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CC-REGULAR TOPOLOGICAL SPACES

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Abstract. A topological space X is called CC-regular if there exists a regular space Y and a bijective function $f : X \to Y$ such that $f_{|C} : C \to f(C)$ is homeomorphism for any countably compact subspace C of X. We investigate this definition. Some relations with weaker versions of regularity have been studied, as L-regular and C-regular spaces.

Keywords: C-regular; L-regular; CC-regular; metrizable.

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1. Introduction

In his visit to Saudi Arabia, Arhangel'skii defined the new concept of C-normality in a seminar in the Department of Mathematics at King Abdulaziz University in 2012. By the definition, a space X is C-normal if there exists a normal space Y and a bijection $f : X \to Y$ such that $f_{|C} : C \to f(C)$ is homeomorphism for any compact subspace C of X. In 2017, AlZahrani and Kalantan investigate C-normal property [1]. In the same year, CC-normality [5] has been presented as a weaker version of normality but stronger than C-normality. After that, C-regular is defined in [2]. We use the idea of this definition to introduce another new weaker version of regularity and we call it CC-regularity. We investigate some topological properties of this space

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and we study the relationships between CC-regular spaces and other spaces such as C-regular, L-regular, and submetrizable spaces. Also, some examples are presented to show that there is no relation between CC-regular spaces and CC-normal spaces.

We provide a necessary condition which is important to prove that every L-regular space is CC- regular space. Also, it is proved that if X is submetrizable then X is CC-regular. We present some examples to show that there is no relation between CC-regularity and CC-normality. Moreover, some properties of CC-regularity have been investigated in this paper.

2. CC-REGULARITY

First, we recall the definition of C-regular spaces.

Definition 2.1. *[2] A space X is called C-regular if there exists a regular space Y and a bijective function $f : X \to Y$ such that $f_{|C} : C \to f(C)$ is homemorphism for any compact subspace C of X.*

Now, we define a new topological property CC-regularity as follows which is analogous to the above definition.

Definition 2.2. *A space X is called CC-regular if there exists a regular space Y and a bijective function $f : X \to Y$ such that $f_{|C} : C \to f(C)$ is homeomorphism for any countably compact subspace C of X.*

It is clear that by taking $X = Y$ and the identity function on X in the above definition, we deduce that any regular space X is CC-regular. The converse is not true in general. The half-disc topology [**?**] is defined on $X = P \cup L$ where $P = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ equipped by the Euclidean metric topology on P and $L = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ where the neighborhood of any point $(x,0) \in L$ is given by $\{\ (x,0)\} \cup (P \cap U)$ where U is an open set with respect to the Euclidean metric topology. The half-disc topology is not regular but it is submetrizable the it is CC-regular by Theorem 2.5. On the other hand, every CC-regular space is C-regular and that is clear because every compact subspace is countably compact. Obviously, any countably compact CC-regular is regular.

Definition 2.3. *[5] A space X is called CC-normal if there exists a normal space Y and a bijective function $f : X \to Y$ such that $f_{|C} : C \to f(C)$ is homeomorphism for any countably compact subspace C of X.*

The following examples show that CC-normality and CC-regularity are independent.

Example 2.1. *Let $(\mathbb{R}, \mathfrak{L})$ be the left ray topological space. Since there are no non-empty disjoint closed sets in $(\mathbb{R}, \mathfrak{L})$, then $(\mathbb{R}, \mathfrak{L})$ is normal space and so it is CC-normal. In [2], it is proved that $(\mathbb{R}, \mathfrak{L})$ is not C-regular and so it is not CC-regular.*

Example 2.2. *Let $M = G \times H$, where $G = \prod_{\alpha \in \omega_1} D$ where $D = \{0, 1\}$ given with the discrete topology, and H is the set of all points of G with at most countably many non-zero coordinates. We consider H as a subspace of G. The space M is Tychonoff. Thus M is regular and so it is CC-regular. It is proved that M is not C-normal in [2]. Thus it is not CC-normal.*

Theorem 2.1. *If X is countably compact non-regular space, then X can not be CC-regular.*

Proof. Let X be countably compact non-regular space. Assume that X is CC-regular. Then there is a regular space Y and a bijective function $f : X \to Y$ such that $f_{|C} : C \to f(C)$ is homeomorphism for any countably compact subspace C of X. Since X is countably compact, then X is homeomorphic to Y. This contradicts the fact that X is not regular and Y is regular. So by contradiction, the space X is not CC-regular. □

As a result of the above theorem, and since the co-finite topological space $(\mathbb{R}, \mathfrak{C})$ is countably compact non-regular, then $(\mathbb{R}, \mathfrak{C})$ is not CC-regular.

The following theorem gives a condition to answer the question when the C-regular space is CC-regular.

Theorem 2.2. *If X is C-regular and every countably compact subspace of X is contained in a compact subspace, then X is CC-regular.*

Proof. Let X be a C-regular and every countably compact subspace of X is contained in a compact subspace of X. Then there is a regular space Y and a bijective function $f : X \to Y$ such that any restriction of f with respect to any compact subspace is homeomorphism. Now if C is countably compact subspace of X then by assumption, C is contained in a compact subspace G such that $f_{|G} : G \to f(G)$ is homeomorphism and this completes the proof. □

Recall that a space X is called L-regular if there is a regular space Y and a bijective function $f : X \to Y$ such that $f_{|L} : L \to f(L)$ is homeomorphism for any Lindelöf subspace L of X [2]. The proof of Theorem 2.3 and the proof of Theorem 2.4 are similar to Theorem 2.2.

Theorem 2.3. *If X is CC-regular and every Lindelöf subspace of X is contained in a countably compact subspace, then X is L-regular.*

Theorem 2.4. *If X is L-regular and every countably compact subspace of X is contained in a Lindelöf subspace, then X is CC-regular.*

Lemma 2.1. *[3] If a function $f : X \to Y$ is continuous and bijection where X is a countably compact space, and Y is a first countable Hasudorff space, then f is a homeomorphism.*

Recall that a topological space (X, τ) is submetrizable if there exists a topology τ_d on X generated by a metric d such that $\tau_d \subseteq \tau$ see [4].

Theorem 2.5. *Every submetrizable space is CC-regular.*

Proof. Let (X, τ) be submetrizable. Then there is a metrizable topology τ_d on X such that $\tau_d \subseteq \tau$. Since (X, τ_d) is metrizable then it is regular. Now, if C is a countably compact subspace of (X, τ) and $id : (X, \tau) \to (X, \tau_d)$ is the identity function on X then $id_{|C} : C \to id(C)$ is continuous and bijection. Since (X, τ_d) is metrizable, the subspace $id(C)$ is first countable Hasudorff. So, $id_{|C}$ is homeomorphism by Lemma 2.1. Thus (X, τ) is CC-regular. □

Remark 2.1. *Countably compactness and compactness are coincide in metrizable spaces.*

It is noted from the proof of Theorem 2.5, $id(C)$ is continuous image of a countably compact and so $id(C)$ is a countably compact in a metrizable space (X, τ_d), which means that $id(C) = C$ is compact in (X, τ).

As a direct proof of Theorem 2.5, it is proved in [2] that every submetrizable is C-regular. By Remark 2.1, every countably compact subset is compact in submetrizable, then every submetrizable is CC-regular.

The converse of Theorem 2.5 is not true in general. Take the modified Dieudonné Plank example [6]. We apply Theorem 2.4 to show that the example is CC-regular. In the example, $X = ((\omega_1+1)\times(\omega_0+1))\backslash\{\langle\omega_1,\omega_0\rangle\}$. See that $(\omega_1+1)\times\{0\}\subseteq X$ where $(\omega_1+1)\times\{0\}\cong(\omega_1+1)$ and (ω_1+1) is not submetrizable. Since submetrizability is hereditary, then X is not submetrizable. Now, write $X = A\cup B\cup N$, where $A=\{\langle\omega_1,n\rangle, n<\omega_0\}$, $B=\{\langle\alpha,\omega_0\rangle,\alpha<\omega_1\}$, and $N=\{\langle\alpha,n\rangle,\alpha<\omega_1,n<\omega_0\}$. The basic open sets of $\langle\alpha,n\rangle\in N$ are given by $\mathfrak{B}(\langle\alpha,n\rangle)=\{\{\langle\alpha,n\rangle\}\}$, the basic open sets of $\langle\omega_1,n\rangle\in A$ are $\mathfrak{B}(\langle\omega_1,n\rangle)=\{(\alpha,\omega_1]\times\{n\}:\alpha<\omega_1\}$ and the basic open sets of $\langle\alpha,\omega_0\rangle\in B$ are $\mathfrak{B}(\langle\alpha,\omega_0\rangle)=\{\{\alpha\}\times(n,\omega_0]:n<\omega_0\}$. A subset C is countably compact if C satisfies all of these conditions: 1) $C\cap A$ and $C\cap A$ are finite, 2) $\{\langle\alpha,n\rangle\in C\cap N:\langle\omega_1,n\rangle\in C\cap A\}$ is finite and 3) $\{\langle\alpha,n\rangle\in C\cap N:\langle\alpha,\omega_0\rangle\notin C\cap B,\ \langle\omega_1,n\rangle\notin C\cap A\}$ is finite. This proves that any countably compact subspace is countable and so Lindelöf. Now, take $Y=X=A\cup B\cup N$ where the basic open sets on $B\cup N$ are the same as in X but we modify the basic open sets of $\langle\omega_1,n\rangle\in A$ by $\mathfrak{B}(\langle\omega_1,n\rangle)=\{\{\langle\omega_1,n\rangle\}\}$. The space Y is paracompact so it is T_4 . By taking $id: X\longrightarrow Y$, it was proved in [6] that X is L-normal but in the same technique we get X is L-regular that because Y is T_4 and so regular. By Theorem 2.4, we get that X is CC-regular.

We know that if $A\subseteq X$, then any countably compact subspace of A is countably compact subspace of X. So, the proof of the following theorem is clear and so it is omitted.

Theorem 2.6. *CC-regularity is hereditary property.*

Theorem 2.7. *CC-regularity is topological property.*

Proof. Let X be a CC-regular and $X\cong Z$. From the definition of the CC-regularity, there is a regular space Y such that and a bijective function $f: X\to Y$ such that $f_{|C}: C\to f(C)$ is homeomorphism for any countably compact subspace C of X. Since $X\cong Z$, then there is a homeomorphism $g: Z\to X$. So, $f\circ g: Z\to Y$ is bijective. Let K be a countably compact subspace of Z. Now $g_{|K} : K \to g(K)$ is homeomorphism being $g : Z \to X$ is homeomorphism. Since K is countably compact subspace of Z and countably compactness is topological property, then $g(K)$ is countably compact subspace of X. Thus $f \circ g : Z \to Y$ is bijective and $(f \circ g)_{|K} = f_{|g(K)} : K \to (f(g(K))$ is homeomorphism for every K is a countably compact subspace of Z. Thus Z is CC-regular. □

The proof of the following theorem is clear and so it is omitted.

Theorem 2.8. *CC-regularity is an additive property.*

Recall that a space X is Fréchet if for any $A \subseteq X$ and $x \in \bar{A}$, then there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n \to x$ where $a_n \in A$ $\forall n \in \mathbb{N}$ [3]. The proof of the following theorem is clear and so it is omitted.

Theorem 2.9. *If X is CC-regular Fréchet space, then any function f witnessing it's CC-regularity is continuous.*

Also, it is noted that any first countable space is Fréchet, so we get the following result.

Corollary 2.1. *If X is CC-regular, first countable space and a function f witnessing it's CC-regularity, then f is continuous.*

Conflict of Interests

The author declares that there is no conflict of interests.

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