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NUMERICAL APPROXIMATIONS OF FREDHOLM INTEGRAL EQUATIONS WITH ABEL KERNEL USING LEGENDRE AND CHEBYCHEV POLYNOMIALS

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Abstract. In this study, Legendre and Chebychev collocation method are presented to solve numerically the Fredholm Integral Equations with Abel kernel. This method is based on replacement of the unknown function by truncated series of well known Legendre and Chebychev expansion of functions. This lead to a system of algebraic equations with Legendre and Chebychev coefficients. Thus, by solving the matrix equation, Legendre and Chebychev coefficients are obtained. Some numerical examples are included to demonstrate the validity and applicability of the proposed technique.

Keywords: Fredholm Integral Equations with Abel kernel, Integral equation, collocation matrix method, Legendre and Chebychev polynomials.

2000 AMS Subject Classification: 47H17; 47H05; 47H09

1. Introduction

An integral equation is an equation in which an unknown function appears within an integral, just as a differential equation is an equation in which an unknown function appears within a derivative. Just as the solution to a differential equation is a function, so too is the solution to an integral equation a function.

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This paper look for the Fredholm integral equation of the second kind with the Abel kernel:

$$\psi(x) - \lambda \int_{-1}^{1} |x-y|^{-\alpha}\psi(y)dy = f(x),\ -1 \leq x \leq 1 \tag{1}$$

with $\alpha \in]0,1[$. The elements $\mathcal{K}(x,y) = |x-y|^{-\alpha}$ is the Abel kernel. The kernel is unbounded at $x = y$, so we have a weak singularity.

Every integral equation has a kernel. Kernels are important because they are at the heart of the solution to integral equations. There are many methods used to solve the Fredholm integral equation. In [2], the authors solve a Volterra Integral Equations using Laguerre Polynomials. The Volterra equations has been solved by many numerically methods. In [3, 4] the authors used Bernstein polynomials in approximation techniques and in [5] the author solved the probleme by Block-Pulse functions and Taylor expansion.

Let us recall, that the Lagrange and Chebychev interpolation polynomial plays an important role in functional approximation and in many numerical methods, such as numerical integration, numerical solutions for differential equations, and so on. We know that the Lagrange interpolation polynomial does not converge to arbitrary continuous function f uniformly. An example for which the Lagrange interpolation does not converge is provided by $f(x) = |x|$ in the interval $[-1,1]$, for which equidistant interpolation diverges for $0 < |x| < 1$ as has been proved by Bernstein. In 1930's, Bernstein gives some methods for improving the uniform convergence of the known Lagrange interpolation polynomial, see [1]. Thereafter, many works based on these methods are presented, see for example [6, 7, 8, 9, 10, 11, 12, 13, 14].

Here, we will give an approximate method for solving (1) using Legendre and Chebychev polynomials. We will consider a polynomial approximation problem of finding a polynomial close to a given (true) function ψ and have the freedom to pick up the target points $\{x_0, x_1, \ldots, x_N\}$. we will think about how to choose the target points for better approximation, rather than taking equidistant points along the x-axis. Noting that the error tends to get bigger in the parts close to both ends of the interval when we chose the equidistant target points, it may be helpful to set the target points denser in the parts close to both ends than in the middle part.

2. Legendre Methods

2.1. Fundamental

Orthogonal polynomials are widely used in applications in mathematics, mathematical physics, engineering and computer science. One of the most common set of orthogonal polynomials is the Legendre polynomials. The Legendre polynomials P_n satisfy the recurrence formula:

$$\begin{aligned} (n+1)P_{n+1}(x) &= (2n+1)xP_n(x) - nP_{n-1}(x),\ n \in \mathbb{N}^* \\ P_0(x) &= 1, \\ P_1(x) &= x \end{aligned} \tag{2}$$

An important property of the Legendre polynomials is that they are orthogonal with respect to the L^2 inner product on the interval $[-1, 1]$:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}$$

where δ_{nm} denotes the Kronecker delta.

2.2. Approximate solution

We choose $x_k, k \in [\![0, n]\!]$ the zeros of the Legendre polynomial of degree equal $n+1$, P_{n+1}. We determine a suitable interpolating elements $\phi_j(x), j = 0, 1, \ldots, n$, such that

$$\psi_n(x) = \sum_{j=0}^{n} \phi_j(x)\psi(x_j) \tag{3}$$

is the unique interpolating polynomial of degree n, which interpolates ψ at the points $x_i, i = 0, 1, \ldots, n$.

The elements $\phi_j(x), j = 0, 1, \ldots, n$ are called the basic functions associated with the Legendre interpolation polynomial and they satisfy $\phi_j(x_i) = \delta_{ij}$.

Then we get an approximation of the exactly integral, let say:

$$I_n(\psi) = \int_{-1}^{1} \mathcal{K}(x,y)\psi_n(y)dy \tag{4}$$

This type of approximation must be chosen so that the integral (4) can be evaluated (either explicitly or by an efficient numerical technique).

The functions $P_0(x), P_1(x), \ldots, P_n(x)$ will be called interpolating elements. In this dissertation, the interpolating function ψ_n will be assumed to be the interpolating polynomial

$$\psi_n(x) = \sum_{j=0}^{n} \beta_j P_j(x) \tag{5}$$

where P_j are Legendre polynomials of degree j, n is the number of Legendre polynomials, and β_j are unknown parameters, to be determined.

The coefficients β_j are obtained by multiplying both sides of Eq. (5) by $P_m, m \leq n$ (as weight functions), and integrating the resulting equation with respect to x over the interval $[-1,1]$ to obtain

$$\int_{-1}^{1} P_m(x)\psi_n(x)dx = \sum_{j=0}^{n} \beta_j \int_{-1}^{1} P_m(x)P_j(x)dx = \beta_m \frac{2}{2m+1}$$

Therefore,

$$\beta_m = \frac{2m+1}{2} \int_{-1}^{1} P_m(x)\psi_n(x)dx \tag{6}$$

Here the integrand $P_m\psi_n$ is a polynomial of degree $n+m \leq 2n$ then its integration in (6) can exactly be obtained from just $n+1$ point Gauss-Legendre method, by using the following formula

$$\beta_m = \frac{2m+1}{2} \sum_{j=0}^{n} w_j P_m(x_j)\psi(x_j) \tag{7}$$

where $w_j, j = 0, \ldots, n$ are the $(n+1)$-point Gauss-Legendre weights.

The $n+1$ grid points $(x_i\ )$ of Gauss Legendre integration in formula (7) giving us the exact integral of an integrand polynomial of degree $n+m \leq 2n$ can be obtained as the zeros of the $n+1$-th-degree Legendre polynomial. Then, given the $n+1$ grid point x_i, we can get the corresponding weight w_i of the i point Gauss Legendre integration formula by solving the system of linear equations. Now, the interpolating polynomial ψ_n can be written as:

$$\begin{aligned}\psi_n(x) &= \sum_{m=0}^{n}\Big(\frac{2m+1}{2}\sum_{j=0}^{n} w_j P_m(x_j)\psi(x_j)\Big)P_m(x)\\ &= \sum_{j=0}^{n}\Big(w_j\sum_{m=0}^{n}\frac{2m+1}{2}P_m(x_j)P_m(x)\Big)\psi(x_j)\end{aligned} \tag{8}$$

Using (3) and (8) we get

$$\phi_j(x) = w_j\sum_{m=0}^{n}\frac{2m+1}{2}P_m(x_j)P_m(x),\ j=0,\dots,n \tag{9}$$

Substituting ψ_n into Eq. (1) and collocating at the points x_i, we obtain:

$$\psi(x_i) - \lambda\sum_{j=0}^{n}\psi(x_j)\int_{-1}^{1}\mathcal{K}(x_i,y)\phi_j(y)dy = f(x_i), i=0,\dots,n \tag{10}$$

2.3. Matrix Form

To simplify the presentation let us define

$$a_{i,j} = \int_{-1}^{1}\mathcal{K}(x_i,y)\phi_j(y)dy \tag{11}$$

Then a $(n+1)\times(n+1)$ linear system is obtained:

$$(Id-\lambda A)\psi = F \tag{12}$$

where $A=(a_{i,j})_{(i,j)\in[\![0,n]\!]^2}$ is square matrix, $\psi=(\psi(x_0),\dots,\psi(x_n))^T$ and $F=(f(x_0),\dots,f(x_n))^T$, capital T indicate the transpose. Obviously, the system (12) has a unique solution if the determinant of the matrix $Id-\lambda A$ is nonzero, which also depends on the choice of collocation point.

Substituting (9) into (11) we obtain

$$a_{i,j} = w_j \sum_{k=0}^{n} \frac{2k+1}{2} P_k(x_j) u_k(x_i)$$

where $u_k(x_i), (i,k) \in [\![0,n]\!]^2$ are defined

$$u_k(x_i) = \int_{-1}^{1} |x_i - y|^{-\alpha} P_k(y) dy$$

The constants $u_k(x_i), (i,k) \in [\![0,n]\!]^2$, can be evaluated from the recurrence relation:

$$(k+3-\alpha)u_{k+2}(x_i) = (2k+3)x_i u_{k+1}(x_i) - (k+\alpha)u_k(x_i), k = 0, \ldots, n$$

with the starting values for this recurrence relation are:

$$u_0(x_i) = \frac{1}{1-\alpha}\Big((1-x_i)^{1-\alpha} + (1+x_i)^{1-\alpha}\Big) \tag{13}$$

$$u_1(x_i) = x_i u_0(x_i) + \frac{1}{2-\alpha}\Big((1-x_i)^{2-\alpha} + (1+x_i)^{2-\alpha}\Big) \tag{14}$$

3. Chebyshev Methods

Like Legendre Methods, we will use here the Chebyshev polynomials T_n of the first kind given by the formula

$$T_n(x) = \cos(n \arccos(x)), \forall x \in [-1, 1] \tag{15}$$

Explicit algebraic expressions for T_n are obtained from the recurrence

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+2}(x) &= 2xT_{n+1}(x) - T_n(x), n \in \mathbb{N}^*, x \in [-1, 1] \end{aligned}$$

The polynomial T_{n+1} has $n+1$ zeros in the interval $[-1; 1]$, which are located at the points

(16)
$$x_k = \cos(\frac{2k+1}{2n+2}\pi), k \in [\![0, n]\!]$$

The Chebyshev polynomials of the first kind of degree n, T_n, satisfy discrete orthogonality relationships on the grid of the $(n+1)$ zeros of T_{n+1}(which are referred to as the Chebyshev nodes):

(17)
$$\sum_{k=0}^{N} T_i(x_k)T_j(x_k) = \begin{cases} 0 & : i \neq j \\ N+1 & : i = j = 0 \\ \frac{N+1}{2} & : i = j \neq 0 \end{cases}$$

For an arbitrary interval $[a, b]$, we can find a mapping that transform $[a, b]$ into $[-1, +1]$:

$$y_k = \frac{b-a}{2}x_k + \frac{a+b}{2} = \frac{b-a}{2}\cos(\frac{2k+1}{2n+2}\pi) + \frac{a+b}{2}, k \in [\![0, n]\!]$$

and the Chebyshev nodes defined by Eq (16) are actually zeros of this Chebyshev polynomial.

Based on the discrete orthogonality relationships of the Chebyshev polynomials, various methods of solving linear and nonlinear ordinary differential equations [16] and integral differential equations [17] were devised at about the same time and were found to have considerable advantage over finite-differences methods. Since then, these methods have become standard [15]. They rely on expanding out the unknown function in a large series of Chebyshev polynomials, truncating this series, substituting the approximation in the actual equation, and determining equations for the coefficients. In our approach we follow closely the procedures like Legendre Method. Let us say, that similar procedures can be applied for a second grid given by the extremas of T_n as nodes.

It is important to stress that our goal is not to approximate a function f on the interval $[-1; 1]$, but rather to approximate the values of the function f corresponding to a given discrete set of points like those given in equation (16).

Here, let $(T_0, T_1, T_2, \ldots, T_n)$ the interpolating elements. The equation (5) becomes

$$\psi_n(x) = {\sum_{j=0}^{n}}' \beta_j T_j(x) \tag{18}$$

where the prime indicates that the first term is to be halved (which is convenient for obtaining a simple formula for all the coefficients β_j). The function ψ_n interpolates ψ at the $n+1$ Chebyshev nodes, we have at these nodes $\psi(x_k) = \psi_n(x_k)$. Hence, using the discrete orthogonality relation (17) we get

$$\beta_j = \frac{2}{n+1} \sum_{k=0}^{n} \psi(x_k) T_j(x_k), j = 0, 1, \ldots, n \tag{19}$$

Now, we get

$$\begin{aligned}
\psi_n(x) &= {\sum_{j=0}^{n}}' \beta_j T_j(x) \\
&= {\sum_{j=0}^{n}}' \frac{2}{n+1} \sum_{k=0}^{n} \psi(x_k) T_j(x_k) T_j(x) \\
&= \sum_{k=0}^{n} \frac{2}{n+1} \Big({\sum_{j=0}^{n}}' T_j(x_k) T_j(x) \Big) \psi(x_k) \qquad (20)
\end{aligned}$$

Using (3) and (20) we get:

$$\phi_k(x) = \frac{2}{n+1} {\sum_{j=0}^{n}}' T_j(x_k) T_j(x) \tag{21}$$

Now,the same system like (12) is obtained with

$$a_{ij} = \frac{2}{n+1} {\sum_{j=0}^{n}}' v_k(x_i) T_k(x_j)$$

where $v_k(x_i), (i,k) \in [\![0, n]\!]^2$ are defined

$$v_k(x_i) = \int_{-1}^{1} |x_i - y|^{-\alpha} T_k(y) dy$$

The constants $v_k(x_i), (i,k) \in [\![0, n]\!]^2$, can be evaluated from the recurrence relation:

$$\Big(1 + \frac{1-\alpha}{m+1}\Big) v_{m+1}(x_i) - 2x_i v_m(x_i) + (1 - \frac{1-\alpha}{m-1}) v_{m-1}(x_i) = \frac{2}{1-m^2} \Big((1-x_i)^{1-\alpha} - (-1)^m (1+x_i)^{1-\alpha} \Big)$$

with the starting values for this recurrence relation are:

$$
\begin{aligned}
(22) \qquad v_0(x_i) &= \frac{1}{1-\alpha}\Big((1-x_i)^{1-\alpha}+(1+x_i)^{1-\alpha}\Big) \\
(23) \qquad v_1(x_i) &= x_i u_0(x_i)+\frac{1}{2-\alpha}\Big((1-x_i)^{2-\alpha}-(1+x_i)^{2-\alpha}\Big) \\
(24) \qquad v_2(x_i) &= 4xv_1(x_i)-(2x_i^2+1)v_0(x_i)+\frac{2}{3-\alpha}\Big((1-x_i)^{3-\alpha}+(1+x_i)^{3-\alpha}\Big)
\end{aligned}
$$

4. Numerical Implementation

In this section, to achieve the validity, the accuracy and support our theoretical discussion of the proposed method, we give some computational results. The computations, associated with the example, are performed by MATLAB 7. Consider the weakly singular Fredholm Integral equation of second kind with:

Example 1. $f(x)=x^2-\frac{1}{\pi}\Big(w_1+w_2)$

where

$$
\begin{aligned}
w_1 &= \frac{1}{1-\alpha}x^2(1+x)^{1-\alpha}-\frac{2}{2-\alpha}x(1+x)^{2-\alpha}+\frac{1}{3-\alpha}(1+x)^{3-\alpha} \\
w_2 &= \frac{1}{1-\alpha}x^2(1-x)^{1-\alpha}+\frac{2}{2-\alpha}x(1-x)^{2-\alpha}+\frac{1}{3-\alpha}(1-x)^{3-\alpha}
\end{aligned}
$$

$\lambda=\frac{1}{\pi}$ *and* $\alpha\in]0,1[$ *then the exact solution is* $\psi(x)=x^2$.

In our computation we will take $\alpha=\frac{1}{2}$. *The Figure 1 shows the exact and approximate solutions using Legendre polynomials with* $n=50$. *We notice that the approximate solutions coincide with the exact solutions even a few of the polynomials are used in the approximation which are shown.*

The error is defined as the difference between the exact solution and approximate solution. The plot of $\log_{10}||Error||_\infty$ *with respect to* n *(see Figure. 2) shows that using Legendre or Chebychev polynomials with* $n=2$ *is necessary in order to reach* $||Error||_\infty\leq$

10^{-15}. *This means that the infinity norm of the error decrease until* $n \simeq 2$ *and it reaches the threshold of machine precision. It can also be seen that increasing the number of the Legendre or Chebychev nodes or, equivalently, increasing the degree of Legendre or Chebychev polynomials makes a substantial contribution towards reducing the approximation error.*

Example 2. $f(x) = x^3 - \frac{2}{\pi}\Big(c_1 + c_2)$

where

$$
\begin{aligned}
c_1 &= x^3(1+x)^{1/2} - x^2(1+x)^{3/2} + \frac{3}{5}x(1+x)^{5/2} - \frac{1}{7}(1+x)^{7/2} \\
c_2 &= x^3(1-x)^{1/2} + x^2(1-x)^{3/2} + \frac{3}{5}x(1-x)^{5/2} + \frac{1}{7}(1-x)^{7/2}
\end{aligned}
$$

then the exact solution is $\psi(x) = x^3$.

Here, we take $\alpha = \frac{1}{2}$. *The Figure 3 shows the exact and approximate solutions using Legendre polynomials with* $n = 50$. *We notice that the approximate solutions coincide with the exact solutions even a few of the polynomials are used in the approximation which are shown.*

The error is defined as the difference between the exact solution and approximate solution. The plot of $\log_{10}||Error||_\infty$ *with respect to* n *(see Figure. 4) shows that using Legendre and Chebychev polynomials with* $n = 10$ *is necessary in order to reach* $||Error||_\infty \leq 10^{-15}$.

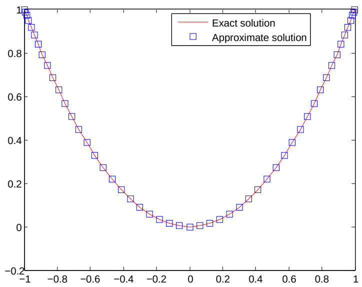

FIGURE 1. Case $\psi(x) = x^2$. Exact and approximate solution with $n = 50$.

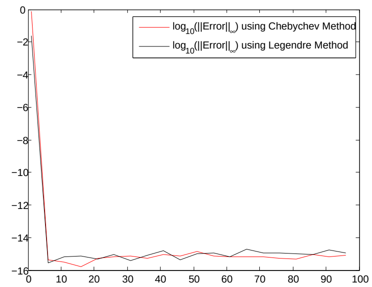

FIGURE 2. Case $\psi(x) = x^2$. Infinity Norm of the Error for $n \in [\![1, 100]\!]$.

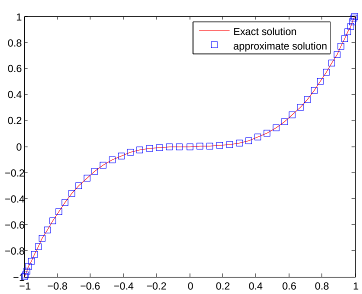

FIGURE 3. Case $\psi(x) = x^3$. Exact and approximate solution with $n = 50$.

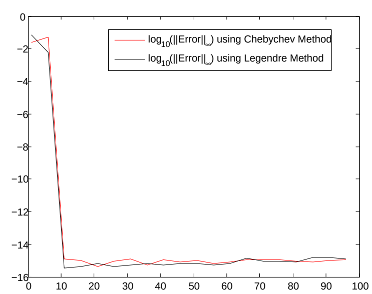

FIGURE 4. Case $\psi(x) = x^3$. Infinity Norm of the Error for $n \in [\![1, 100]\!]$.

6. Conlusion

- The Legendre and Chebychev polynomials basis has been developed to solve singular Fredholm integral equations.
- Numerical results have been obtained with great accuracy.
- This method may be applied to solve Volterra Fredholm integral equations with singular Kernels and a nonlinear Volterra integral equation.
- Other type of singular Kernels can be investigate using the same method.
- Don't take a big n: as the size of the matrix grows, the round-off errors are apt to accumulate and propagated in matrix operations.

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