1)M, a difference method of $O(k^2+h^4)$ for the differential equations (8.1) and (8.2) due to Numerov is given by

$$(U_{l+1,m}-2U_{l,m}+U_{l-1,m})+\frac{h^2}{12}\left[\overline{U}_{yy\,l+1,m}+\overline{U}_{yy\,l-1,m}+10\overline{U}_{yy\,l,m}\right]$$

$$=\frac{h^2}{12}\left[V_{l+1,m}+V_{l-1,m}+10V_{l,m}\right]+O(k^2h^2+h^6),$$

$$(V_{l+1,m}-2V_{l,m}+V_{l-1,m})+\frac{h^2}{12}\left[\overline{V}_{yy\,l+1,m}+\overline{V}_{yy\,l-1,m}+10\overline{V}_{yy\,l,m}\right]$$

$$=\frac{h^2}{12}\left[f_{l+1,m}+f_{l-1,m}+10f_{l,m}\right]+O(k^2h^2+h^6),$$

or,

$$\lambda_1(U_{l+1,m}+U_{l-1,m})+\lambda_2(U_{l,m+1}+U_{l,m-1})$$

$$+\ \lambda_3\left[U_{l+1,m+1}+U_{l+1,m-1}+U_{l-1,m+1}+U_{l-1,m-1}-(24p^2+20)U_{l,m}\right]$$

$$=\frac{k^2}{12}\left[V_{l+1,m}+V_{l-1,m}+10V_{l,m}\right]+O(k^4+k^2h^4), \qquad (11.1)$$

$$\lambda_1(V_{l+1,m}+V_{l-1,m})+\lambda_2(V_{l,m+1}+V_{l,m-1})$$

$$+\ \lambda_3\left[V_{l+1,m+1}+V_{l+1,m-1}+V_{l-1,m+1}+V_{l-1,m-1}-(24p^2+20)V_{l,m}\right]$$

$$=\frac{k^2}{12}\left[f_{l+1,m}+f_{l-1,m}+10f_{l,m}\right]+O(k^4+k^2h^4) \qquad (11.2)$$

where $\lambda_1=p^2-\frac{2}{12}$, $\lambda_2=\frac{10}{12}$, $\lambda_3=\frac{1}{12}$.

Similarly, a difference method of $O(k^2+h^4)$ for the triharmonic equation (9.1)-(9.3) due to Numerov is given by

$$\lambda_1(U_{l+1,m}+U_{l-1,m})+\lambda_2(U_{l,m+1}+U_{l,m-1})$$

$$+\ \lambda_3\left[U_{l+1,m+1}+U_{l+1,m-1}+U_{l-1,m+1}+U_{l-1,m-1}-(24p^2+20)U_{l,m}\right]$$

$$=\frac{k^2}{12}\left[V_{l+1,m}+V_{l-1,m}+10V_{l,m}\right]+O(k^4+k^2h^4), \qquad (12.1)$$

$$\lambda_1(V_{l+1,m}+V_{l-1,m})+\lambda_2(V_{l,m+1}+V_{l,m-1})$$
$$+\ \lambda_3\left[V_{l+1,m+1}+V_{l+1,m-1}+V_{l-1,m+1}+V_{l-1,m-1}-(24p^2+20)V_{l,m}\right]$$
$$=\frac{k^2}{12}\left[W_{l+1,m}+W_{l-1,m}+10W_{l,m}\right]+O(k^4+k^2h^4) \quad (12.2)$$

$$\lambda_1(W_{l+1,m}+W_{l-1,m})+\lambda_2(W_{l,m+1}+W_{l,m-1})$$
$$+\ \lambda_3\left[W_{l+1,m+1}+W_{l+1,m-1}+W_{l-1,m+1}+W_{l-1,m-1}-(24p^2+20)W_{l,m}\right]$$
$$=\frac{k^2}{12}\left[f_{l+1,m}+f_{l-1,m}+10f_{l,m}\right]+O(k^4+k^2h^4) \quad (12.3)$$

For convergence, the condition which is usually imposed on equations (11.1)-(11.2), (12.1)-(12.3) is that $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 > 0$, that is, $0 < \sqrt{6}p < 1$.

By combining the difference equations at each internal grid points, we obtain a large sparse system of matrix to solve. At each interior mesh point, we have unknowns u, $\nabla^2 u \equiv v$ and $\nabla^2 v \equiv w$ that is, the number of bands with non-zero entries is increased, and so is the size of the final matrix for the same mesh size. However, by this new method, the value of the Laplacian, which is often of interest, is also computed.

3. Block iterative methods and numerical results

To solve the system (11) or (12) and indeed to demonstrate the existence of a solution, one can use block successive over relaxation (BSOR) iterative method (see [14-21]).

To define BSOR method, we first write (11.1) and (11.2) in the form

$$\boldsymbol{A}_1\boldsymbol{u}+\boldsymbol{B}_1\boldsymbol{v}=\boldsymbol{0} \quad (13.1)$$

$$\boldsymbol{A}_2\boldsymbol{u}+\boldsymbol{B}_2\boldsymbol{v}=\boldsymbol{d} \quad (13.2)$$

where $\boldsymbol{A}_{1L}=\frac{1}{12}[1, 10, 1]$, $\boldsymbol{A}_{1D}=\frac{1}{6}[6p^2-1, -12p^2-10, 6p^2-1]$, $\boldsymbol{A}_{1U}=\frac{1}{12}[1, 10, 1]$ represent lower, main and upper diagonal tri-diagonal matrices of the tri-block diagonal matrix $\boldsymbol{A}_1=[\boldsymbol{A}_{1L}, \boldsymbol{A}_{1D}, \boldsymbol{A}_{1U}]$ and $\boldsymbol{B}_{1L}=[0, 0, 0]$, $\boldsymbol{B}_{1D}=[1, 10, 1]$, $\boldsymbol{B}_{1U}=[0, 0, 0]$ are tri-diagonal matrices of tri-block diagonal matrix $\boldsymbol{B}_1 = \frac{-k^2}{12}$ [$\boldsymbol{B}_{1L}$, $\boldsymbol{B}_{1D}$, $\boldsymbol{B}_{1U}$]. $\boldsymbol{A}_2$ = [***0, 0, 0***] is a zero matrix and $\boldsymbol{B}_{2L}$ = $\frac{1}{12}$[1, 10, 1], $\boldsymbol{B}_{2D}$ = $\frac{1}{6}$[$6p^2$-1, -$12p^2$-10, $6p^2$-1], $\boldsymbol{B}_{2U}$ = $\frac{1}{12}$[1, 10, 1] represent lower, main and upper diagonal tri-diagonal matrices of the tri-block diagonal matrix $\boldsymbol{B}_2$ = [$\boldsymbol{B}_{2L}$, $\boldsymbol{B}_{2D}$, $\boldsymbol{B}_{2U}$] , and ***d*** is the vector consisting of right hand side functions and associated boundary conditions.

Relative to the partitioning (13.1) and (13.2), the BSOR method is defined by

$$\boldsymbol{A}_{1D}\, \boldsymbol{u}^{(k+1)} = \omega\,[-(\boldsymbol{A}_{1L}+\boldsymbol{A}_{1U})\, \boldsymbol{u}^{(k)} - \boldsymbol{B}_1 \boldsymbol{v}^{(k)}] + (1-\omega)\, \boldsymbol{A}_{1D} \boldsymbol{u}^{(k)} \tag{14.1}$$

$$\boldsymbol{B}_{2D}\, \boldsymbol{v}^{(k+1)} = \omega\,[-(\boldsymbol{B}_{2L}+\boldsymbol{B}_{2U})\, \boldsymbol{v}^{(k)} - \boldsymbol{A}_2 \boldsymbol{u}^{(k+1)} + \boldsymbol{d}] + (1-\omega)\, \boldsymbol{B}_{2D} \boldsymbol{v}^{(k)} \tag{14.2}$$

where $0<\omega<2$ is a relaxation parameter. The above system of equations can be solved by using a line solver. For $\omega=1$, the BSOR method reduces to block-Gauss Seidel iterative method. In a similar manner, we can write block iterative methods for (12.1)-(12.3).

4. Numerical results

The second order approximations for the coupled system of differential equations (8.1) and (8.2) are straightforward and can be written in a coupled manner

$$p^2(U_{l+1,m}+U_{l-1,m})-(2+2p^2)U_{l,m}+(U_{l,m+1}+U_{l,m-1})=k^2V_{l,m}+O(k^4+k^2h^2)\,, \tag{15.1}$$

$$p^2(V_{l+1,m}+V_{l-1,m})-(2+2p^2)V_{l,m}+(V_{l,m+1}+V_{l,m-1})=k^2 f_{l,m}+O(k^4+k^2h^2)\,, \tag{15.2}$$

and the second order approximations for the system of poisons equations (9.1)-(9.3) may be written as

$$p^2(U_{l+1,m}+U_{l-1,m})-(2+2p^2)U_{l,m}+(U_{l,m+1}+U_{l,m-1})=k^2V_{l,m}+O(k^4+k^2h^2)\,, \tag{16.1}$$

$$p^2(V_{l+1,m}+V_{l-1,m})-(2+2p^2)V_{l,m}+(V_{l,m+1}+V_{l,m-1})=k^2W_{l,m}+O(k^4+k^2h^2)\,, \tag{16.2}$$

$$p^2(W_{l+1,m}+W_{l-1,m})-(2+2p^2)W_{l,m}+(W_{l,m+1}+W_{l,m-1})=k^2 f_{l,m}+O(k^4+k^2h^2)\,. \tag{16.3}$$

Note that, the second order approximations (15.1)-(15.2) and (16.1)-(16.3) require only 5-grid points on a single computational cell (see Fig.1). In a similar manner, we can discuss the block iterative methods for the systems (15.1)-(15.2) and (16.1)-(16.3).

In order to validate the proposed fourth order method and test its robustness, we solve the following two test problems in the region $0<x,y<1$, whose exact solutions are known. The

Dirichlet boundary conditions and right hand side homogeneous functions are obtained by using the exact solutions. We have solved the linear systems by using block Gauss-Seidel iterative method. We have also compared the numerical results obtained by proposed high order approximations with the numerical results obtained by corresponding $O(k^2+h^2)$ approximations (15.1)-(15.2) and (16.1)-(16.3). In all cases, we have considered $\boldsymbol{u}^{(0)}=\boldsymbol{0}$ as the initial approximation and the iterations were stopped when the absolute error tolerance $\left|\boldsymbol{u}^{(k+1)}-\boldsymbol{u}^{(k)}\right|\le 10^{-12}$ was achieved. In all cases, we have calculated maximum absolute errors (l_∞-norm) for different grid sizes. All computations were performed using double precision arithmetic for a fixed value of $\gamma=(k/h^2)$.

Example 1 (2D Biharmonic problem)

$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}=4\pi^4\sin\pi x\,\sin\pi y\,,\qquad 0<x,y<1 \tag{17}$$

The exact solution is $u(x,y)=\sin\pi x\,\sin\pi y$.

The maximum absolute errors are tabulated in Table 1 for a fixed value of γ =20.

Example 2 (2D Triharmonic problem)

$$\frac{\partial^6 u}{\partial x^6}+3\left(\frac{\partial^6 u}{\partial x^4\partial y^2}+\frac{\partial^6 u}{\partial x^2\partial y^4}\right)+\frac{\partial^6 u}{\partial y^6}=-8\pi^6\sin\pi x\,\sin\pi y\,,\quad 0<x,y<1 \tag{18}$$

The exact solution is $u(x,y)=\sin\pi x\,\sin\pi y$.

The maximum absolute errors are tabulated in Table 2 for a fixed value of γ =20.

Table 1

Example 1: The maximum absolute errors (γ =20)

h		$O(k^2+h^4)-$Method	$O(k^2+h^2)-$Method
1/10	u	0.3168(-01)	0.3988(-01)
	$\nabla^2 u$	0.3101(+00)	0.3896(+00)
	u	0.2060(-02)	0.4121(-02)

1/20 $\nabla^2 u$	0.2032(-01)	0.4063(-01)
1/40 u	0.1286(-03)	0.6427(-03)
$\nabla^2 u$	0.1269(-02)	0.6342(-02)

Table 2

Example 2: The maximum absolute errors (γ =20)

h	$O(k^2+h^4)-$Method	$O(k^2+h^2)-$Method
$\frac{1}{10}$ u	0.4792(-01)	0.6045(-01)
$\nabla^2 u$	0.6255(+00)	0.7873(+00)
$\nabla^4 u$	0.6123(+01)	0.7691(+01)
$\frac{1}{20}$ u	0.3091(-02)	0.6188(-02)
$\nabla^2 u$	0.4066(-01)	0.8135(-01)
$\nabla^4 u$	0.4011(+00)	0.8021(+00)
$\frac{1}{40}$ u	0.1929(-03)	0.9642(-03)
$\nabla^2 u$	0.2539(-02)	0.1268(-01)
$\nabla^4 u$	0.2506(-01)	0.1251(+00)

5. Conclusions

In this paper, we discuss a class of new compact finite difference discretizations of order two in *y*- and four in *x*-directions for the solution of 2D biharmonic and triharmonic partial differential equations. The methods are derived on a 9-point compact stencil using the values of u, $\nabla^2 u$ and $\nabla^4 u$ as unknowns. We have obtained the numerical solution of $\nabla^2 u$ and $\nabla^4 u$ as by-product of the solution, which are quite often of interest in many applied mathematics problems. We have solved biharmonic and triharmonic problems and obtained high accuracy solutions with great efficiency. We have shown that for a fixed value of (k/h^2), the proposed methods are fourth order accurate. We are currently working to extend this technique to solve non-linear biharmonic and triharmonic elliptic and time dependent parabolic partial differential equations.

REFERENCES

[1] J. Smith, The coupled equation approach to the numerical solution of the biharmonic equation by finite differences, SIAM. J. Num. Anal., 5 (1970) 104-111.

[2] L.W. Ehrlich, Solving the Biharmonic equation as coupled finite difference equations, SIAM. J. Num. Anal., 8 (1971) 278-287.

[3] L.W. Ehrlich, Point and block SOR applied to a coupled set of difference equations, Computing, 12 (1974) 181-194.

[4] L. Bauer and E.L. Riess, Block five diagonal matrices and the fast numerical solution of the biharmonic equation, Math. Comp., 26 (1972) 311-326.

[5] Y. Kwon, R. Manohar and J.W. Stephenson, Single cell fourth order methods for the biharmonic equation, Congress Numerantium, 34 (1982)475-482.

[6] J.W. Stephenson, Single cell discretization of order two and four for biharmonic problems, J. Comput. Phys., 55 (1984) 65-80.

[7] R.K. Mohanty and P.K. Pandey, Difference methods of order two and four for systems of mildly non-linear biharmonic problems of second kind in two space dimensions, Numer. Meth. Partial Diff. Eq., 12 (1996) 707-717.

[8] D.J. Evans and R.K. Mohanty, Block iterative methods for the numerical solution of two-dimensional non-linear biharmonic equations, Int. J. Computer Math., 69 (1998) 371-390.

[9] M. Dehghan and A. Mohebbi, Solution of the two dimensional second biharmonic equation with high-order accuracy, Kybernetes, 37 (2008) 1165-1179.

[10] R.K. Mohanty, A new high accuracy finite difference discretization for the solution of 2D non-linear biharmonic equations using coupled approach, Numer. Meth. Partial Diff. Eq., 26 (2010) 931-944.

[11] R.K. Mohanty , Single cell compact finite difference discretizations of order two and four for multi-dimensional triharmonic problems, Numer. Meth. Partial Diff. Eq., 26 (2010) 1420-1426.

[12] R.K. Mohanty, M.K. Jain and B.N. Mishra, A compact discretization of $O(h^4)$ for two-dimensional non-linear triharmonic equations, Physica Scripta, 84 (2011), ID: 025002.

[13] R.K. Mohanty, Samir Karaa and U. Arora, , Fourth order nine point unequal mesh discretization for the solution of 2D nonlinear elliptic partial differential equations, Neural Parallel and Scientific Computations,14 (2006) 453-470.

[14] L.A. Hageman and D.M. Young, Applied Iterative Methods, Academic Press, New York (1981).

[15] S.V. Parter, Block Iterative Methods, Elliptic Problem Solvers, Schultz, M.H., Editors, 247-253, Academic Press, New York (1981).

[16] M.K. Jain, Numerical Solution of Differential Equations, Wiley Eastern Limited, New Delhi (1984).

[17] C.T. Kelly, Iterative Methods for Linear and Non-linear Equations, SIAM publications, Philadelphia, PA (1995).

[18] G. Meurant, Computer Solution of Large Linear Systems, North-Holland Publishing Co., Amsterdam (1999).

[19] J.W. Thomas, Numerical Partial Differential Equations: Conservation Laws and Elliptic Equations, Springer Verlag, New York (1999).

[20] Y. Saad, Iterative Methods for Sparse Linear Systems, SIAM publications, Philadelphia, PA (2003).

[21] J.C. Strikwerda, Finite Difference Schemes and Partial Differential Equations, SIAM, Philadelphia (2004).