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ON Sg^*w-CLOSED SETS AND $ST^*_{\frac{1}{2}}$ SPACES IN WEAK STRUCTURES

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Abstract. In 2011, Császár introduced the concept of weak structure, in the present paper we introduce some new concepts of Strongly g^*w-closed (g^*w-closed, for short) sets in weak structures. Also, we find the relation between this class and the classes of $w$closed, gw-closed, and gsw-closed sets in weak structures. We also characterize their basic properties via gw-closed. Finally, we introduce the concepts of $ST^*_{\frac{1}{2}}$ spaces by using the concepts of gw-closed and Sg^*w-closed sets.

Keywords: weak structures; gw-closed sets; strongly g^*w-closed sets; gsw-closed sets; $ST^*_{\frac{1}{2}}$ spaces.

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1. Introduction

Levine [8] introduced the concept of generalized closed sets in topological space Y (A subset U of Y is called generalized closed (g-closed, for short) set if $cl(U) \subseteq V$, whenever $U \subseteq V$ and V is an open).

Császár [5]studied certain ideas, including continuity and generalized open sets, and introduced the concept of generalized topology in 2002. Moreover, Császár [6] developed the idea of a weak structure.

Let Y be a non-empty set and $P(Y)$ its power set. A class $w \subset P(Y)$ is said to be a weak

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structure (WS, for short) on Y if and only if $\phi \in w$, a subset U is said to be w-open if $U \in w$ and its complement is called w-closed. He also defined two operations $i_w(U)$ and $c_w(U)$ in WS on Y as the union of all w-open subsets contained in U and the intersection of all $w-closed$ sets containing U. Moreover, he gave some characteristics $c_w(U)$ and $i_w(U)$. The concept of gw-closed sets (in the sense of Al Omari and Noiri [1]) is a special case of gw-closed sets presented here.

In the present paper we introduce some new concepts of Strongly g^*w-closed (Sg^*w-closed, for short) sets in weak structures. Also, we find the relation between this class and the class of gw-closed sets and the class of gsw-closed sets in weak structures. We also characterize their basic properties via gw-closed. Finally, we introduce the concepts of $ST^*_{\frac{1}{2}}$ spaces by using the concepts of gw-closed and Sg^*w-closed sets.

2. Preliminaries

Let WS be a weak structure on a nonempty set Y. For a subset U of Y, $c_w(U)$ and $i_w(U)$ represent the closure of U with respect to w, the interior of U with respect to w respectively and the complement of U in Y is denoted by $(U)^c$.

Theorem 2.1. [7] *Let WS be a weak structure on a nonempty set Y and $U,V \subset Y$. Then the following hold.*

(1) $c_w(U) \cup c_w(V) = c_w(U \cup V)$.

(2) $i_w(U \cap V) = i_w(U) \cap i_w(V)$

(3) $H \cap c_w(U) \subset c_w(H \cap U)$ *for every* $H \in w$ *and* $U \subset Y$.

(4) $c_w(G \cap c_w(U)) = c_w(H \cap U)$ *for every* $G \in w$ *and* $U \subset Y$.

(5) $c_w(H) = c_w(H \cap U)$ *for every* $H \in w$ *and* U *is w-dense.*

Theorem 2.2. [6] *Let WS be a weak structures on Y and $U,V \subseteq Y$. Then the following statements are true:*

(1) $U \subseteq c_w(U)$

(2) *If* $U \subseteq V$, *then* $c_w(U) \subset c_w(U)$,

(3) *If* U *is w-closed, then* $U = c_w(U)$,

(4) $c_w(c_w(U)) = c_w(U)$,

(5) $y \in c_w(U)$ *if and only if* $U \cap V \neq \phi$ *for each w-open set* V *containing* y.

Theorem 2.3. [6] *Let* WS *be a weak structures on* Y *and* $U, V \subseteq Y$. *Then the following statements are true:*

(1) $U \supseteq i_w(U)$,

(2) *If* $U \subset V$, *then* $i_w(U) \subset i_w(V)$,

(3) $i_w(i_w(U)) = i_w(U)$,

(4) *If* U *is w-open, then* $U = i_w(U)$,

(5) $c_w(Y - U) = Y - i_w(U)$,

(6) $i_w(Y - U) = Y - c_w(U)$,

(7) $i_w(c_w(i_w(c_w(U)))) = i_w(c_w(U))$,

(8) $c_w(i_w(c_w(i_w(U)))) = c_w(i_w(U))$,

(9) $y \in i_w(U)$ *if and only if there is a w-open set* V *where* $y \in V \subset U$,

Definition 2.4. [10] *Let* WS *be a weak structures on* Y. *The set* U *is said to be generalized w-closed (gw-closed, for short) if* $c_w(U) \subset G$, *whenever* $U \subset G$ *and* G *is w-open.*
The complement of a generalized w-closed set is said to be generalized w-open (gw-open, for short). We will be denoted the family of all gw-closed (resp. gw-open) sets in a WS *on* Y *by* $gwC(Y)$ *(resp.* $gwO(Y)$*).*

Theorem 2.5. [10] *Let* WS *be a weak structures on* Y. *A subset* U *is gw-open if and only if* $H \subset i_w(U)$, *whenever* $H \subset U$ *and* H *is w-closd*

Definition 2.6. *Let* WS *be a weak structures on* Y. *A subset* U *is said to be* g^*w*-closed if* $c_w(U) \subset G$, *whenever* $U \subset G$ *and* G *is gw-open*

Definition 2.7. [7] *Let* w *be a weak structures on* Y. *a subset* V *is called:*

(i) *a semi-w-open (semi-w-closed) (sw-open (sw-closed), for short) set if* $V \subseteq c_w(i_w(V))(i_w(c_w(V)) \subseteq V)$.

(ii) *a regular w-open (regular w-closed) (rw-open (rw-closed), for short) set if* $V = i_w(c_w(V))(V = c_w(i_w(V)))$.

Definition 2.8. [7] *Let w be a weak structures on Y. The intersection of all sw-closed sets containing V is called the sw-closure of V and it is denoted by $sc_w(V)$.*

3. Strongly g^*w-Closed and Strongly g^*w-Open Sets

Definition 3.1. *Let WS be a weak structures on Y. A subset U is said to be strongly g^*w-closed (Sg^*w-closed, for short) set if $c_w(i_w(U)) \subseteq G$ whenever $U \subseteq G$ and G is gw-open.*
The complement of a Sg^*w-closed set is said to be Sg^*w-open.

Theorem 3.2. *Every w-closed set is Sg^*w-closed set.*

Proof. Let WS be a weak structures on Y. Suppose that U be w-closed subset and $U \subset G$ where G is gw-open set. Since U is w-closed, then $c_w(U) = U$ for every subset U of Y. Therefore, $c_w(i_w(U)) \subseteq G$ and hence U is Sg^*w-closed set. □

Remark 3.3. *The converse of the above theorem need not be true, we show that by the following example.*

Example 3.4. *Let $Y = \{r_1, r_2, r_3\}$, $w = \{\phi, \{r_1\}, \{r_1, r_3\}\}$, and $U = \{r_3\}$ we can see that U is a Sg^*w-closed set but not a w-closed set.*

Theorem 3.5. *Let WS be a weak structures on Y. If a subset U is g^*w-closed, then U is Sg^*w-closed.*

Proof. Suppose that U is g^*w-closed and let G be w-open set containing U. Then G contains $c_w(U)$ and $G \supseteq c_w(U) \supseteq c_w(i_w(U))$. Thus U is Sg^*w-closed. □

Remark 3.6. *The converse of the above theorem need not be true as seen from the following example.*

Example 3.7. *Let $Y = \{r_1, r_2, r_3\}$, $w = \{\phi, \{r_1\}, \{r_1, r_2\}\}$. Then the set $U = \{r_2\}$ is Sg^*w-closed but not g^*w-closed set.*

Theorem 3.8. *Let WS be a weak structures on Y. If a subset U is w-open and Sg^*-closed, then it is w-closed.*

Proof. Suppose that U be a subset of Y which is both w-open and Sg^*w-closed. Thus $U \supseteq c_w(i_w(U)) \supseteq c_w(U)$ and $U \supseteq c_w(U)$. Since $c_w(U) \supseteq U$, we have $U = c_w(U)$. Thus U is w-closed. □

Theorem 3.9. *Let WS be a weak structures on Y. If a set U is Sg^*w-closed, then $c_w(i_w(U)) - U$ contains no non empty gw-closed set.*

Proof. Suppose that U is non empty Sg^*w-closed and H be gw-closed subset contained in $c_w(i_w(U)) - U$. Now $H \subseteq c_w(i_w(U)) - U$, this implies $H \subseteq c_w(i_w(U)) \cap U^c$. Since $c_w(i_w(U)) - U = c_w(i_w(U)) \cap U^c$. Thus $H \subseteq c_w(i_w(U))$. Now $U \subseteq H^c$, this implies $H \subseteq U^c$. Here H^c is gw-open and U is Sg^*w-closed, we have $c_w(i_w(U)) \subseteq H^c$. Thus $H \subseteq (c_w(i_w(U)))^c$. Hence $H \subseteq (c_w(i_w(U))) \cap (c_w(i_w(U)))^c = \phi$. Therefore $H = \phi$ implies $c_w(i_w(U)) - U$ contains no non empty gw-closed sets. □

Remark 3.10. *The converse of the above theorem need not be true as seen from the following example.*

Example 3.11. *Let $Y = \{r_1, r_2, r_3\}$, $w = \{\phi, \{r_1\}, \{r_1, r_2\}\}$. Then the set $U = \{r_2\}$ is Sg^*w-closed but not g^*w-closed set.*

Theorem 3.12. *Let H is Sg^*w-closed set relative to V in weak structures WS and that both w-open and Sg^*w-closed subset of Y where $H \subseteq V \subseteq Y$, then H is Sg^*w-closed set relative to Y.*

Proof. . Let $H \subseteq G$ and G be a gw-open set. If $H \subseteq V \subseteq Y$, then $H \subseteq V$ and $H \subseteq G$. This implies $H \subseteq V \cap G$ and since H is Sg^*w-closed relative to V, $c_w(i_w(H)) \subseteq V \cap G$. This mean $V \cap c_w(i_w(H)) \subseteq V \cap G$ and this implies $V \cap (c_w(i_w(H))) \subseteq G$. Thus $(V \cap (c_w(i_w(H)))) \cup (c_w(i_w(H)))^c \subseteq G \cup (c_w(i_w(H)))^c$ implies $V \cup (c_w(i_w(H)))^c \subseteq G \cup (c_w(i_w(H)))^c$. Since V is Sg^*w-closed, we have $(c_w(i_w(V))) \subseteq G \cup (c_w(i_w(H)))^c$. Also $H \subseteq V$ implies $c_w(i_w(H)) \subseteq c_w(i_w(V))$. Thus $c_w(i_w(H)) \subseteq c_w(i_w(V)) \subseteq G \cup (c_w(i_w(H)))^c$. Therefore H is Sg^*w-closed set relative to Y. □

Corrollary 3.13. *Let V be Sg^*w-closed and suppose that U is gw-closed then $V \cap U$ is Sg^*w-closed set.*

Proof. To show that $V \cap U$ is Sg^*w-closed, we have to show $c_w(i_w(V \cap U)) \subseteq G$ whenever $V \cap U \subseteq G$ and G is gw-open. $V \cap U$ is gw-closed in V and so Sg^*w-closed in V. By the above theorem $V \cap U$ is Sg^*w-closed in Y. Since $V \cap U \subseteq V \subseteq Y$. □

Theorem 3.14. *If V is Sg^*w-closed and $V \subseteq H \subseteq c_w(i_w(V))$, then H is Sg^*w-closed.*

Proof. Suppose that G is gw-open such that $H \subseteq G$, then $V \subseteq G$. Since $V \subseteq G$ and V is Sg^*w-closed, then $c_w(i_w(V)) \subseteq G$. Now $H \subseteq c_w(i_w(V))$, and so $c_w(i_w(H) \subseteq c_w(i_w(V)) \subseteq V$. Thus H is Sg^*w-closed set. □

Definition 3.15. *Let WS be a weak structures on Y. A subset V is called a generalized semi-w-closed (gsw-closed, for short) set if $sc_w(V) \subseteq U$ whenever $V \subseteq U$ and U is w-open. The complement of gsw-closed set is gsw-open set.*

Theorem 3.16. *Let WS be a weak structures on Y. Every Sg^*w-closed set in Y is gsw-closed in Y.*

Proof. Let WS be a weak structures on Y and V be a Sg^*w-closed set in Y. Suppose that U be a w-open set, then $V \subseteq U$. Since V is Sg^*w-closed, and every w-open set is gw-open set, this implies $c_w(i_w(V)) \subseteq U$. Thus $V \cup c_w(i_w(V)) \subseteq V \cup U$, and so $sc_w(V) \subseteq U$. Hence V is gsw-closed set in Y. □

Remark 3.17. *Let WS be a weak structures on Y, gsw-closed set is not always be Sg^*w-closed set in Y, as we can see in the following example.*

Example 3.18. *Let $Y = \{r_1, r_2, r_3\}$ with $w = \{\phi, \{r_2\}\}$. Then $U = \{r_2\}$ is gsw-closed sets but not Sg^*w-closed.*

Definition 3.19. *Let WS be a weak structures on Y. A subset V is called a regular generalized-w-closed (rgw-closed, for short) set if $c_w(V) \subseteq U$ whenever $V \subseteq U$ and U is rw-open. The complement of rgw-closed set is rgw-open set.*

Theorem 3.20. *Let WS be a weak structures on Y. Every Sg^*w-closed set in Y is rgw-closed in Y.*

Proof. The proof is obvious since every gw-closed set is rgw-closed set. □

Remark 3.21. *Let WS be a weak structures on Y, rgw-closed set is not always be Sg^*w-closed set in Y, as we can see in the following example.*

Example 3.22. *Let $Y = \{r_1, r_2, r_3\}$ with $w = \{\phi, \{r_1\}, \{r_2\}, \{r_1, r_2\}\}$. Then $U = \{r_2\}$ is rgw-closed set but it is not Sg^*w-closed.*

Remark 3.23. *Let WS be a weak structures on Y. The intersection of two Sg^*w-closed sets need not be Sg^*w-closed set as seen from the following example.*

Example 3.24. *Let $Y = \{r_1, r_2, r_3, r_4\}$ with $w = \{\phi, \{r_1, r_2, r_3\}, \{r_1, r_2, r_4\}, \{r_1, r_3, r_4\}, \{r_3, r_4\}\}$. Then $U = \{r_3\}$ and $V = \{r_1, r_2\}$ are Sg^*w-closed sets but $U \cup V = \{r_1, r_2, r_3\}$ it is not Sg^*w-closed.*

Remark 3.25. *Let WS be a weak structures on Y. The intersection of two Sg^*w-closed sets need not be Sg^*w-closed set as seen from the following example.*

Example 3.26. *Let $Y = \{r_1, r_2, r_3\}$ with $w = \{\phi, \{r_2\}, \{r_3\}, \{r_1, r_3\}, \{r_2, r_3\}\}$. Then $U = \{r_1, r_3\}$ and $V = \{r_2, r_3\}$ are Sg^*w-closed sets but $U \cap V = \{r_3\}$ it is not Sg^*w-closed.*

Theorem 3.27. *Let WS be a weak structures on Y. For each $y \in Y$, the singleton $\{y\}$ is gw-closed set or $\{y\}^c$ is Sg^*w-closed set.*

Proof. Let WS be a weak structures on Y and $\{y\}$ is not gw-closed, then $\{y\}^c$ will not be gw-open. Then Y is the only gw-open set containing $\{y\}^c$ and $c_w(i_w(\{y\}^c)) \subseteq Y$. Therefore $\{y\}^c$ is Sg^*w-closed set and $\{y\}$ is gw-open set. □

Theorem 3.28. *Let WS be a weak structures on Y. If V is Sg^*w-closed set, then $c_w(y_i) \cap V \neq \phi$ for each $y_i \in c_w(i_w(V))$.*

Proof. Let WS be a weak structures on Y. Suppose that $y_i \in c_w(i_w(V))$ and V is Sg^*w-closed set. If $c_w(\{y_i\}) \cap V = \phi$, then $V \subset Y - c_w(y_i)$ such that $Y - c_w(y_i)$ is gw-open set. Thus $y_i \in Y - c_w(y_i)$ which is contradiction. □

Remark 3.29. *Let WS be a weak structures on Y. The converse of Theorem 3. 26, is not true in general. If a subset $V = \{r_2\}$ in Example 4.11 is not Sg^*w-closed. However $c_w(y_i) \cap V \neq \phi$ for each $y_i \in c_w(i_w(V))$.*

4. Separation Axioms on Weak Structures

Definition 4.1. *Let WS be a weak structures on Y. A WS is said to be w-$ST^*_{\frac{1}{2}}$ if each Sg^*w-closed set V of Y, $c_w(i_w(V)) = V$.*

Remark 4.2. *Let WS be a weak structures on Y. If WS is w-$ST^*_{\frac{1}{2}}$, then there exists a singleton $\{y\} \in Y$ such that $\{y\}$ is neither gw-closed nor $\{y\} \neq c_w(i_w(\{y\})$.*

Example 4.3. *Let WS be a weak structures on Y and let $Y = \{r_1, r_2, r_3\}$, $w = \{\phi, \{r_2\}, \{r_3\}, \{r_2, r_3\}\}$. Clear that each singleton is gw-open or gw-closed. But we have $V = \{r_1, r_2\}$ and V is Sg^*w-closed and $c_w(i_w(V) = Y \neq V$. So, w is not w-$ST^*_{\frac{1}{2}}$.*

Theorem 4.4. *Let WS be a weak structures on Y. A WS on Y is w-$ST^*_{\frac{1}{2}}$-space if and only if each singleton $\{y\}$ is either w-open or gw-closed.*

Proof. Let WS be a weak structures on Y. Let $y \in Y$. Suppose that $\{y\}$ is not gw-closed. Then $\{y\}^c$ is not gw-open set. Thus $\{y\}^c$ is Sg^*w-closed, by Theorem 3.27. Since Y is $w - ST^*_{\frac{1}{2}}$-space, $\{y\}^c$ is gw-closed set of Y, i.e $\{y\}$ is w-open set of Y.
Conversely, suppose that V be a Sg^*w-closed set of Y. Now, put $y \in c_w(i_w(V)$, then by the fact in the first side, $\{y\}$ is either w-open or gw-closed, so we have two cases:
case (i) Let $\{y\}$ be a w-open. Since $y \in c_w(i_w(V))$, $\{y\} \cap V \neq \phi$. This shows that $y \in V$.
case (ii) Let $\{y\}$ be a gw-open. Now, we suppose that $y \notin V$, then we would have $y \in c_w(i_w(V))$ - V, which is contradiction with Theorem 3.9. Hence $y \in V$. Therefore, in both cases we have that $\{y\}^c$ is w-closed. Hence Y is a w-$ST^*_{\frac{1}{2}}$-space. □

Definition 4.5. *A weak structure WS on Y is said to be gw -T_1 if for any points $r_1, r_2 \in Y$ with $r_1 \neq r_2$, there exist two gw-open sets U and V such that $r_1 \in U, r_2 \notin U, r_1 \notin V$ and $r_2 \in V$.*

Theorem 4.6. *Let WS be a weak structures on Y and every gw-closed in Y is w-closed. A WS on Y is gw-T_1 if every singleton in Y is gw-closed.*

Remark 4.7. *The converse of the above theorem need not be true in general, and the following example show that.*

Example 4.8. *Let* $Y = \{r_1, r_2, r_3\}$, $w = \{\phi, \{r_1\}, \{r_2\}, \{r_3\}\}$. *It is clear that:*
WS is gw - T_1, but the singleton $\{r_2\}$ is not gw-closed.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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