Available online at http://scik.org
J. Math. Comput. Sci. 2023, 13:2
https://doi.org/10.28919/jmcs/7869
ISSN: 1927-5307

ON THE NULL CONTROLLABILITY OF SWITCHED POSITIVE AND PERIODIC POSITIVE SYSTEMS

ILYAS SITLI*, FOUAD LAHMIDI

Laboratory of Analysis, Modeling and Simulation, Faculty of Sciences Ben M'sik, Hassan II University, Casablanca, Morocco

Abstract. In this paper, we study the null controllability of switched positive systems and periodic positive systems. We solve this problem by a technique based on the graph theory.

Keywords: switched positive system; periodic positive system; null controllability; graph Theory.

2020 AMS Subject Classification: 93B05, 93C30, 93C28, 97K30.

1. Introduction

Switched systems, which are a type of hybrid dynamical systems, have piqued the interest of engineers and applied mathematicians in recent years. A switched system is made up of a finite number of differential or difference subsystems and a switching law that determines which subsystem is active at any given time. Switched systems are of particular importance because they are useful for mathematical modeling of a variety of systems, including network control systems, near-space vehicle control systems [1], biological systems [2], ac/dc converters, oscillators [3], chaos generators [4], and so on. The early research on this type of system concentrated almost entirely on the concepts of stability and stabilization [5, 6]. So far, several

*Corresponding author

E-mail address: ilyassitli22@gmail.com

Received December 27, 2022

issues regarding switched systems have been addressed, for example, the problems of reachability and controllability [7, 8, 9, 10].

In this paper we deal with positive discrete-time linear control systems in the state-space model, i.e., systems whose states and inputs are nonnegative. Such systems can be found in a variety of real-world systems, including engineering, economics, pharmacology and medicine, biology, and other disciplines (see Farina and Rinaldi [11], Kaczorek [12], and Luenberger [13]).

The periodic system, possibly the most basic class of time-varying system, has attracted a lot of attention in recent decades. The study of periodic systems is motivated by the fact that many practical systems possess periodic characteristics. A periodic system, for example, can be used to model a multirate sampled data system [14], a pendulum system with cyclic behavior [15]. Positive and periodic linear systems have been investigated in a variety of applications, including biology and chemistry. Indeed, periodic processes occur frequently in nature and engineering, and hence, linear periodic systems can be found in a wide range of domains. A lot of different results on linear periodic systems have been published in the literature. Bougatef et al. [16] and Rami and Napp [17] studied the stability and stabilization problems of discrete-time periodic positive systems. [18] studied periodic positive systems with time delays for the stability analysis problem. See also [19, 20] for those interested in periodic systems research results. The concept of controllability has gotten a lot of attention in control theory. It has been studied by several researchers. Among the most recent studies on controllability [21, 22, 23].

In this paper, we introduce the characterization of the null controllability property in digraph form for two classes (switched and periodic) of discrete-time positive linear systems. The paper is organized as follows. Section 2 contains some definitions and reminders about the digraph of positive matrices. In section 3, the null controllability characterization for switched positive systems is presented in subsection 1, and for periodic positive systems in subsection 2. Some examples are given to verify the theoretical results.

Notation : The notations used in this work are : $\mathbb{Z}_+$ set of nonnegative integers, $\mathbb{Z}_+^0$ set of positive integers, $\mathbb{R}$ set of real numbers, $\mathbb{R}^n$ set of n-dimensional real vectors, $\mathbb{R}^{m\times n}$ set of $m\times n$ real matrices, $\mathbb{R}_+^n$ positive orthant of $\mathbb{R}^n$, $\mathbb{R}_+^{n\times m}$ set of $m\times n$ nonnegative real matrices, $\sigma_s^k=\{s,s+1,...,k\}$ the finite subset of $\mathbb{Z}_+$ with $s\leq k$.

2. PRELIMINARIES

In this section we will give some definitions and reminders on digraph of positive matrices :

Definition 1. *Let $A \in \mathbb{R}_+^{n\times n}$. The directed graph $G = (N,U)$, with the set of vertices $N = \sigma_1^n$ and U the set of arcs defined by:*

$$\text{an arc } (i,j) \in U \text{ if and only if } a_{ij} > 0,$$

is called digraph of the matrix A and denoted by $D(A)$.

- A path in $D(A)$ is an alternating sequence of vertices and arcs of $D(A)$, i.e. $(i_1,(i_1,i_2),i_2,...,(i_k,i_{k+1}),i_{k+1})$.
- The path length is defined to be equal to the number of arcs in the path.
- Let $D_1 = (N_1,U_1)$ and $D_2 = (N_2,U_2)$ be two digraphs. The operation union $D_1 \cup D_2$ produces the digraph $(N_1 \cup N_2, U_1 \cup U_2)$.
- Let $A_k \in \mathbb{R}_+^{n\times n}$, for $k \in \sigma_1^m$. We define a joint digraph $D(A_1,A_2,...,A_m)$ as $D(A_1) \cup D(A_2) \cup ... \cup D(A_m)$ in which each arc is labelled (coloured) with a subset of σ_1^m depending upon which of the digraphs $D(A_1),D(A_2),...,D(A_m)$ includes that arc.

Lemma 1. *The digraph $D(A_1A_2...A_m)$ contains an arc (i,j) if and only if there is a path of length m from i to j in the joint digraph $D(A_1,A_2,...,A_m)$ coloured with $1,2,...,m$, in that order(The arcs that correspond to A_i are coloured with i).*

Proof. Denote $A_k = \left(a_{i,j}^{(k)}\right)_{i,j\in\sigma_1^n}$. There is an arc $(i,j) \in D(A_1A_2...A_m)$ if and only if $(A_1A_2...A_m)_{i,j} = \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}...\sum_{k_{m-1}=1}^{n} a_{i,k_1}^{(1)}a_{k_1,k_2}^{(2)}...a_{k_{m-1},j}^{(m)} > 0$. This, in its turn, is verified if and only if there exist $k_1,k_2,...,k_{m-1} \in \sigma_1^n$ such that $a_{i,k_1}^{(1)} > 0$, $a_{k_1,k_2}^{(2)} > 0$, ..., $a_{k_{m-1},j}^{(m)} > 0$. Finaly, this is equivalent to the existence of a path of length m from i to j, namely $(i,(i,k_1),k_1,...,(k_{m-1},j))$ in $D(A_1,A_2,...,A_m)$ coloured with $1,2,...,m$ in that order. □

3. MAIN RESULTS

3.1. Switched positive system.

Consider the discrete-time switched linear system described by :

$$\begin{cases} x_{t+1} = A_{\psi(t)}x_t + B_{\psi(t)}u_t,\ t \in \mathbb{Z}_+ \\ x_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state vector, $u_t \in \mathbb{R}^m$ is the control input, and $\psi : \mathbb{Z}_+ \longrightarrow \sigma_1^M$ is a switching sequence, $M \geq 1$ is the number of subsystems. Subsystem (A_i, B_i), $i \in \sigma_1^M$, is activated if and only if $\psi(t) = i$.

The solution of system (1), with the initial condition x_0, at time k, is given by:

$$x_k = \Delta_{\psi,0}^k x_0 + \sum_{t=0}^{k-1} \Delta_{\psi,t+1}^k B_{\psi(t)} u_t \tag{2}$$

where

$$\Delta_{\psi,s}^t = \begin{cases} I_n, & t = s \geq 0 \\ A_{\psi(t-1)}A_{\psi(t-2)}\cdots A_{\psi(s)}, & t > s. \end{cases} \tag{3}$$

The definition of positivity for system (1) is given by :

Definition 2. *System (1) is said to be positive if for any initial condition $x_0 \in \mathbb{R}_+^n$ and for any control $u_t \in \mathbb{R}_+^m$, $t \in \mathbb{Z}_+$, we have $x_t \in \mathbb{R}_+^n$ for $t \in \mathbb{Z}_+$ under arbitrary switching sequence.*

Now, we introduce the characterization of positivity for a switched system :

Proposition 2. *System (1) is positive if and only if, $A_i \in \mathbb{R}_+^{n\times n}, B_i \in \mathbb{R}_+^{n\times m}$ for all $i \in \sigma_1^M$.*

Proof. We assume that system (1) is positive. Let $\psi(t) = i \in \sigma_1^M$, so the system (1) can be written in the form :

$$x_{t+1} = A_i x_t + B_i u_t \tag{4}$$

Let $u_t = 0$ for $t \in \mathbb{Z}_+$. Then for $t = 0$ we have $x_1 = A_i x_0 \in \mathbb{R}_+^n$. That implies $A_i \in \mathbb{R}_+^{n\times n}$, since $x_0 \in \mathbb{R}_+^n$ may be arbitrary. Assuming $x_0 = 0$, then for $t = 0$, we obtain $x_1 = B_i u_0 \in \mathbb{R}_+^n$, which implies that $B_i \in \mathbb{R}_+^{n\times m}$ since $u_0 \in \mathbb{R}_+^m$ is arbitrary. So $A_i \in \mathbb{R}_+^{n\times n}, B_i \in \mathbb{R}_+^{n\times m}$ for all $i \in \sigma_1^M$.

Conversely, if $A_i \in \mathbb{R}_+^{n\times n}, B_i \in \mathbb{R}_+^{n\times m}$ for all $i \in \sigma_1^M$. then (2) implies that for all $x_0 \in \mathbb{R}_+^n$ and $u_t \in \mathbb{R}_+^m$, $t \in \mathbb{Z}_+$, we have $x_t \in \mathbb{R}_+^n$ for $t \in \mathbb{Z}_+$.

□

In the rest of this paper, we assume that system (1) is positive. The definition of null controllability for system (1) will be formulated as follows :

Definition 3. *The positive switched system (1) is said to be null controllable if, for any* $x_0 \in \mathbb{R}^n_+$*, there exist* $k \geq 1$*, a switching sequence* $\psi : \sigma_0^{k-1} \longrightarrow \sigma_1^M$*, and nonnegative control inputs* $u_t \in \mathbb{R}^m_+$ *,* $t \in \sigma_0^{k-1}$*, such that* $x_k = 0$*.*

The aim of the following is to establish a necessary and sufficient condition for the null controllability of system (1) based on the graph theory :

Proposition 3. *System (1) is null controllable if and only if there exists* $T \in \mathbb{Z}^0_+$ *and* $k_1, k_2, ..., k_T \in \sigma_1^M$ *such that there is no path of length* T *in* $D(A_{k_1}, A_{k_2} ..., A_{k_T})$ *coloured with* $k_1, k_2, ..., k_T$ *in that order.*

Proof. (Necessity) Let the system (1) be null controllable. Then for $x_0 = (1, \cdots, 1)^T$, there exists $T \in \mathbb{Z}^0_+$, a switching sequence $\psi : \sigma_0^{T-1} \longrightarrow \sigma_1^M$, and control inputs $u_t \in \mathbb{R}^m_+$, $t \in \sigma_0^{T-1}$ such that $x_T = 0$. From (2), we get $A_{\psi(T-1)} A_{\psi(T-2)} ... A_{\psi(0)} = 0$. By posing $k_i = \psi(T-i)$, $i \in \sigma_1^T$. we obtain $\prod_{i=1}^{T} A_{k_i} = 0$, and by lemma 1 there is no path of length T in $D(A_{k_1}, A_{k_2} ..., A_{k_T})$ coloured with $k_1, k_2, ..., k_T$ in that order.

(Sufficiency) Conversely, we assume that there exists $T \in \mathbb{Z}^0_+$ and $k_1, k_2, ..., k_T \in \sigma_1^M$ such that there is no path of length T in $D(A_{k_1}, A_{k_2} ..., A_{k_T})$ coloured with $k_1, k_2, ..., k_T$ in that order. From lemma 1, we get $\prod_{i=1}^{T} A_{k_i} = 0$. Let $\psi : i \in \sigma_0^{T-1} \longrightarrow k_{T-i} \in \sigma_1^M$ and $u_i = 0$, $i \in \sigma_0^{T-1}$. Then, for any $x_0 \in \mathbb{R}^n_+$, we have : $x_T = \Delta^T_{\psi,0} x_0 = \prod_{i=1}^{T} A_{\psi(T-i)} x_0 = \prod_{i=1}^{T} A_{k_i} x_0 = 0$. □

Remark 1. *The null controllability of system (1) does not depend on the* B_i*,* $i \in \sigma_1^M$*.*

Example 1. *Consider a switched system composed of three subsystems with state matrices given by :*

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

There exists no path of length 2 in $D(A_1, A_3)$ *(Fig. 1) coloured with 1, 3 in that order, so the system is null controllable.*

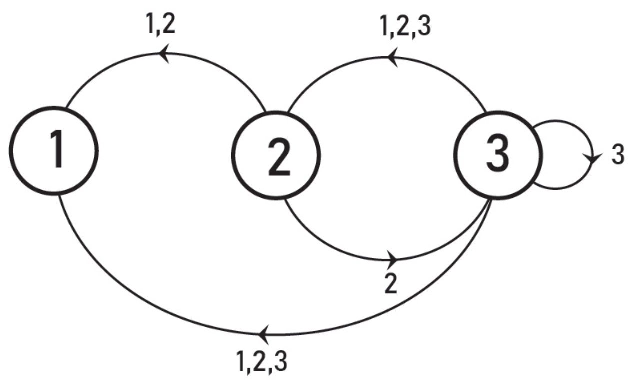

FIGURE 1. $D(A_1, A_3)$

3.2. Periodic positive system.

Consider the discrete-time periodic linear system described by:

$$\begin{cases} x_{t+1} = A_t x_t + B_t u_t,\ t \in \mathbb{Z}_+ \\ x_0 \in \mathbb{R}^n_+ \end{cases} \tag{5}$$

where $x_t \in \mathbb{R}^n$ is the state vector, $u_t \in \mathbb{R}^m$ is the control input, A_t and B_t are real matrices with appropriate dimensions and we assume that there exists $T \in \mathbb{Z}^0_+$ such that, for all $t \in \mathbb{Z}_+$, we have $A_t = A_{t+T}$ and $B_t = B_{t+T}$.

The solution of system (3), with the initial condition x_0, at time k, is given by :

$$x_k = \Delta^k_0 x_0 + \sum_{t=0}^{k-1} \Delta^k_{t+1} B_t u_t \tag{6}$$

where

$$\Delta^t_s = \begin{cases} I_n, & t = s \geq 0 \\ A_{t-1} A_{t-2} \cdots A_s, & t > s. \end{cases} \tag{7}$$

Definition 4. *System* (3) *is said to be positive if for any initial condition* $x_0 \in \mathbb{R}^n_+$ *and for any control* $u_t \in \mathbb{R}^m_+$, $t \in \mathbb{Z}_+$, *we have* $x_t \in \mathbb{R}^n_+$ *for all* $t \in \mathbb{Z}_+$.

Remark 2. *System* (3) *is positive if* $A_t \in \mathbb{R}^{n\times n}_+, B_t \in \mathbb{R}^{n\times m}_+$ *for all* $t \in \sigma_0^{T-1}$.

In the rest of this paper, we assume that system (3) is positive.

Definition 5. *The positive periodic system (3) is said to be null controllable if, for all* $x_0 \in \mathbb{R}_+^n$*, there exist* $k \geq 1$*, and control inputs* $u_t \in \mathbb{R}_+^m$ *,* $t \in \sigma_0^{k-1}$ *such that* $x_k = 0$

The following result gives a characterization of the null controllability for system (3):

Proposition 4. *The system (3) is null controllable if and only if there exists* $r \in \sigma_1^{T-1}$*, and* $l \in \mathbb{Z}_+^0$ *, such that there is no path of length* $r+1$ *in the joint diagraph* $D(A_0,...,A_{r-1},\left(\Delta_0^T\right)^l)$ *coloured with* $0,1,\ldots,r-1,r$ *in that order(The arcs that correspond to* A_i *are coloured with i, and those that correspond to* $\left(\Delta_0^T\right)^l$ *are coloured with r).*

Proof. System (3) is null controllable if and only if there exist $t \in \mathbb{Z}_+^0$ such that $\Delta_0^t = 0$. Using the periodicity of A_t we can write :

$$\begin{aligned}
\Delta_0^t &= A_{t-1}A_{t-2}...A_0 \\
&= A_{r-1}...A_0\left(A_{T-1}...A_1A_0\right)...\left(A_{T-1}...A_1A_0\right) \\
&= A_{r-1}...A_0\left(\Delta_0^T\right)^l
\end{aligned}$$

where $l \in \mathbb{Z}_+$, $r \in \sigma_0^{T-1}$ such that $t = lT + r$. By lemma 1 the matrix $A_{r-1}...A_0\left(\Delta_0^T\right)^l$ is null if and only if there is no path of length $r+1$ in the joint diagraph $D(A_0,...,A_{r-1},\left(\Delta_0^T\right)^l)$ coloured with $0,1,\ldots,r-1,r$ in that order. □

Example 2. *Consider system (3) with period T=3 and:*

$$A_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have : $\Delta_0^3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

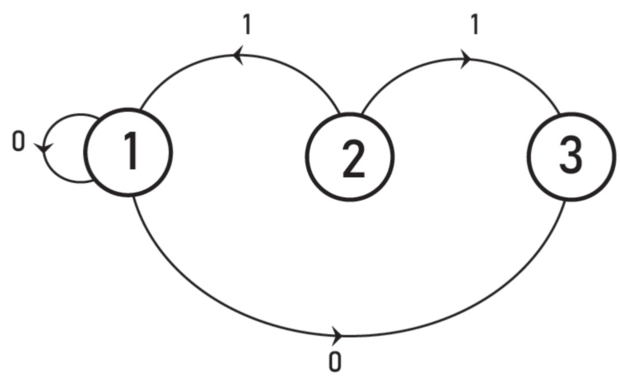

FIGURE 2. $D(A_0, \Delta_0^3)$

In digraph $D(A_0, \Delta_0^3)$(Fig. 2) there is no path of length 2 coloured with 0 and 1 in that order so the system (3) is null controllable. Indeed the matrix $\Delta_0^4 = A_0\Delta_0^3$ is null.

CONCLUSION

In this work, we have established the characterization of the null controllability for positive switched system (Proposition 2), and for positive periodic system (Proposition 3). Examples have been used to verify the theoretical results.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] Y. Wang, C. Jiang, Q. Wu, Attitude tracking control for variable structure near space vehicles based on switched nonlinear systems, Chinese J. Aeronautics, 26 (2013), 186–193.

[2] N.H. El-Farra, A. Gani, P.D. Christofides, A switched systems approach for the analysis and control of mode transitions in biological networks, in: Proceedings of the 2005, American Control Conference, IEEE, Piscataway, NJ, USA, pp. 3247–3252, June 2005.

[3] H. Torikai, T. Saito, Synchronization of chaos and its itinerancy from a network by occasional linear connection, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. 45 (1998), 464–472.

[4] K. Mitsubori, T. Saito, Dependent switched capacitor chaos generator and its synchronization, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. 44 (1997), 1122–1128.

[5] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag. 19 (1999), 59-70.

[6] M.A. Wicks, P. Peleties, R.A. de Carlo, Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems, Eur. J. Control. 4 (1998), 140-147.

[7] S.S. Ge, Z. Sun, T.H. Lee, Reachability and controllability of switched linear discrete-time systems, IEEE Trans. Automat. Control 46 (2001), 1437-1441.

[8] Z. Sun, D. Zheng, On reachability and stabilization of switched linear systems, IEEE Trans. Automat. Control 46 (2001), 291-295.

[9] Y. Wang, G. Xie, L. Wang, Reachability of switched discrete-time systems under constrained switching, in: Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, Hawaii, 2003,pp. 5765-5770.

[10] G. Xie, L.Wang, Reachability realization and stabilizability of switched linear discrete-time systems, J. Math. Anal. Appl. 280 (2003), 209-220.

[11] L. Farina, S. Rinaldi, Positive linear systems: theory and applications, John Wiley and Sons, New York, (2000).

[12] T. Kaczorek, Positive 1D and 2D systems. Springer, London, (2002).

[13] D.G. Luenberger, Introduction to dynamic systems: theory, models and applications, Academic Press, New York, (1979).

[14] T. Chen, B. Francis, Optimal sampled-data control systems. Springer: London, 1995.

[15] S. Bittanti, D. B. Hernandez, G. Zerbi, The simple pendulum and the periodic LQG control problem, J. Franklin Inst. 328 (1991), 299–315.

[16] N. Bougatef, M. Chaabane, O. Bachelier, D. Mehdi, S. Cauet. On the stabilization of a class of periodic positive discrete time systems. In: Proc. 49th IEEE Conf. Decis. Control, 2010, pp. 4311-4316.

[17] M. A. Rami, D. Napp. Discrete-time positive periodic systems with state and control constraints. IEEE Trans. Autom. Control, 61 (2016), 234-239.

[18] X. Liu. Stability analysis of periodic positive systems with delays. In: Proc. 31st Chinese Control Conf., 2012, pp. 95-100.

[19] S. Bittanti, P. Colaneri, Periodic systems: filtering and control, Springer, London, 2008.

[20] I. Sitli, F. Lahmidi, A. Namir, M. Naim. Output reachable set analysis for periodic positive systems. Int. J. Math. Math. Sci. 2021 (2021), 8345694.

[21] I. Sitli, M. Naim, F. Lahmidi, A. Namir, On the controllability and observability of positive nonlinear continuous systems, Commun. Math. Biol. Neurosci. 2021 (2021), 52.

[22] A. Babiarz, A. Czornik, J. Klamka, M. Niezabitowski, Controllability of discrete-time linear switched systems with constrains on switching signal, Springer, Switzerland, 2015.

[23] L. Tian, Y. Guan, L. Wang. Controllability and observability of switched multi-agent systems, Int. J. Control, 92 (2019), 1742–1752.