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TENSOR PRODUCT OF SCHAUDER FRAMES AND BESSELIAN SCHAUDER FRAMES OF BANACH SPACES

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Abstract. We introduce the tensor product of besselian sequences and besselian Schauder frames in tensor product of separable Banach spaces. We prove that the tensor product of two besselian sequences (resp. besselian Schauder frames) is a besselian sequence (resp. a besselian Schauder frame) if one of the two sequences is rectangular. On the other hand we give an example for which the tensor product of two besselain Schauder frames is not besselain Schauder frame.

Keywords: tensor product; Schauder frames; besselian Schauder frames.

2020 AMS Subject Classification: 47A80.

1. Introduction

In 1946, Gabor [7] performed a new method for the signal reconstruction from elementary signals. In 1952, Duffin and Schaeffer [5] developped, in the field of nonharmonic series, a similar tool and introduced frame theory for Hilbert spaces. A frames for a Banach space was introduced in (1991) by Gröchenig [13]. In light of the works of Cassaza, Han and Larson [17] (1999), Han and Larson [4] (2000). Cassaza [16] (2008) introduced the notion of Schauder frame of a given Banach space. In 2021 Karkri and Zoubeir [23] introduced the notion of besselian Schauder frame of Banach spaces.

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In 1956 Grothendieck [8] Developed in his interesting paper a theory of tensor products on Banach spaces. Gelbaum and Gil de Lamadrid [3] proved that the tensor product of basis (resp. unconditional basis) in tensor product of Banach spaces is a basis (resp. is not necessarily an unconditional basis). The tensor product of frames in tensor product of Hilbert spaces is introduced by Khosravi and Asgari in 2003 [1].

In this paper, we introduce the notion of tensor product of besselian sequences and besselian Schauder frames in tensor product of separable Banach spaces. We prove also, that the tensor product of two besselian sequences (resp. besselian Schauder frames) is a besselian sequence (resp. a besselian Schauder frame) if one of the two sequences is rectangular.

2. Main Definitions and Notations

Let X and Y be separable Banach spaces on $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and X^*, Y^* (resp. X^{**}, Y^{**}) its topological dual (resp. bidual).

(1) The Banach space X is said to be weakly sequentially complete if for each sequence $(x_n)_{n\in\mathbb{N}^*}$ of X such that $\lim\limits_{n\to+\infty} x^*(x_n)$ exists for every $x^* \in X^*$, there exists $x \in X$ such that $\lim\limits_{n\to+\infty} x^*(x_n) = x^*(x)$ for every $x^* \in X^*$ [22, page 218, definition 2.5.23] [6, pages 37-38].

(2) We denote by $\mathbb{B}_X$ the closed unit ball of X :

$$\mathbb{B}_X := \{x \in X : \|x\|_X \leq 1\}$$

(3) We denote by $L(X)$ the set of all bounded linear operators $f : X \to X$. It is well-known that $L(X)$ is a Banach space for the norm $\|\cdot\|_{L(X)}$ defined by the formula:

$$\|f\|_{L(X)} := \sup_{x\in\mathbb{B}_X} \|f(x)\|_X$$

(4) We denote by $B(X\times Y)$ the set of all bounded bilinear forms $B : X\times Y \to \mathbb{K}$. It is well-known that $B(X\times Y)$ is a Banach space for the norm $\|\cdot\|_{B(X\times Y)}$ defined by the formula:

$$\|B\|_{B(X\times Y)} := \sup_{(x,y)\in\mathbb{B}_X\times\mathbb{B}_Y} |B(x,y)|$$

(5) We denote by $X \otimes_\pi Y$ the linear space of linear functionals on $B(X,Y)$ endowed with the projective norm π. That is

$$X \otimes_\pi Y := span\{x \otimes y : (x,y) \in X \times Y\}$$

where

$$x \otimes y : B(X,Y) \to \mathbb{K}$$

$$B \mapsto (x \otimes y)(B) = B(x,y)$$

and

$$\pi(u) = inf\left\{\sum_{j=1}^{n} \|x_j\|\|y_j\| : u = \sum_{j=1}^{n} x_j \otimes y_j\right\}$$

(6) We denote by $X\widehat{\otimes}_\pi Y$ the completion of $X \otimes_\pi Y$.

(7) If $S \in L(X)$ and $T \in L(Y)$, we denote by $S \otimes_\pi T$ the unique bounded operator of $L(X\widehat{\otimes}_\pi Y)$ such that $S \otimes_\pi T(x \otimes y) = S(x) \otimes T(y)$ for every $x \in X, y \in Y$ [2, page 18, Proposition 2.3].

(8) If $f \in X^*$ and $g \in Y^*$, we denote by $f \otimes_\pi g$ the bounded operator of $(X\widehat{\otimes}_\pi Y)^*$ such that $f \otimes_\pi g(x \otimes y) = f(x)g(y)$ for every $x \in X, y \in Y$.

(9) A sequence $\mathscr{F} = ((x_n, y_n^*))_{n\in\mathbb{N}^*} \subset X \times X^*$ is called a paire of X.

(10) The paire $\mathscr{F}$ of X is called a Schauder frame (resp. unconditional Schauder frame) of X if for all $x \in X$, the serie $\sum y_n^*(x)x_n$ is convergent (resp. unconditionally convergent) in X to x.

(11) If $\mathscr{F}$ is a Schauder frame of X, we denote by $K_{\mathscr{F}}$ the finite quantity

$$K_{\mathscr{F}} := \sup_{x\in\mathbb{B}_E,\, n\in\mathbb{N}^*} \left(\left\|\sum_{k=1}^{n} b_k^*(x)\, a_k\right\|\right)$$

(12) $((x_n, y_n^*))_{n\in\mathbb{N}^*}$ is said to be a besselian paire of X if there exists a constant $A > 0$ such that

$$\sum_{k=1}^{+\infty} |y_n^*(x)|\,|y^*(x_n)| \le A\,\|x\|_X\,\|y^*\|_{X^*}$$

for each $(x, y^*) \in X \times X^*$.

(13) $((x_n, y_n^*))_{n\in\mathbb{N}^*}$ is said to be a besselian Schauder frame of X if it is both a Schauder frame and a besselian paire.

(14) Let $\theta : \mathbb{N}^* \longrightarrow \mathbb{N}^* \times \mathbb{N}^*$ be the bijective mapping defined by:

$$\theta(n) = \begin{cases} (\theta_1(n), \theta_2(n)) = (k,k), \ \ if \ n = k^2 \\ (\theta_1(n), \theta_2(n)) = (n-k^2, k+1), \ \ if \ k^2 < n \leq k^2+k+1 \\ (\theta_1(n), \theta_2(n)) = (k+1, (k+1)^2+1-n), \ \ if \ k^2+k+1 < n < (k+1)^2 \end{cases}$$

Then the order $\theta(1), \theta(2), ...$ is called the square ordiring in $\mathbb{N}^* \times \mathbb{N}^*$.

(15) We denote by $l^1(\mathbb{K})$ the $\mathbb{K}$-vector space of sequences $\lambda := (\lambda_n)_{n\in\mathbb{N}^*}$ such that $\lambda_n \in \mathbb{K}$ for each $n \in \mathbb{N}^*$ and $\sum_{n=1}^{+\infty} |\lambda_n| < +\infty$. It is a classical result that $l^1(\mathbb{K})$ is a Banach space for the norm:

$$\begin{aligned} \|.\|_{l^1(\mathbb{K})} : \quad & l^1(\mathbb{K}) \longrightarrow \mathbb{R}^+ \\ & (\lambda_n)_{n\in\mathbb{N}^*} \longmapsto \sum_{n=1}^{+\infty} |\lambda_n| \end{aligned}$$

(16) We denote by $l^1(X)$ the $\mathbb{K}$-vector space of sequences $x := (x_n)_{n\in\mathbb{N}^*}$ such that $x_n \in X$ for each $n \in \mathbb{N}^*$ and $\sum_{n=1}^{+\infty} \|x_n\|_X < +\infty$. It is a classical result that $l^1(X)$ is a Banach space for the norm:

$$\begin{aligned} \|.\|_{l^1(X)} : \quad & l^1(X) \longrightarrow \mathbb{R}^+ \\ & (x_n)_{n\in\mathbb{N}^*} \longmapsto \sum_{n=1}^{+\infty} \|x_n\|_X \end{aligned}$$

(17) We denote by $L_1(\mu, \mathbb{R})$ the Banach space of equivalence classes of Lebesgue integrable functions $f : [0,1] \longrightarrow \mathbb{R}$, with the norm

$$\|f\|_{L_1(\mu,\mathbb{R})} = \int_0^1 |f| d\mu$$

(18) A function $f : [0,1] \longrightarrow X$ is said to be μ-measurable simple function if there is a partition of $[0,1]$ into disjoint measurable subsets A_1, ..., A_n and distinct nonzero elements x_1, ..., x_n of X such that $f(\omega) = x_i$ if $\omega \in A_i$. We may then write

$$f = \sum_{j=1}^{n} \chi_{A_j} x_j$$

(19) Let $f = \sum_{j=1}^{n} \chi_{A_j} x_j$ be a μ-measurable simple function and suppose that each of the set E_j has finite μ-measure. The integral of f over a measurable subset $A \subset [0,1]$ is defined

to be

$$\int_A f d\mu = \sum_{j=1}^{n} \mu(A_j \cap A) x_j$$

(20) A function $f : [0,1] \longrightarrow X$ is said to be μ-measurable function if there is a functions $(f)_n$ of μ-measurable simple functions that converges almost everywhere to f.

(21) A function $f : [0,1] \longrightarrow X$ is said to be Bochner integrable if there is a functions $(f)_n$ of μ-measurable simple functions that converges almost everywhere to f and satisfing

$$\lim_{n\to+\infty} \int_0^1 \|f - f_n\| d\mu = 0$$

The Bochner integral of f over a measurable subset A of $[0,1]$ is

$$\int_A f d\mu = \lim_{n\to+\infty} \int_A f_n d\mu$$

(22) We denote by $L_1(\mu, X)$ the Banach space of equivalence classes of Bochner integrable functions $f : [0,1] \longrightarrow X$, with the norm

$$\|f\|_{L_1(\mu,X)} = \int_0^1 \|f\| d\mu$$

Remark 1. *For a besselian paire $\mathscr{F}$ of X, the quantity*

$$\mathscr{L}_{\mathscr{F}} := \sup_{(u,v^*)\in\mathbb{B}_X\times\mathbb{B}_{X^*}} \left(\sum_{n=1}^{+\infty} |y_n^*(u)| \, |v^*(x_n)| \right)$$

is finite and for each $(x,y) \in X \times X^*$, the following inequality holds

$$\sum_{n=1}^{+\infty} |y_n^*(x)| \, |y^*(x_n)| \leq \mathscr{L}_{\mathscr{F}} \|x\|_X \|y^*\|_{X^*}$$

The quantity $\mathscr{L}_{\mathscr{F}}$ is then called the constant of the besselian paire $\mathscr{F}$.

For all the material on Banach spaces, Hilbertian frames or tensor product of Banach spaces, one can refer to [22], [11], [12], [20], [19], [9], [15], [18], [10], [4], [24], [2] and [14].

3. Tensor Product of Schauder Frames

Definition 1. *Let $(E_1, \|.\|_{E_1})$ and $(E_2, \|.\|_{E_2})$ be two Banach spaces over $\mathbb{K}$ and $\mathscr{F}_1 := ((a_n, b_n^*))_{n\in\mathbb{N}^*}$ and $\mathscr{F}_2 := ((x_n, y_n^*))_{n\in\mathbb{N}^*}$ be Schauder frames of E_1 and E_2 respectively. The sequence $\left(\left(a_{\theta_1(n)} \otimes x_{\theta_2(n)}, b^*_{\theta_1(n)} \otimes_\pi y^*_{\theta_2(n)}\right)\right)_{n\in\mathbb{N}^*}$ is called the tensor product of $\mathscr{F}_1$ and $\mathscr{F}_2$ (in this order) under the square ordering and is denoted by $\mathscr{F}_1 \otimes_\pi \mathscr{F}_2$.*

The sequence $\left(\left(a_{\theta_1(n)}\otimes x_{\theta_2(n)}, b^*_{\theta_1(n)}\otimes_\pi y^*_{\theta_2(n)}\right)\right)_{n\in\mathbb{N}^*}$ is represented as in [2, page 88] by the following diagram:

$$\begin{array}{ccccccc}
(a_1\otimes x_1, b_1^*\otimes_\pi y_1^*) & \rightarrow & (a_1\otimes x_2, b_1^*\otimes_\pi y_2^*) & & (a_1\otimes x_3, b_1^*\otimes_\pi y_3^*) & & (a_1\otimes x_4, b_1^*\otimes_\pi y_4^*) \\
 & & \downarrow & & \downarrow & & \downarrow \\
(a_2\otimes x_1, b_2^*\otimes_\pi y_1^*) & \leftarrow & (a_2\otimes x_2, b_2^*\otimes_\pi y_2^*) & & (a_2\otimes x_3, b_2^*\otimes_\pi y_3^*) & & (a_2\otimes x_4, b_2^*\otimes_\pi y_4^*) \\
 & & & & \downarrow & & \downarrow \\
(a_3\otimes x_1, b_3^*\otimes_\pi y_1^*) & \leftarrow & (a_3\otimes x_2, b_3^*\otimes_\pi y_2^*) & \leftarrow & (a_3\otimes x_3, b_3^*\otimes_\pi y_3^*) & & (a_3\otimes x_4, b_3^*\otimes_\pi y_4^*) \\
 & & & & & & \downarrow \\
 & & & \dots & (a_4\otimes x_3, b_4^*\otimes_\pi y_3^*) & \leftarrow & (a_4\otimes x_4, b_4^*\otimes_\pi y_4^*)
\end{array}$$

Thus the square ordering is: $(a_1\otimes x_1, b_1^*\otimes_\pi y_1^*)$, $(a_1\otimes x_2, b_1^*\otimes_\pi y_2^*)$, $(a_2\otimes x_2, b_2^*\otimes_\pi y_2^*)$, $(a_2\otimes x_1, b_2^*\otimes_\pi y_1^*)$, $\left(a_1\otimes x_3, b_1^*\otimes_\pi y_3^*\right)$, $\left(a_2\otimes x_3, b_2^*\otimes_\pi y_3^*\right)$, $\left(a_3\otimes x_3, b_3^*\otimes_\pi y_3^*\right)$, $\left(a_3\otimes x_2, b_3^*\otimes_\pi y_2^*\right)$, $\left(a_3\otimes x_1, b_3^*\otimes_\pi y_1^*\right)$, $(a_1\otimes x_4, b_1^*\otimes_\pi y_4^*)$, $(a_2\otimes x_4, b_2^*\otimes_\pi y_4^*)$,...

We denote, for each $n\in\mathbb{N}^*$, by $T_{n,1}$ and $T_{n,2}$ the linear mappings $T_{n,1}: E_1\longrightarrow E_1$ and $T_{n,2}: E_2\longrightarrow E_2$ defined by the formulas:

$$\begin{cases} T_{n,1}(x) = \sum\limits_{j=1}^{n} b_j^*(x)a_j, \;\; x\in E_1 \\ T_{n,2}(y) = \sum\limits_{j=1}^{n} y_j^*(y)x_j, \;\; y\in E_2 \end{cases}$$

For each $n\in\mathbb{N}^*$, we denote by

$$\begin{aligned}\sigma_n &:= \left((a_1\otimes x_1, b_1^*\otimes_\pi y_1^*), (a_1\otimes x_2, b_1^*\otimes_\pi y_2^*), (a_2\otimes x_2, b_2^*\otimes_\pi y_2^*), \dots\right) \\ &= \left(\left(a_j\otimes x_k, b_j^*\otimes_\pi y_k^*\right)\right)_{(j,k)\in I_n}\end{aligned}$$

the intervall of the previous square ordering which consists of the n first comples. It follows from [2, page 18, Proposition 2.3] that: for each $n\in\mathbb{N}^*$, the operator $T_n: E_1\widehat{\otimes}_\pi E_2 \longrightarrow E_1\widehat{\otimes}_\pi E_2$ defined by:

$$T_n := \sum_{(j,k)\in I_n} \left(b_j^*\otimes a_j\right)\otimes_\pi \left(y_k^*\otimes x_k\right)$$

is continuous. As in [2, page 88], we have for each $n\in\mathbb{N}^*$:

$$T_n = \begin{cases} T_{k,1}\otimes_\pi T_{k,2}, \;\; if\; n=k^2 \\ T_{k,1}\otimes_\pi T_{k,2} + T_{n-k^2,1}\otimes_\pi \left(y_{k+1}^*\otimes_\pi x_{k+1}\right), \;\; if\; k^2 < n \le k^2+k+1 \\ T_{k+1,1}\otimes_\pi T_{k+1,2} - \left(b_{k+1}^*\otimes_\pi a_{k+1}\right)\otimes_\pi T_{(k+1)^2-n,2}, \;\; if\; k^2+k+1 < n < (k+1)^2 \end{cases}$$

where $k\in\mathbb{N}^*$.

Proposition 1. *Let E_1 and E_2 be Banach spaces and $\mathscr{F}_1 := ((a_n, b_n^*))_{n\in\mathbb{N}^*}$ and $\mathscr{F}_2 := ((x_n, y_n^*))_{n\in\mathbb{N}^*}$ be Schauder frames of E_1 and E_2 respectively. Then the paire $\mathscr{F}_1 \otimes_\pi \mathscr{F}_2$ is a Schauder frame of $E_1 \widehat{\otimes}_\pi E_2$.*

Proof: Firstly we denote for each $n \in \mathbb{N}$ by r_n the closest integer to $\sqrt{n}$. We subdivide the proof of the theorem into 4 steps:

(1) Proof that: For each $u_1 \in E_1$ and $u_2 \in E_2$, we have for each $n \in \mathbb{N}^*$:

$$\pi\left(T_n(u_1\otimes u_2) - T_{r_n,1}(u_1)\otimes T_{r_n,2}(u_2)\right) \leq K_{\mathscr{F}_1}\|u_1\|_{E_1}\left\|y^*_{r_n+1}(u_2)x_{r_n+1}\right\|_{E_2} + K_{\mathscr{F}_2}\|u_2\|_{E_2}\left\|b^*_{r_n}(u_1)a_{r_n}\right\|_{E_1}$$

Indeed, there exists three cases for $n \in \mathbb{N}^*$:

(a) If $n = k^2$ with $k \in \mathbb{N}^*$. Then $k = \sqrt{n}$, that is $k = r_n = \sqrt{n}$. It follows that:

$$\begin{aligned}(1)\qquad &\pi\left[T_n(u_1\otimes u_2) - T_{r_n,1}(u_1)\otimes T_{r_n,2}(u_2)\right]\\ &= \pi\left[\left(T_{k,1}\otimes_\pi T_{k,2}\right)(u_1\otimes u_2) - T_{k,1}(u_1)\otimes T_{k,2}(u_2)\right]\\ &= 0\\ &\leq K_{\mathscr{F}_1}\|u_1\|_{E_1}\left\|y^*_{r_n+1}(u_2)x_{r_n+1}\right\|_{E_2} + K_{\mathscr{F}_2}\|u_2\|_{E_2}\left\|b^*_{r_n}(u_1)a_{r_n}\right\|_{E_1}\end{aligned}$$

Hence the inequality (1) holds in this case.

(b) If $k^2 < n \leq k^2+k+1$ with $k \in \mathbb{N}^*$. Then $k = r_n$. It follows that:

$$\begin{aligned}&\pi\left[T_n(u_1\otimes u_2) - T_{r_n,1}(u_1)\otimes T_{r_n,2}(u_2)\right]\\ &= \pi\left[\left[T_{n-k^2,1}\otimes_\pi\left(y^*_{k+1}\otimes x_{k+1}\right)\right](u_1\otimes u_2)\right]\\ &= \left\|T_{n-k^2,1}(u_1)\right\|_{E_1}\left\|\left(y^*_{k+1}\otimes x_{k+1}\right)(u_2)\right\|_{E_2}\\ &= \left\|T_{n-k^2,1}(u_1)\right\|_{E_1}\left\|y^*_{k+1}(u_2)x_{k+1}\right\|_{E_2}\end{aligned}$$

But

$$\begin{aligned}\left\|T_{n-k^2,1}(u_1)\right\|_{E_1} &= \left\|\sum_{j=1}^{n-k^2} b^*_j(u_1)a_j\right\|_{E_1}\\ &\leq K_{\mathscr{F}_1}\|u_1\|_{E_1}\end{aligned}$$

It follows that:

$$\begin{aligned}
&\pi\left[T_n(u_1\otimes u_2)-T_{r_n,1}(u_1)\otimes T_{r_n,2}(u_2)\right]\\
&\leq K_{\mathscr{F}_1}\|u_1\|_{E_1}\left\|y^*_{k+1}(u_2)x_{k+1}\right\|_{E_2}\\
&\leq K_{\mathscr{F}_1}\|u_1\|_{E_1}\left\|y^*_{r_n+1}(u_2)x_{r_n+1}\right\|_{E_2}+K_{\mathscr{F}_2}\|u_2\|_{E_2}\left\|b^*_{r_n}(u_1)a_{r_n}\right\|_{E_1}
\end{aligned}$$

Hence the inequality (1) holds in this case.

(c) If $k^2+k+1<n<(k+1)^2$ with $k\in\mathbb{N}^*$. In this cas $r_n=k+1$. It follows that:

$$\begin{aligned}
&\pi\left[T_n(u_1\otimes u_2)-T_{r_n}(u_1)\otimes T_{r_n}(u_2)\right]\\
&=\pi\left[\left[\left(b^*_{k+1}\otimes a_{k+1}\right)\otimes T_{(k+1)^2-n,2}\right](u_1\otimes u_2)\right]\\
&=\left\|T_{(k+1)^2-n,2}(u_2)\right\|_{E_2}\left\|b^*_{k+1}(u_1)a_{k+1}\right\|_{E_1}
\end{aligned}$$

But

$$\begin{aligned}
\left\|T_{(k+1)^2-n,2}(u_2)\right\|_{E_2}&=\left\|\sum_{j=1}^{(k+1)^2-n}y^*_j(u_2)x_j\right\|_{E_2}\\
&\leq K_{\mathscr{F}_2}\|u_2\|_{E_2}
\end{aligned}$$

It follows that:

$$\begin{aligned}
&\pi\left[T_n(u_1\otimes u_2)-T_{r_n}(u_1)\otimes T_{r_n}(u_2)\right]\\
&\leq K_{\mathscr{F}_2}\|u_2\|_{E_2}\left\|b^*_{k+1}(u_1)a_{k+1}\right\|_{E_1}\\
&\leq K_{\mathscr{F}_2}\|u_2\|_{E_2}\left\|b^*_{r_n}(u_1)a_{r_n}\right\|_{E_1}\\
&\leq K_{\mathscr{F}_1}\|u_1\|_{E_1}\left\|y^*_{r_n+1}(u_2)x_{r_n+1}\right\|_{E_2}+K_{\mathscr{F}_2}\|u_2\|_{E_2}\left\|b^*_{r_n}(u_1)a_{r_n}\right\|_{E_1}
\end{aligned}$$

Hence the inequality (1) holds in this case.

Consequently, the inequality (1) holds for each $n\in\mathbb{N}^*$, $u_1\in E_1$ and $u_2\in E_2$.

(2) Proof that the following relation holds in $(E_1\otimes_\pi E_2,\pi)$ for every $u\in E_1\otimes_\pi E_2$:

$$\lim_{n\longrightarrow+\infty}T_n(u)=u$$

Indeed, for each $u \in E_1 \otimes_\pi E_2$, there exists $N \in \mathbb{N}^*$ and $u_1,...,u_N$ in E_1, $v_1,...,v_N$ in E_2 such that

$$u = \sum_{j=1}^{N} v_j \otimes w_j$$

Hence we have for each $n \in \mathbb{N}^*$:

$$\begin{aligned}
\pi\left[T_n(u) - u\right] &= \pi\left[\sum_{j=1}^{N}\left(T_n\left(v_j \otimes w_j\right) - v_j \otimes w_j\right)\right] \\
&\leq \sum_{j=1}^{N} \pi\left[T_n\left(v_j \otimes w_j\right) - v_j \otimes w_j\right] \\
&\leq \sum_{j=1}^{N} \pi\left[T_n\left(v_j \otimes w_j\right) - T_{r_n,1}(v_j) \otimes T_{r_n,2}(w_j)\right] \\
&+ \sum_{j=1}^{N} \pi\left[T_{r_n,1}(v_j) \otimes T_{r_n,2}(w_j) - v_j \otimes w_j\right] \\
&\leq \sum_{j=1}^{N} K_{\mathscr{F}_1} \left\|v_j\right\|_{E_1} \left\|y^*_{r_n+1}(w_j) x_{r_n+1}\right\|_{E_2} \\
&+ \sum_{j=1}^{N} K_{\mathscr{F}_2} \left\|w_j\right\|_{E_2} \left\|b^*_{r_n}(v_j) x_{r_n}\right\|_{E_1} \\
&+ \sum_{j=1}^{N} \pi\left[\left(T_{r_n,1}(v_j) - v_j\right) \otimes T_{r_n,2}(w_j) + v_j \otimes \left(T_{r_n,2}(w_j) - w_j\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^{N} K_{\mathscr{F}_1} \left\|v_j\right\|_{E_1} \left\|y^*_{r_n+1}(w_j) x_{r_n+1}\right\|_{E_2} \\
&+ \sum_{j=1}^{N} K_{\mathscr{F}_2} \left\|w_j\right\|_{E_2} \left\|b^*_{r_n}(v_j) x_{r_n}\right\|_{E_1} \\
&+ \sum_{j=1}^{N} \left(\left\|T_{r_n,1}(v_j) - v_j)\right\|_{E_1} \left\|T_{r_n,2}(w_j)\right\|_{E_2} + \left\|v_j\right\|_{E_1} \left\|T_{r_n,2}(w_j) - w_j\right\|_{E_2}\right)
\end{aligned}$$

But we have, for each $n \in \mathbb{N}^*$ and $j \in \{1,...,N\}$:

$$\left\|T_{r_n,2}(w_j)\right\|_{E_2} \leq K_{\mathscr{F}_2} \left\|w_j\right\|_{E_2}$$

It follows that:

$$\begin{aligned}\pi\left[T_n(u)-u\right] \leq & \sum_{j=1}^{N} K_{\mathscr{F}_1}\left\|v_j\right\|_{E_1}\left\|y^*_{r_n+1}(w_j)x_{r_n+1}\right\|_{E_2} \\ & + \sum_{j=1}^{N} K_{\mathscr{F}_2}\left\|w_j\right\|_{E_2}\left\|b^*_{r_n}(v_j)x_{r_n}\right\|_{E_1} \\ & + \sum_{j=1}^{N}\left(K_{\mathscr{F}_2}\left\|w_j\right\|_{E_2}\left\|T_{r_n,1}(v_j)-v_j)\right\|_{E_1}+\left\|v_j\right\|_{E_1}\left\|T_{r_n,2}(w_j)-w_j\right\|_{E_2}\right)\end{aligned}$$

But we know that:

$$\begin{cases} \lim\limits_{n\to+\infty} r_n = +\infty \\ \lim\limits_{n\to+\infty}\left\|y^*_{r_n}(w_j)x_{r_n}\right\|_{E_2} = \lim\limits_{n\to+\infty}\left\|b^*_{r_n}(v_j)a_{r_n}\right\|_{E_1} = 0 \\ \lim\limits_{n\to+\infty}\left\|T_{r_n,1}(v_j)-v_j\right\|_{E_1} = \lim\limits_{n\to+\infty}\left\|T_{r_n,2}(w_j)-w_j\right\|_{E_2} = 0 \end{cases}$$

where $j\in\{1,...,N\}$. Consequently, we have

$$\lim_{n\to+\infty}\pi\left[T_n(u)-u\right]=0,\ \ u\in E_1\otimes_\pi E_2$$

(3) Proof that the following inequality holds for each $n\in\mathbb{N}^*$ and $u\in E_1\otimes_\pi E_2$:

$$\pi\left[T_n(u)\right]\leq 9K_{\mathscr{F}_1}K_{\mathscr{F}_2}\pi(u) \tag{2}$$

Indeed, let $u\in E_1\otimes_\pi E_2$ and let;

$$u=\sum_{j=1}^{N} v_j\otimes_\pi w_j,\ \ v_j\in E_1,;w_j\in E_2$$

be an arbitrary representation of u. We have then for each $n\in\mathbb{N}^*$ thanks to the computation of stap 2:

$$\pi\left[T_n(u)\right]\leq\pi(u)+\pi\left[T_n(u)-u\right]$$

But we have

$$\begin{cases} \left\|y^*_{r_n+1}(w_j)x_{r_n+1}\right\|_{E_2}\leq 2K_{\mathscr{F}_2}\left\|w_j\right\|_{E_2} \\ \left\|b^*_{r_n+1}(v_j)a_{r_n+1}\right\|_{E_1}\leq 2K_{\mathscr{F}_1}\left\|v_j\right\|_{E_1} \\ \left\|T_{r_n,1}(v_j)-v_j)\right\|_{E_1}\leq 2K_{\mathscr{F}_1}\left\|v_j\right\|_{E_1} \\ \left\|T_{r_n,2}(w_j)-w_j)\right\|_{E_2}\leq 2K_{\mathscr{F}_1}K_{\mathscr{F}_2}\left\|w_j\right\|_{E_2} \end{cases}$$

It follows that

$$\pi\left[T_n(u)\right] \leq \pi(u) + 8K_{\mathscr{F}_1}K_{\mathscr{F}_2}\sum_{j=1}^{N}\left\|v_j\right\|_{E_1}\left\|w_j\right\|_{E_2}$$

Taking the infinum over all the possibel representations of u we obtain that:

$$\pi\left[T_n(u)\right] \leq \pi(u) + 8K_{\mathscr{F}_1}K_{\mathscr{F}_2}\pi(u)$$

(4) Let $u \in E_1\widehat{\otimes}_\pi E_2$ and $(\xi_m)_{m\in\mathbb{N}^*} \subset E_1 \otimes_\pi E_2$ such that $\underset{m\to+\infty}{lim}\xi_m = u$. For each $n \in \mathbb{N}^*$ and $m \in \mathbb{N}^*$ we have:

$$\begin{aligned}\pi\left[T_n(u)-u\right] &\leq \pi\left[T_n(u)-T_n(\xi_m)\right] + \pi\left[T_n(\xi_m)-\xi_m\right] + \pi\left[\xi_m - u\right]\\ &\leq \pi\left[T_n(u-\xi_m)\right] + \pi\left[T_n(\xi_m)-\xi_m\right] + \pi\left[\xi_m - u\right]\\ &\leq 10K_{\mathscr{F}_1}K_{\mathscr{F}_2}\pi\left[\xi_m - u\right] + \pi\left[T_n(\xi_m)-\xi_m\right]\end{aligned}$$

It follows that:

$$\overline{\underset{n\to+\infty}{lim}}\,\pi\left[T_n(u)-u\right] \leq 10K_{\mathscr{F}_1}K_{\mathscr{F}_2}\pi\left[\xi_m - u\right]$$

Hence

$$\overline{\underset{n\to+\infty}{lim}}\,\pi\left[T_n(u)-u\right] = 0,$$

that is,

$$\underset{n\to+\infty}{lim}\,\pi\left[T_n(u)-u\right] = 0.$$

Consequently, $\mathscr{F}_1 \otimes_\pi \mathscr{F}_2$ is a Schauder frame of $E_1\widehat{\otimes}_\pi E_2$.

4. Tensor Product of Besselian Paires

Definition 2. *Let $S := (y_n)_{n\in\mathbb{N}^*}$ be a sequence of vectors of a Banach space $(E, \|.\|_E)$. We say that S is a rectangular sequence of E if there exists a constant $C > 0$ such that the following condition holds for each $n \in \mathbb{N}^*$ and $(\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$:*

$$\sum_{j=1}^{j=n}\left|\alpha_j\right|\left\|y_j\right\|_E \leq C\left\|\sum_{j=1}^{j=n}\alpha_j y_j\right\|_E \tag{3}$$

Example 1. *It is clear that the sequence $(e_n)_{n\in\mathbb{N}^*}$ of vectors of $l^1(\mathbb{K})$ defined by $e_n := (\delta_{mn})_{m\in\mathbb{N}^*}$ is a rectangular sequence of $l^1(\mathbb{K})$.*

Proposition 2. *Let $(E_1, \|.\|_{E_1})$ and $(E_2, \|.\|_{E_2})$ be Banach spaces over $\mathbb{K}$ and $\mathscr{F}_1 = ((a_n, b_n^*))_{n\in\mathbb{N}^*}$ and $\mathscr{F}_2 = ((x_n, y_n^*))_{n\in\mathbb{N}^*}$ be besselian sequences of E_1 and E_2 respectively. We assume that one of the sequence $S_1 = (a_n)_{n\in\mathbb{N}^*}$ or $S_2 = (x_n)_{n\in\mathbb{N}^*}$ is a rectangular sequence in its ambiant Banach space. Then $\left(\left(a_{\theta_1(n)} \otimes x_{\theta_2(n)}, b^*_{\theta_1(n)} \otimes_\pi y^*_{\theta_2(n)}\right)\right)_{n\in\mathbb{N}^*}$ is a besselian sequence of $E_1\widehat{\otimes}_\pi E_2$.*

Proof: To make our reasoning clear, we assume that S_2 is such a rectangular sequence. Let B be a continuous linear form on $E_1\widehat{\otimes}_\pi E_2$. Then, accorging to [2, page 22, Theorem 2.9], there exists a continuous bilinear form $\widetilde{B} : E_1 \times E_2 \longrightarrow \mathbb{K}$ such that:

(1) $B(x\otimes y) = \widetilde{B}(x,y),\ \ x \in E_1, y \in E_2$
(2) $\pi^*(B) = \left\|\widetilde{B}\right\|_{B(E_1\times E_2)}$

For each $x \in E_1, y \in E_2$, the mapping:

$$\begin{aligned}\widetilde{B}(.,y) : E_1 &\longrightarrow \mathbb{K}\\ z &\longmapsto \widetilde{B}(z,y)\end{aligned}$$

is a continuous linear mapping on E_1. It follows that:

$$\sum_{n=1}^{+\infty} \left|b_n^*(x)\widetilde{B}(a_n,y)\right| \leq \mathscr{L}_{\mathscr{F}_1} \|x\|_{E_1} \left\|\widetilde{B}(.,y)\right\|_{E_1^*}$$

Hence, we obtain that:

$$\sum_{n=1}^{+\infty} \left|b_n^*(x)\widetilde{B}(a_n,y)\right| \leq \mathscr{L}_{\mathscr{F}_1} \|x\|_{E_1} \|y\|_{E_2} \left\|\widetilde{B}\right\|_{B(E_1\times E_2)} \tag{4}$$

So we can write, by means of (4)

$$\begin{aligned}\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \left|\left(b_n^*\otimes_\pi y_m^*\right)(x\otimes y)\right| \left|B\left(a_n\otimes x_m\right)\right| &\leq \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} |b_n^*(x)|\,|y_m^*(y)| \left|\widetilde{B}\left(a_n,x_m\right)\right|\\
&\leq \sum_{m=1}^{+\infty} |y_m^*(y)| \sum_{n=1}^{+\infty} |b_n^*(x)| \left|\widetilde{B}\left(a_n,x_m\right)\right|\\
&\leq \sum_{m=1}^{+\infty} |y_m^*(y)|\, \mathscr{L}_{\mathscr{F}_1} \|x\|_{E_1} \|x_m\|_{E_2} \left\|\widetilde{B}\right\|_{B(E_1\times E_2)}\\
&\leq \mathscr{L}_{\mathscr{F}_1} \|x\|_{E_1} \left\|\widetilde{B}\right\|_{B(E_1\times E_2)} \sum_{m=1}^{+\infty} |y_m^*(y)|\, \|x_m\|_{E_2}\end{aligned}$$

But $S_2 = (x_m)_{m\in\mathbb{N}^*}$ is a rectangular sequence in E_2, hence there exists a constant $C(S_2) > 0$ such that:

$$\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \left|\left(b_n^* \otimes_\pi y_m^*\right)(x\otimes y)\right| \left|B\left(a_n \otimes x_m\right)\right| \leq \mathscr{L}_{\mathscr{F}_1} C(S_2) \left\|x\right\|_{E_1} \left\|y\right\|_{E_2} \pi^*(B) \tag{5}$$

Now let $u \in E_1 \widehat{\otimes}_\pi E_2$, $u = \sum\limits_{n=1}^{+\infty} u_1 \otimes v_2$ be an arbitrary tensor of $E_1 \widehat{\otimes}_\pi E_2$. Using the ineqality (5) we have:

$$\begin{aligned}\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \left|\left(b_n^* \otimes_\pi y_m^*\right)(u)\right| \left|B\left(a_n \otimes x_m\right)\right| &\leq \sum_{j=1}^{+\infty}\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \left|\left(b_n^* \otimes_\pi y_m^*\right)\left(\left(u_j \otimes v_j\right)\right)\right| \left|B\left(a_n \otimes x_m\right)\right| \\ &\leq \pi^*(B) \mathscr{L}_{\mathscr{F}_1} C(S_2) \sum_{j=1}^{+\infty} \left\|u_j\right\|_{E_1} \left\|v_j\right\|_{E_2}\end{aligned}$$

Taking the infimum of $\sum\limits_{j=1}^{+\infty} \left\|u_j\right\|_{E_1} \left\|v_j\right\|_{E_2}$ over all the representation $u = \sum\limits_{j=1}^{+\infty} u_j \otimes v_j$ of u, we obtain the following inequality

$$\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \left|\left(b_n^* \otimes_\pi y_m^*\right)(u)\right| \left|B\left(a_n \otimes x_m\right)\right| \leq \mathscr{L}_{\mathscr{F}_1} C(S_2) \pi(u) \pi^*(B)$$

□

Corollary 1. *Let E be a Banach space with a besselian Schauder frame. Then the Banach space $l^1(E)$ has a besselian Schauder frame.*

Proof:

(1) Firstly we prove that the paire $\left((e_n, e_n^*)\right)_{n\in\mathbb{N}^*}$ (where $e_m^*(e_n) = \delta_{mn}$ for each m and n in $\mathbb{N}^*$) is a besselian Schauder frame of the Banach space $l^1(\mathbb{K})$ [23]. indeed, let $y^* \in \left(l^1\left(\mathbb{K}\right)\right)^*$ and $x = (x_n)_{n\in\mathbb{N}^*} \in l^1\left(\mathbb{K}\right)$. We have:

$$x = \sum_{n=1}^{+\infty} e_n^*\left(x\right) e_n$$

Furthermore we have :

$$\begin{aligned}\sum_{n=1}^{+\infty} \left|e_n^*\left(x\right)\right| \left|y^*\left(e_n\right)\right| &\leq \sum_{n=1}^{+\infty} \left|x_n\right| \left\|y^*\right\|_{\left(l^1(\mathbb{K})\right)^*} \left\|e_n\right\|_{l^1(\mathbb{K})} \\ &\leq \left\|x\right\|_{l^1(\mathbb{K})} \left\|y^*\right\|_{\left(l^1(\mathbb{K})\right)^*}\end{aligned}$$

It follows that $\left((e_n, e_n^*)\right)_{n\in\mathbb{N}}$ is a besselian Schauder frame of $l^1\left(\mathbb{K}\right)$.

(2) Since $(e_n)_{n\in\mathbb{N}^*}$ is rectangular sequence in $l^1(\mathbb{K})$, it follows thanks to the propositions (1) and (2) that the Banach space $l^1(\mathbb{K})\widehat{\otimes}_\pi E$ has a besselian Schauder frame. But we know, according to [2, page 20] that there exists an isometric isomorphism

$$J : l^1(\mathbb{K})\widehat{\otimes}_\pi E \longrightarrow l^1(E)$$

It follows that the Banach space $l^1(E)$ has a besselian Schauder frame.

□

Example 2. *Let X be a separable Banach space with a Schauder frame $((x_n, y_n^*))_{n\in\mathbb{N}^*}$. The Banach space $L_1(\mu, X)$ has a Schauder frame.*

Proof. We known that $L_1(\mu,\mathbb{R})$ has a Schauder basis [22, page 359, Example 4.1.27]. Then $L_1(\mu,\mathbb{R})$ has a Schauder frame $((a_n, b_n^*))_{n\in\mathbb{N}^*}$. The proposition (1) implies that the sequence $\left(\left(a_{\theta_1(n)}\otimes x_{\theta_2(n)}, b^*_{\theta_1(n)}\otimes\pi\, y^*_{\theta_2(n)}\right)\right)_{n\in\mathbb{N}^*}$ is a Schauder frame of $L_1(\mu,\mathbb{R})\widehat{\otimes}_\pi X$. According to [2, page 29], there is an isometry

$$J : L_1(\mu,\mathbb{R})\widehat{\otimes}_\pi X \longrightarrow L_1(\mu, X)$$

such that $J(f\otimes x)(\omega) = f(\omega)x$, for each $f\in L_1(\mu,\mathbb{R})$ and $x\in X$.
Then $\left(\left(J\left(a_{\theta_1(n)}\otimes x_{\theta_2(n)}\right), \left(b^*_{\theta_1(n)}\otimes\pi\, y^*_{\theta_2(n)}\right)\circ J^{-1}\right)\right)_{n\in\mathbb{N}^*}$ is a Schauder frame of $L_1(\mu, X)$. Indeed, let $f\in L_1(\mu, X)$ then there exists $u\in L_1(\mu,\mathbb{R})\widehat{\otimes}_\pi X$ such that $f = J(u)$

$$\begin{aligned}
f &= J\left(\sum_{n=1}^{+\infty}\left(b^*_{\theta_1(n)}\otimes\pi\, y^*_{\theta_2(n)}\right)(u)\, a_{\theta_1(n)}\otimes x_{\varphi_2(n)}\right)\\
&= \sum_{n=1}^{+\infty}\left(b^*_{\theta_1(n)}\otimes\pi\, y^*_{\theta_2(n)}\right)(u)\, J\left(a_{\theta_1(n)}\otimes x_{\theta_2(n)}\right)\\
&= \sum_{n=1}^{+\infty}\left(b^*_{\theta_1(n)}\otimes\pi\, y^*_{\theta_2(n)}\right)\circ J^{-1}(f)\, J\left(a_{\theta_1(n)}\otimes x_{\theta_2(n)}\right)
\end{aligned}$$

Lemma 1. [23] *We assume that E is a weakly sequentially complete Banach space and that $\mathscr{F} := ((a_n, b_n^*))_{n\in\mathbb{N}^*}$ is a besselian paire of E. Then for each $x\in E$, the series $\sum b_n^*(x)\, a_n$ is unconditionally convergent in E.*

Proof. For each $x \in E$, $y^* \in E^*$ we have:

$$\begin{aligned}\sum_{n=1}^{+\infty} |y^*(b_n^*(x) a_n)| &= \sum_{k=1}^{+\infty} |b_n^*(x) y^*(a_n)| \\ &\leq \mathscr{L}_{\mathscr{F}} \|x\|_E \|y^*\|_{E^*} \\ &< +\infty\end{aligned}$$

Hence the series $\sum b_n^*(x) a_n$ is weakly unconditionally convergent. Then, since E is weakly sequentially complete, the well-known Orlicz's theorem (1929) [20, Proposition.4 ,page 59 and page 66], entails that the series $\sum b_n^*(x) a_n$ is unconditionally convergent. □

Proposition 3.
Let X and Y be Banach spaces such that X is weakly sequentially complete. If $\mathscr{F} = ((A_n, B_n^*))_{n\in\mathbb{N}^*}$ is a besselian Schauder frame of $X\widehat{\otimes}_\pi Y$ then, X has a besselian Schauder frame.

Proof. Let $(x,y) \in X \times Y$, $f \in X^*$ and $g \in Y^*$ such that $g(y) = 1$. We have $x \otimes y \in X\widehat{\otimes}_\pi Y$ and $f \otimes_\pi g \in \left(X\widehat{\otimes}_\pi Y\right)^*$.

(1) Assume that $((A_n, B_n^*))_{n\in\mathbb{N}^*}$ is a besselian paire of $X\widehat{\otimes}_\pi Y$. Then

$$\begin{aligned}\sum_{n=1}^{+\infty} |B_n^*(x\otimes y)|\,|(f \otimes_\pi g)(A_n)| &\leq \mathscr{L}_{\mathscr{F}} \pi(x\otimes y)\, \|f \otimes_\pi g\|_{\left(X\widehat{\otimes}_\pi Y\right)^*} \\ &\leq \mathscr{L}_{\mathscr{F}} \|x\|_X \|y\|_Y \|f\|_{X^*} \|g\|_{Y^*}\end{aligned}$$

Let $n \in \mathbb{N}^*$. According to [2, page 21] there exists sequences $(x_i^n)_{i\in\mathbb{N}^*}$ in X and $(y_i^n)_{i\in\mathbb{N}^*}$ in Y such that

$$A_n = \sum_{i=1}^{+\infty} x_i^n \otimes y_i^n \;\; and \;\; \sum_{i=1}^{+\infty} \|x_i^n\|_X \|y_i^n\|_Y < +\infty$$

consequently, we have

$$\begin{aligned} f\otimes g(A_n) &= \sum_{i=1}^{+\infty} f(x_i^n)g(y_i^n) \\ &= \sum_{i=1}^{+\infty} f(g(y_i^n)x_i^n) \\ &= f\left(\sum_{i=1}^{+\infty} g(y_i^n)x_i^n\right) \\ &= f(a_n) \end{aligned}$$

where $a_n = \sum_{i=1}^{+\infty} g(y_i^n)x_i^n$. Finally, if b_n^* is the linear form $B_n^*(.,y)$ on X then

$$\begin{aligned} \sum_{n=1}^{+\infty} |b_n^*(x)|\,|f(a_n)| &= \sum_{n=1}^{+\infty} |B_n^*(x\otimes y)|\,|(f\otimes_\pi g)(A_n)| \\ &\leq (\mathscr{L}_{\mathscr{F}}\,\|y\|_Y\,\|g\|_{Y^*})\,\|x\|_X\,\|f\|_{X^*} \end{aligned}$$

That is, $((a_n,b_n^*))_{n\in\mathbb{N}^*}$ is a besselian paire of X.

(2) Assume that $((A_n,B_n^*))_{n\in\mathbb{N}^*}$ is a Schauder frame of $X\widehat{\otimes}_\pi Y$. Since $((a_n,b_n^*))_{n\in\mathbb{N}^*}$ is a besselian paire of X then accorging to the lemma 1, the series $\sum b_n^*(x)a_n$ is unconditionally convergent in X. Consequently we have

$$\begin{aligned} f(x) &= f(x)g(y) \\ &= f\otimes_\pi g(x\otimes y) \\ &= f\otimes_\pi g\left(\sum_{n=1}^{+\infty} B_n^*(x\otimes y)A_n\right) \\ &= \sum_{n=1}^{+\infty} B_n^*(x\otimes y) f\otimes_\pi g\,(A_n) \\ &= \sum_{n=1}^{+\infty} b_n^*(x) f\,(a_n) \\ &= f\left(\sum_{n=1}^{+\infty} b_n^*(x)a_n\right) \end{aligned}$$

That is, $((a_n,b_n^*))_{n\in\mathbb{N}^*}$ is a Schauder frame of X.

□

Proposition 4. *Let $((a_n,b_n^*))_{n\in\mathbb{N}^*}$ (resp. $((x_n,y_n^*))_{n\in\mathbb{N}^*}$) a besselian Schauder frame of a Banach space X (resp. of a Banach space Y). The sequence $\mathscr{F}_1\otimes_\pi\mathscr{F}_2$ is not necessarily a besselian Schauder frame of $X\widehat{\otimes}_\pi Y$.*

Example 3. *Let $\mathscr{F}=((e_n,u_n^*))_{n\in\mathbb{N}^*}$ be the standard orthonormal basis for $l_2(\mathbb{K})$. $\mathscr{F}$ is a besselian Schauder frame of $l_2(\mathbb{K})$ but $\mathscr{F}\otimes_\pi\mathscr{F}$ is not a besselian Schauder frame of $l_2(\mathbb{K})\widehat{\otimes}_\pi l_2(\mathbb{K})$.*

Proof. Thanks to Q. Bu and J. Diestel [21] the Banach space $l_2(\mathbb{K})\widehat{\otimes}_\pi l_2(\mathbb{K})$ is weakly sequentially complete. Assume that $\mathscr{F}\otimes_\pi\mathscr{F}$ is a besselian Schauder frame of $l_2(\mathbb{K})\widehat{\otimes}_\pi l_2(\mathbb{K})$. According to lemma 1, $\mathscr{F}\otimes_\pi\mathscr{F}$ is a besselian unconditionally Schauder frame of $l_2(\mathbb{K})\widehat{\otimes}_\pi l_2(\mathbb{K})$. which would contradict the fact that the basis $\mathscr{F}\otimes_\pi\mathscr{F}$ is not unconditionally [2, page 90].

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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