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ON CONFORMAL BI-SLANT SUBMERSION FROM COSYMPLECTIC MANIFOLD

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Abstract. The study of conformal bi-slant submersions from cosymplectic manifolds onto Riemannian manifold are the subject of this article. We deal with the integrability of slant and anti-invariant distributions and investigate each distribution's totally geodesicness condition as well. Additionally, we identify the conditions for the maps to be totally geodesic. There are also some decomposition theorems for total space and fibers addressed.

Keywords: Bi-slant submersion; Conformal submersion; Distribution; Integrability; Contact metric manifold.

2020 AMS Subject Classification: 53C25, 30C20, 53C42

1. INTRODUCTION

The concept of Riemannian submersion is an anlogue of isometric immersion which was initially proposed by B. O'Neill [21] and also researched over same time period by A. Gray [12]. Later, in 1976, B. Watson [30], considered the submersion between almost Hermitian manifolds and called it, "Almost Hermitian submersions". Riemannian submersions have several uses in both mathematics and physics including in super gravity and superstring theories ([16], [20]), Yang-Mills theory ([6], [31]), Kaluza-Klein theory ([17], [20]). While exploring the Riemannian manifold with differentiable structure, Riemannian submersions are powerful

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tool in differential geometry. B. Sahin, in [24], presented the idea of anti-invariant Riemannian submersions from almost Hermitian manifold onto a Riemannian manifold. He studied fibres, base space and total space with differential geometric point of view. Later, several other kind of Riemannain submersions such as semi-invariant submersion [25], semi-slant submersions [23] and generic Riemannian submersions[5] etc. are defined. For the detailed study of Riemannian submersions and its applications, readers may through [27], where B. sahin has presented a collective study of different kind of Riemannian submersions. Furthermore, the notion of almost contact Riemannian submersions from almost contact manifold was introduced by D. Chinea in [7].

As a generalization of Riemannain submersion, B. Fuglede [11] and T. Ishihara [18] separately studied horizontally conformal submersions. Number of scholars later explored several new types of conformal Riemannian submersion from almost Hermitian as well as from contact metric manifold onto a Riemannian manifold such as conformal anti-invariant submersions ([1], [24]), conformal slant submersions ([4], [14]), Conformal semi-slant submersions ([3], [13]) and conformal hemi-slant submersions [19] etc.

As a generalization of conformal semi-slant submersion [3] and conformal hemi-slant submersion [19], in this paper, we investigate conformal bi-slant submersion from a cosymplectic manifold onto a Riemannian manifold. The paper has the following structure: Section 2 presents the fundamental information and definitions of conformal Riemannian submersion and contact metric manifolds, particularly cosymplectic manifolds with properties relevant to this paper. In section 3, we define the conformal bi-slant submersion and obtain some basic results. Section 4 contains the main research findings of this paper such as the condition of integrability and totally geodesicness of the distributions. Decomposition theorems for the fibres as well as for the total manifolds are covered in section 5.

Note: We will use abbreviation CBSS- conformal bi-slant submersion, throughout the paper.

2. Preliminaries

A $(2n+1)$-dimensional manifold M which having an almost contact structures (ϕ, ξ, η), where a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1, \tag{1}$$

where I is the identity tensor. If $N \oplus d\eta \otimes \xi = 0$, with Nijenhuis tensor N related to ϕ then almost contact structure turns into normal. There is also a Riemannian metric g which holds

$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V), \eta(U) = g(U, \xi). \tag{2}$$

Then (ϕ, ξ, η, g)-structure is called an almost contact metric structure. An almost contact metric manifold with almost contact structure (ϕ, ξ, η, g) is called a cosymplectic manifold if

$$(\nabla_U \phi)V = 0, \tag{3}$$

where ∇ is the Levi-Civita connection of g. From above formula, we have for cosymplectic manifold

$$\nabla_U \xi = 0. \tag{4}$$

The covariant derivative of ϕ defined as

$$(\nabla_U \phi)V = \nabla_U \phi V - \phi \nabla_U V. \tag{5}$$

Example 2.1. *Consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_i, y_i, z)\,(i = 1, \ldots, n)$ and its usual contact form*

$$\eta = dz.$$

The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$ and its Riemannian metric g and tensor field ϕ are given by

$$g = \sum_{i=1}^{n} \left((dx_i)^2 + (dy_i)^2\right) + (dz)^2, \quad \phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, n.$$

It can easily be seen that the structure (φ, ξ, η, g) defines a cosymplectic structure on $\mathbb{R}^{2n+1}$.

Now, we provide a definition for conformal submersion and discuss some useful results that provide background for our discussion.

Definition 2.2. A smooth map φ from (M, g_M) onto (N, g_N) where M and N are Riemannian manifolds with m_1 and m_2 be the dimensions of manifolds respectively, is called horizontally weakly conformal or semi-conformal at $p \in M$ if, either

(i) $\varphi_{*p} = 0$ or

(ii) φ_{*p} is surjective and there always have a number $\Omega(p) \neq 0$ satisfying

$$g_N(\varphi_{*p}U, \varphi_{*p}V) = \Omega(p) g_M(U,V),$$

for any $U, V \in \Gamma(ker\varphi_*)^\perp$.

In this case, we label a point p satisfying type (i) as a critical point and rank of φ_{*p} is 0 at this point and type (ii) as a regular point at which the rank of φ_{*p} is m_2. Also, the number $\Omega(p)$ is called the square dilation. Its square root $\lambda(p) = \sqrt{\Omega(p)}$ is called the dilation. If the map φ is horizontally weakly conformal at each point on M, it is referred to as horizontally weakly or semi-conformal on M. If φ has no critical point, it is said to be a (horizontally) conformal submersion.

Let $\varphi : M \to N$ be a submersion. A vector field X on M is called a basic vector field if $X \in \Gamma(\ker \varphi_*)^\perp$ and φ-related with a vector field X on N i.e $\varphi_*(X(q)) = X\varphi(q)$ for $q \in M$.

The given formulae provide $(1,2)$ tensor fields $\mathscr{T}$ and $\mathscr{A}$ are

$$\mathscr{T}(E_1,E_2) = \mathscr{T}_{E_1}E_2 = \mathscr{H}\nabla_{\mathscr{V}E_1}\mathscr{V}E_2 + \mathscr{V}\nabla_{\mathscr{V}E_1}\mathscr{H}E_2, \tag{6}$$

$$\mathscr{A}(E_1,E_2) = \mathscr{A}_{E_1}E_2 = \mathscr{V}\nabla_{\mathscr{H}E_1}\mathscr{H}E_2 + \mathscr{H}\nabla_{\mathscr{H}E_1}\mathscr{V}E_2, \tag{7}$$

for any $E_1, E_2 \in \Gamma(TM)$.

Note that a Riemannian submersion $\varphi : M \to N$ has totally geodesic fibers if and only if $\mathscr{T}$ vanishes identically. From equations (6) and (7), we can deduce

$$\nabla_U V = \mathscr{T}_U V + \hat{\nabla}_U V, \tag{8}$$

(9) $$\nabla_U X = \mathcal{T}_U X + \mathcal{H}\nabla_U X,$$

(10) $$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U,$$

(11) $$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y,$$

for any $U,V \in \Gamma(ker\varphi_*)$ and $X,Y \in \Gamma(ker\varphi_*)^\perp$, where $\hat{\nabla}_U V = \mathcal{V}\nabla_U V$. It is clear that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric, i.e.,

(12) $$g(\mathcal{A}_X E_1, E_2) = -g(E_1, \mathcal{A}_X E_2),\ g(\mathcal{T}_V E_1, E_2) = -g(E_1, \mathcal{T}_V E_2),$$

for all $E_1, E_2 \in \Gamma(T_pM)$.

The following results holds for the particular case, where φ is horizontally conformal:

Proposition 2.3. *Let $\varphi : M \to N$ be horizontally conformal submersion with dilation λ and $X,Y \in \Gamma(ker\varphi_*)^\perp$, then*

(13) $$A_X Y = \frac{1}{2}\big(\mathcal{V}[X,Y] - \lambda^2 g_M(X,Y)\, grad_{\mathcal{V}}(\frac{1}{\lambda^2})\big).$$

The second fundamental form of smooth map φ is given by the formula

(14) $$(\nabla\varphi_*)(X,Y) = \nabla^{\varphi}_X \varphi_* Y - \varphi_* \nabla_X Y$$

and the map be totally geodesic if $(\nabla\varphi_*)(X,Y) = 0$ for all $X,Y \in \Gamma(T_pM)$, where ∇ and ∇^{φ} are Levi-Civita and pullback connections.

Lemma 2.4. *Let M and N be Riemannian manifolds and φ be horizontal conformal submersion. Then, for any vector fields $X,Y \in \Gamma(ker\varphi_*)^\perp$ and $U,V \in \Gamma(ker\varphi_*)$, we have*

(i) $(\nabla\varphi_*)(X,Y) = X(\ln\lambda)\varphi_*(Y) + Y(\ln\lambda)\varphi_*(X) - g_M(X,Y)\varphi_*(grad \ln\lambda)$,
(ii) $(\nabla\varphi_*)(U,V) = -\varphi_*(\mathcal{T}_U V)$,
(iii) $(\nabla\varphi_*)(X,U) = -\varphi_*(\nabla^M_X U) = -\varphi_*(\mathcal{A}_X U)$.

3. Conformal Bi-Slant Submersions

Definition 3.1. Let φ be a conformal submersion from a cosymplectic manifold (M,ϕ,ξ,η,g_M) onto Riemannian manifold (N,g_N). Then φ defines a conformal bi-slant submersion (CBSS) if D^{θ_1} and D^{θ_2} are slant distributions with corresponding slant angles θ_1 and θ_2, respectively, such that $ker\varphi_* = D^{\theta_1} \oplus D^{\theta_2} \oplus \xi$. If θ_1, θ_2 are neither equal to 0 nor $\dfrac{\pi}{2}$, then φ is proper.

Now, let m and n be the dimensions of D^{θ_1} and D^{θ_2}, respectively, then we observe that

(i) If $m = 0$ and $\theta_2 = \frac{\pi}{2}$, then φ is a conformal anti-invariant submersion,

(ii) If $m,n \neq 0$, $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then φ is a conformal semi-invariant submersion.

(iii) If $m,n \neq 0$, $\theta_1 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then φ is a conformal semi-slant submersion.

(iv) If $m,n \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$, then φ is a conformal hemi-slant submersion.

We now give the following example of proper CBSS from a cosymplectic manifold to a Riemannian manifold using the same structure (ϕ,ξ,η,g) as in Example 2.1.

Example 3.2. *Define a conformal Riemannian submersion $\varphi : \mathbb{R}^9 \to \mathbb{R}^4$ as follows:*

$$\varphi(x_1,\ldots,x_8,z) = \pi^{11}\left(x_1,(\cos\alpha)x_2 + (\sin\alpha)x_4,(-\cos\beta)x_5 + (\sin\beta)x_7,x_6\right),$$

where $(x_1,\ldots,x_8,z)$ are natural coordinates of $\mathbb{R}^9$. Then, by the direct calculation, we obtain

$$D^{\theta_1} = \left\{V_1 = \frac{\partial}{\partial x_3}, V_2 = \sin\beta\frac{\partial}{\partial x_5} + \cos\beta\frac{\partial}{\partial x_7}\right\}$$
$$D^{\theta_2} = \left\{V_3 = \frac{\partial}{\partial x_8}, V_4 = \sin\alpha\frac{\partial}{\partial x_2} - \cos\alpha\frac{\partial}{\partial x_4}\right\} \text{ and}$$
$$\xi = \frac{\partial}{\partial z}.$$

Thus, φ is conformal bi-slant submersion with the slant angels θ_1, θ_2 as β and α, respectively., where dilation is π^{11}. It can be seen that the vector field ξ is vertical.

Now, for CBSS from cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N), i.e., $\varphi : (M,\phi,\xi,\eta,g_M) \to (N,g_N)$, take $U \in \Gamma(ker\varphi_*)$, we can write

$$U = P_1U + P_2U + \eta(U)\xi, \tag{15}$$

where $P_1U \in \Gamma(D^{\theta_1})$ and $P_2U \in \Gamma(D^{\theta_2})$.

Also, for $U \in \Gamma(ker\varphi_*)$

$$\phi U = \psi U + \omega U, \tag{16}$$

where $\psi U \in \Gamma(ker\varphi_*)$ and $\omega U \in \Gamma(ker\varphi_*)^\perp$. For any $X \in \Gamma(ker\varphi_*)^\perp$, we have

$$\phi X = tX + fX \tag{17}$$

where $tX \in \Gamma(ker\varphi_*)$ and $fX \in \Gamma(ker\varphi_*)^\perp$. On using equations (15), (16) and (17), we have

$$\psi D^{\theta_1} = D^{\theta_1}, \ \psi D^{\theta_2} = D^{\theta_2}, \ t\omega D^{\theta_1} = D^{\theta_1}, \ t\omega D^{\theta_2} = D^{\theta_2}. \tag{18}$$

The horizontal distribution $(ker\varphi_*)^\perp$ is decomposed as

$$(ker\varphi_*)^\perp = \omega D^{\theta_1} \oplus \omega D^{\theta_2} \oplus \mu, \tag{19}$$

where μ is distribution which is complementary to $\omega D^{\theta_1} \oplus \omega D^{\theta_2}$ in $(ker\varphi_*)^\perp$.

Lemma 3.3. *Let $(M, \phi, \xi, \eta, g_M)$ be cosymplectic manifold and (N, g_N) be a Riemannian manifold. If $\varphi : M \to N$ is a conformal bi-slant submersion, then we have*

$$\omega tX + f^2X = -X, \ \psi tX + tfX = 0,$$

$$\psi^2 U + t\omega U = -U + \eta(U)\xi, \ \omega\psi U + f\omega U = 0$$

for $U \in \Gamma(\ker\varphi_*)$ and $X \in \Gamma((ker\varphi_*)^\perp)$.

Proof. On using equations (1), (16) and (17), we get the desired results. □

Lemma 3.4. *Let $(M, \phi, \xi, \eta, g_M)$ be cosymplectic manifold and (N, g_N) be a Riemannian manifold. If $\varphi : M \to N$ is a conformal bi-slant submersion, then we have*

$$\mathscr{A}_X tY + \mathscr{H}\nabla_X fY = f\mathscr{H}\nabla_X Y + \omega\mathscr{A}_X Y, \tag{20}$$

$$\mathscr{V}\nabla_X tY + \mathscr{A}_X fY = t\mathscr{H}\nabla_X Y + \psi\mathscr{A}_X Y, \tag{21}$$

$$\mathscr{V}\nabla_X \psi V + \mathscr{A}_X \omega V = t\mathscr{A}_X V + \psi\mathscr{V}\nabla_X V, \tag{22}$$

$$\mathscr{A}_X \psi V + \mathscr{H}\nabla_X \omega V = f\mathscr{A}_X V + \omega\mathscr{V}\nabla_X V, \tag{23}$$

$$\mathscr{T}_V tX + \mathscr{H}\nabla_V fX = \omega\mathscr{T}_V X + f\mathscr{H}\nabla_V X, \tag{24}$$

$$\mathscr{T}_V fX + \mathscr{V}\nabla_V tX = \psi\mathscr{T}_V X + t\mathscr{H}\nabla_V X, \tag{25}$$

$$\mathscr{V}\nabla_V \psi U + \mathscr{T}_V \omega U = t\mathscr{T}_V U + \psi\mathscr{V}\nabla_V U, \tag{26}$$

$$\mathscr{T}_V \psi U + \mathscr{H}\nabla_V \omega U = f\mathscr{T}_V U + \omega\mathscr{V}\nabla_V U, \tag{27}$$

for any $U,V \in \Gamma(ker\varphi_*)$ *and* $X,Y \in \Gamma(\ker\varphi_*)^{\perp}$.

Now we define the following:

$$(\nabla_U \psi)V = \mathscr{V}\nabla_U \psi V - \psi\mathscr{V}\nabla_U V, \tag{28}$$

$$(\nabla_U \omega)V = \mathscr{H}\nabla_U \omega V - \omega\mathscr{V}\nabla_U V, \tag{29}$$

$$(\nabla_X t)Y = \mathscr{V}\nabla_X tY - t\mathscr{H}\nabla_X Y, \tag{30}$$

$$(\nabla_X f)Y = \mathscr{H}\nabla_X fY - f\mathscr{H}\nabla_X Y, \tag{31}$$

for any $U,V \in \Gamma(\ker\varphi_*)$ and $X,Y \in \Gamma(\ker\varphi_*)^{\perp}$.

Lemma 3.5. *Let* (M,ϕ,ξ,η,g_M) *be cosymplectic manifold and* (N,g_N) *be a Riemannian manifold. If* $\varphi : M \to N$ *is a conformal bi-slant submersion, then we have*

$$(\nabla_U \psi)V = t\mathscr{T}_U V - \mathscr{T}_U \omega V,$$

$$(\nabla_U \omega)V = f\mathscr{T}_U V - \mathscr{T}_U \psi V,$$

$$(\nabla_X t)Y = \psi\mathscr{A}_X Y - \mathscr{A}_X fY,$$

$$(\nabla_X f)Y = \omega\mathscr{A}_X Y + \mathscr{A}_X tY,$$

for any $U,V \in \Gamma(\ker\varphi_*)$ *and* $X,Y \in \Gamma((ker\varphi_*)^{\perp}$.

Proof. On using equations (3), (5) with (8)- (11) and equations (28)-(31), we get the result of lemma. □

If the tenors ψ and ω are parallel with respect to the connection ∇ of M, then we have

$$t\mathscr{T}_U V = \mathscr{T}_U \omega V,$$

$$f\mathscr{T}_U V = \mathscr{T}_U \psi V,$$

for any $U,V \in \Gamma(TM)$.

Theorem 3.6. *Let $\varphi : (M,\phi,\xi,\eta,g_M) \to (N,g_N)$ be CBSS from cosymplectic manifold onto a Riemannian manifold with slant angel θ_1 and θ_2. Then we have*

$$\psi^2 = -\cos^2\theta_i(I-\eta\otimes\xi),\ i=1,2. \tag{32}$$

4. Integrability and Totally Geodesicness

We will start the integrability of slant distributions as follows:

Theorem 4.1. *Let φ be the CBSS from the cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with slant angles θ_1 and θ_2. Then*

(i) the distribution D^{θ_1} is integrable if and only if

$$\begin{aligned}\lambda^{-2}g_N((\nabla\varphi_*)(U,\omega V),\varphi_*\omega W) =& g_M(T_V\omega\psi U - T_U\omega\psi V, W)\\ &+ g_M(T_U\omega V - T_V\omega U, \psi W)\\ &+ \lambda^{-2}g_N((\nabla\varphi_*)(V,\omega U),\varphi_*\omega W)\\ &+ \lambda^{-2}g_N(\nabla^{\varphi}_U\varphi_*\omega V,\varphi_*\omega W)\\ &- \lambda^{-2}g_N(\nabla^{\varphi}_V\varphi_*\omega U,\varphi_*\omega W),\end{aligned}$$

(ii) the distribution D^{θ_2} is integrable if and only if

$$\begin{aligned}\lambda^{-2}g_N((\nabla\varphi_*)(Z,\omega W),\varphi_*\omega U) =& g_M(T_Z\omega\psi W - T_W\omega\psi Z, U)\\ &+ g_M(T_W\omega Z - T_Z\omega W, \psi U)\\ &+ \lambda^{-2}g_N((\nabla\varphi_*)(Z,\omega W),\varphi_*\omega U)\\ &+ \lambda^{-2}g_N(\nabla^{\varphi}_Z\varphi_*\omega W,\varphi_*\omega U)\\ &- \lambda^{-2}g_N(\nabla^{\varphi}_W\varphi_*\omega Z,\varphi_*\omega U),\end{aligned}$$

for $U,V \in \Gamma(D^{\theta_1})$ and $Z,W \in \Gamma(D^{\theta_2})$.

Proof. (i). For any vector fields $U,V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$ and on using equations (2), (3) and from (16), we have

$$
\begin{aligned}
g_M([U,V],W) &= g_M(\nabla_V \psi^2 U, W) - g_M(\nabla_U \psi^2 V, W) \\
&\quad - g_M(\nabla_U \omega\psi V, W) + g_M(\nabla_V \omega\psi U, W) \\
&\quad + g_M(\nabla_U \omega V, \phi W) - g_M(\nabla_V \omega U, \phi W).
\end{aligned}
$$

Considering Theorem 3.6, we have

$$
\begin{aligned}
\sin^2\theta_1 g_M([U,V],W) &= -g_M(\nabla_U \omega\psi V, W) + g_M(\nabla_V \omega\psi U, W) \\
&\quad + g_M(\nabla_U \omega V, \phi W) - g_M(\nabla_V \omega U, \phi W).
\end{aligned}
$$

On using equation (9), we obtained

$$
\begin{aligned}
\sin^2\theta_1 g_M([U,V],W) &= g_M(\mathscr{T}_V \omega\psi U - \mathscr{T}_U \omega\psi V, W) \\
&\quad - g_M(\mathscr{T}_U \omega V - \mathscr{T}_V \omega U, \psi W) \\
&\quad + g_M(\mathscr{H}\nabla_U \omega V - \mathscr{H}\nabla_V \omega U, \omega W).
\end{aligned}
$$

Now considering Lemma 2.4 and equation (14), we get

$$
\begin{aligned}
\sin^2\theta_1 g_M([U,V],W) &= g_M(\mathscr{T}_V \omega\psi U - \mathscr{T}_U \omega\psi V, W) \\
&\quad - g_M(\mathscr{T}_U \omega V - \mathscr{T}_V \omega U, \psi W) \\
&\quad - \lambda^{-2} g_N((\nabla\varphi_*)(U,\omega V), \varphi_*\omega W) \\
&\quad + \lambda^{-2} g_N((\nabla\varphi_*)(V,\omega U), \varphi_*\omega W) \\
&\quad + \lambda^{-2} g_N(\nabla^{\varphi}_U \varphi_*\omega V, \varphi_*\omega W) \\
&\quad - \lambda^{-2} g_N(\nabla^{\varphi}_V \varphi_*\omega U, \varphi_*\omega W)
\end{aligned}
$$

For part (ii) the calculation is same as (i). □

Theorem 4.2. *Let φ be the CBSS from the cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with slant angles θ_1 and θ_2. Then the distribution D^{θ_1} defines totally geodesic foliation if and only if*

$$
\begin{aligned}
\lambda^{-2} g_N((\nabla\varphi_*)(U,\omega V), \varphi_*\omega Z) &= g_M(\mathscr{T}_U \omega V, \psi Z) - g_M(\mathscr{T}_U \omega\psi V, Z) \\
&\quad + \lambda^{-2} g_N(\nabla^{\varphi}_U \varphi_*\omega V, \varphi_*\omega Z),
\end{aligned} \tag{33}
$$

and

$$\begin{aligned}\lambda^{-2}g_M(\nabla^{\varphi}_X\varphi_*\omega U,\varphi_*\omega V) =& \sin^2\theta_1 g_M([U,X],V)+g_M(\mathscr{A}_X\omega\psi U,V)\\ &+g_M(grad\ln\lambda,X)g_M(\omega U,\omega V)\\ &+g_M(grad\ln\lambda,\omega U)g_M(X,\omega V)\\ &-g_M(grad\ln\lambda,\omega V)g_M(X,\omega U)\\ &-g_M(\mathscr{A}_X\omega U,\psi V),\end{aligned} \tag{34}$$

for $U,V\in\Gamma(D^{\theta_1})$, $Z\in\Gamma(D^{\theta_2})$ *and* $X\in((ker\varphi_*)^{\perp})$.

Proof. For $U,V\in\Gamma(D^{\theta_1})$ and $Z\in\Gamma(D^{\theta_2})$ with using equation (2), (3) and (16), we have

$$g_M(U,V,Z)=g_M(\nabla_U\omega V,\phi Z)-g_M(\nabla_U\omega\psi V,Z)-g_M(\nabla_U\psi^2V,Z).$$

From Theorem 3.6, we can write

$$\sin^2\theta_1 g_M(\nabla_U V,Z)=-g_M(\nabla_U\omega\psi V,Z)+g_M(\nabla_U\omega V,\phi Z)$$

On using (9), we have

$$\begin{aligned}\sin^2\theta_1 g_M(\nabla_U V,Z)=& g_M(\mathscr{T}_U\omega V,\psi Z)-g_M(\mathscr{T}_U\omega\psi V,Z)\\ &+g_M(\mathscr{H}\nabla_U\omega V,\omega Z).\end{aligned}$$

Considering equation (14) and Lemma 2.4, we obtain

$$\begin{aligned}\sin^2\theta_1 g_M(\nabla_U V,Z)=& g_M(\mathscr{T}_U\omega V,\psi Z)-g_M(\mathscr{T}_U\omega\psi V,Z)\\ &-\lambda^{-2}g_N((\nabla\varphi_*)(U,\omega V),\varphi_*\omega Z)\\ &+\lambda^{-2}g_N(\nabla^{\varphi}_U\varphi_*\omega V,\varphi_*\omega Z)\end{aligned}$$

which is the first part of Theorem 4.2.

On the other hand, $U,V\in\Gamma(D_1)$ and $X\in\Gamma(ker\varphi_*)^{\perp}$ with using (2), (3) and (16), we can write

$$g_M(\nabla_U V,X)=-g_M([U,X],V)+g_M(\phi\nabla_X\psi U,V)-g_M(\nabla_X\omega U,\phi V).$$

Considering Theorem 3.6, we obtained

$$\sin^2\theta_1 g_M(\nabla_U V,X)=-g_M([U,V],X)+g_M(\nabla_X\omega\psi U,V)-g_M(\nabla_X\omega U,\phi V).$$

On using equation (11), we have

$$\sin^2\theta_1 g_M(\nabla_U V,X)=\sin^2\theta_1 g_M([U,X],V)+g_M(\mathscr{A}_X\omega\psi U,V)$$
$$-g_M(\mathscr{A}_X\omega U,\psi V)-\lambda^{-2}g_N(\varphi_*\nabla_X\omega U,\varphi_*\omega V).$$

Using Lemma 2.4, we yields

$$\begin{aligned}\sin^2\theta_1 g_M(\nabla_U V,X)=&\sin^2\theta_1 g_M([U,X],V)+g_M(\mathscr{A}_X\omega\psi U,V)\\&-\lambda^{-2}g_N(\nabla_X^{\varphi}\varphi_*\omega U,\varphi_*\omega V)\\&+g_M(grad\ln\lambda,X)g_M(\omega U,\omega V)\\&+g_M(grad\ln\lambda,\omega U)g_M(X,\omega V)\\&-g_M(grad\ln\lambda,\omega V)g_M(X,\omega U)\\&-g_M(\mathscr{A}_X\omega U,\psi V)\end{aligned}$$

This completes the proof of the Theorem. □

Theorem 4.3. *Let φ be the CBSS from the cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with slant angles θ_1 and θ_2. Then the distribution D^{θ_2} defines totally geodesic foliation if and only if*

$$\begin{aligned}\lambda^{-2}g_N((\nabla\varphi_*)(W,\omega Z),\varphi_*\omega U)=&-g_M(\mathscr{T}_W\omega\psi Z,U)+g_M(\mathscr{T}_W\omega Z,\psi U)\\&+\lambda^{-2}g_N(\nabla_W^{\varphi}\varphi_*\omega Z,\varphi_*\omega U)\end{aligned}\tag{35}$$

and

$$\begin{aligned}\lambda^{-2}g_N(\nabla_X^{\varphi}\varphi_*\omega W,\varphi_*\omega Z)=&\sin^2\theta_2 g_M([W,X],Z)+g_M(\mathscr{A}_X\omega\psi W,Z)\\&+g_M(grad\ln\lambda,X)g_M(\omega W,\omega Z)\\&+g_M(grad\ln\lambda\,\omega W)g_M(X,\omega Z)\\&-g_M(grad\ln\lambda,\omega Z)g_M(X,\omega W)\\&-g_M(\mathscr{A}_X\omega W,\psi Z),\end{aligned}\tag{36}$$

for $Z,W\in\Gamma(D^{\theta_2}),U\in\Gamma(D^{\theta_1})$ and $X\in((ker\varphi_*)^{\perp})$.

Theorem 4.4. *Let φ be the CBSS from the cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with slant angles θ_1 and θ_2. Then horizontal distribution $\Gamma((ker\varphi_*)^{\perp})$ defines totally geodesic foliation if and only if*

$$
\begin{aligned}
\lambda^{-2}g_N(\nabla^{\varphi}_X\varphi_*\omega U,\varphi_* fY) &= \lambda^{-2}g_N(grad\ln\lambda,X)g_M(\omega U,fY)\\
&+g_M(grad\ln\lambda,\omega U)g_M(X,fY)\\
&-g_M(grad\ln\lambda,fY)g_M(X,\omega U)\\
&+g_M(grad\ln\lambda,X)g_M(\omega\psi U,Y)\\
&+g_M(grad\ln\lambda,\omega\psi U)g_M(X,fY)\\
&-g_M(grad\ln\lambda,Y)g_M(X,\omega\psi U)\\
&-\lambda^{-2}g_N(\nabla^{\varphi}_X\varphi_*\omega\psi U,\varphi_* Y)\\
&-g_M(\mathscr{A}_X\omega U,tY).
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\lambda^{-2}g_N(\nabla^{\varphi}_X\varphi_*\omega V,\varphi_* fY) &= \lambda^{-2}g_N(grad\ln\lambda,X)g_M(\omega V,fY)\\
&+g_M(grad\ln\lambda,\omega V)g_M(X,fY)\\
&-g_M(grad\ln\lambda,fY)g_M(X,\omega V)\\
&+g_M(grad\ln\lambda,X)g_M(\omega\psi V,Y)\\
&+g_M(grad\ln\lambda,\omega\psi V)g_M(X,fY)\\
&-g_M(grad\ln\lambda,Y)g_M(X,\omega\psi V)\\
&-\lambda^{-2}g_N(\nabla^{\varphi}_X\varphi_*\omega\psi V,\varphi_* Y)\\
&-g_M(\mathscr{A}_X\omega V,tY),
\end{aligned}
\tag{38}
$$

for $X,Y\in\Gamma((ker\varphi_*)^{\perp}),U\in\Gamma(D^{\theta_1})$ and $V\in\Gamma(D^{\theta_2})$.

Proof. For $X,Y\in\Gamma((ker\varphi_*)^{\perp})$ and $U\in\Gamma(D^{\theta_1})$ with using (2), (3) and (16), we have

$$g_M(\nabla_X Y,U)=g_M(\nabla_X\phi\psi U,Y)-g_M(\nabla_X\omega U,\phi Y).$$

Taking into account of the fact from Theorem 3.6, we can write

$$\sin^2\theta_1 g_M(\nabla_X Y, U) = -g_M(\nabla_X \omega\psi U, \phi Y) - g_M(\nabla_X \omega U, Y).$$

From (11), we can obtain

$$\begin{aligned}\sin^2\theta_1 g_M(\nabla_X Y, U) = & -g_M(\mathscr{A}_X \omega U, tY) \\ & -\lambda^{-2} g_N(\varphi_* \nabla_X \omega U, \varphi_* fY) \\ & -\lambda^{-2} g_N(\varphi_* \nabla_X \omega\psi U, \varphi_* Y).\end{aligned}$$

Considering Lemma 2.4, we have

$$\begin{aligned}\sin^2\theta_1 g_M(\nabla_X Y, U) = & \lambda^{-2} g_N(grad \ln\lambda, X) g_M(\omega U, fY) \\ & + g_M(grad \ln\lambda, \omega U) g_M(X, fY) \\ & - g_M(grad \ln\lambda, fY) g_M(X, \omega U) \\ & + g_M(grad \ln\lambda, X) g_M(\omega\psi U, Y) \\ & + g_M(grad \ln\lambda, \omega\psi U) g_M(X, fY) \\ & - g_M(grad \ln\lambda, Y) g_M(X, \omega\psi U) \\ & - \lambda^{-2} g_N(\nabla_X^{\varphi} \varphi_* \omega\psi U, \varphi_* Y) \\ & - \lambda^{-2} g_N(\nabla_X^{\varphi} \varphi_* \omega U, \varphi_* fY) \\ & - g_M(\mathscr{A}_X \omega U, tY)\end{aligned}$$

Similarly, for $X, Y \in \Gamma((ker\varphi_*)^{\perp})$ and $V \in \Gamma(D_2)$, we have

$$\begin{aligned}\sin^2\theta_2 g_M(\nabla_X Y, V) = & \lambda^{-2} g_N(grad \ln\lambda, X) g_M(\omega V, fY) \\ & + g_M(grad \ln\lambda, \omega V) g_M(X, fY) \\ & - g_M(grad \ln\lambda, fY) g_M(X, \omega V) \\ & + g_M(grad \ln\lambda, X) g_M(\omega\psi V, Y)\end{aligned}$$

$$+g_M(grad\ln\lambda,\omega\psi V)g_M(X,fY)$$
$$-g_M(grad\ln\lambda,Y)g_M(X,\omega\psi V)$$
$$-\lambda^{-2}g_N(\nabla_X^{\varphi}\varphi_*\omega\psi V,\varphi_*Y)$$
$$-\lambda^{-2}g_N(\nabla_X^{\varphi}\varphi_*\omega V,\varphi_*fY)$$
$$-g_M(\mathscr{A}_X\omega V,tY).$$

□

Theorem 4.5. *Let φ be the CBSS from the cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with slant angles θ_1 and θ_2. Then vertical distribution $(ker\varphi_*)$ defines totally geodesic foliation if and only if*

$$\begin{aligned}\lambda^{-2}g_N(\nabla_X^{\varphi}\varphi_*\omega U,\varphi_*\omega V)&=(\cos^2\theta_1-\cos^2\theta_2)g_M(\nabla_XP_2U,V)\\&+g_M(\mathscr{A}_X\psi V,\omega U)-g_M(\mathscr{A}_XV,\omega\psi U)\\&+g_M(grad\ln\lambda,X)g_M(\omega U,\omega V)\\&+g_M(grad\ln\lambda,\omega U)g_M(X,\omega V)\\&-g_M(grad\ln\lambda,\omega V)g_M(X,\omega U)\\&-\sin^2\theta_1\,g_M([U,X],V)-\eta(\nabla_XU)\eta(V),\end{aligned}\tag{39}$$

for $U,V\in\Gamma(ker\varphi_*)$ and $X\in\Gamma(ker\varphi_*)^{\perp}$.

Proof. On taking $U,V\in\Gamma(ker\varphi_*)$ and $X\in\Gamma(ker\varphi_*)^{\perp}$ with using (2), (3) and (16), we have

$$\begin{aligned}g_M(\nabla_UV,X)&=-g_M([U,X],V)+g_M(\nabla_X\phi\psi U,V)\\&-g_M(\nabla_X\omega U,\phi V)-\eta(\nabla_XU)\eta(V).\end{aligned}$$

On using decomposition (15) and Theorem 3.6, we obtained

$$\begin{aligned}g_M(\nabla_UV,X)&=-g_M([U,X],V)-\cos^2\theta_1\,g_M(\nabla_XP_1U,V)\\&-\cos^2\theta_2\,g_M(\nabla_XP_2U,V)+g_M(\nabla_X\omega\psi U,V)\\&-g_M(\nabla_X\omega U,\psi V)-g_M(\nabla_X\omega U,\omega V)\\&-\eta(\nabla_XU)\eta(V)\end{aligned}$$

With taking account the fact of equation (11), we can write

$$\sin^2\theta_1 g_M(\nabla_U V, X) = (\cos^2\theta_1 - \cos^2\theta_2) g_M(\nabla_X P_2 U, V)$$
$$- \sin^2\theta_1\ g([U,X],V) + g_M(\mathscr{A}_X \psi V, \omega U)$$
$$- g_M(\mathscr{A}_X V, \omega\psi U) - g_M(\mathscr{H}\nabla_X \omega U, \omega V)$$
$$- \eta(\nabla_X U)\eta(V).$$

Using equation (14), we yields

$$\sin^2\theta_1\ g_M(\nabla_U V, X) = (\cos^2\theta_1 - \cos^2\theta_2) g_M(\nabla_X P_2 U, V)$$
$$+ g_M(\mathscr{A}_X \psi V, \omega U) - g_M(\mathscr{A}_X V, \omega\psi U)$$
$$+ \lambda^{-2} g_N((\nabla\varphi_*)(X, \omega U), \varphi_*\omega V)$$
$$- \lambda^{-2} g_N(\nabla^{\varphi}_X \varphi_*\omega U, \varphi_*\omega V)$$
$$- \sin^2\theta_1\ g_M([U,X],V)$$
$$- \eta(\nabla_X U)\eta(V).$$

Considering Lemma 2.4, have

$$\sin^2\theta_1\ g_M(\nabla_U V, X) = (\cos^2\theta_1 - \cos^2\theta_2) g_M(\nabla_X P_2 U, V)$$
$$+ g_M(\mathscr{A}_X \psi V, \omega U) - g_M(\mathscr{A}_X V, \omega\psi U)$$
$$+ g_M(grad \ln\lambda, X) g_M(\omega U, \omega V)$$
$$+ g_M(grad \ln\lambda, \omega U) g_M(X, \omega V)$$
$$- g_M(grad \ln\lambda, \omega V) g_M(X, \omega U)$$
$$- \lambda^{-2} g_N(\nabla^{\varphi}_X \varphi_*\omega U, \varphi_*\omega V)$$
$$- \sin^2\theta_1\ g_M([U,X],V)$$
$$- \eta(\nabla_X U)\eta(V).$$

This completes the proof of the Theorem. □

Theorem 4.6. *Let φ be the CBSS from the cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then φ is totally geodesic map if and only if*

(i)

$$\begin{aligned}\lambda^{-2}g_N(\nabla^{\varphi}_U\varphi_*\omega\psi V,\varphi_*X) &= (\cos^2\theta_1-\cos^2\theta_2)g_M(\nabla_U P_2V,X)\\ &+g_M(grad\ln\lambda,U)g_M(\omega\psi V,X)\\ &-g_M(grad\ln\lambda,U)g_M(\omega V,fX)\\ &-\lambda^{-2}g_N(\nabla^{\varphi}_U\varphi_*\omega\psi V,\varphi_*X)\\ &-g_M(\mathscr{T}_U\omega V,X),\end{aligned}$$

(ii)

$$\begin{aligned}\lambda^{-2}g_N(\nabla^{\varphi}_X\varphi_*\omega U,\varphi_*fY) &= (\cos^2\theta_2-\cos^2\theta_1)g_M(\mathscr{A}_X P_1U,Y)\\ &+g_M(tX,U)\eta(Y)+g_M(\mathscr{A}_X\omega U,\omega Y)\\ &-g_M(grad\ln\lambda,X)g_M(\omega\psi U,Y)\\ &-g_M(grad\ln\lambda,\omega\psi U)g_M(X,Y)\\ &+g_M(grad\ln\lambda,Y)g_M(X,\omega\psi U)\\ &+g_M(grad\ln\lambda,X)g_M(\omega U,fY)\\ &+g_M(grad\ln\lambda,\omega U)g_M(X,fY)\\ &+g_M(grad\ln\lambda,fY)g_M(X,\omega U)\\ &+\lambda^{-2}g_N(\nabla^{\varphi}_X\varphi_*\omega\psi U,\varphi_*Y),\end{aligned}$$

for $U,V\in\Gamma(ker\varphi_*)$ *and* $X,Y\in\Gamma((ker\varphi_*)^{\perp})$.

Proof. For $U,V\in\Gamma(ker\varphi_*)$ and $X\in\Gamma((ker\varphi_*)^{\perp})$ with using equation (14), we can write

$$\lambda^{-2}g_N((\nabla\varphi_*)(U,V),\varphi_*X)=g_M(\nabla_U V,X) \tag{40}$$

From equations (2), (3) and (16), we have

$$g_M(\nabla_U V,X)=g_M(\nabla_U\omega V,\phi X)-g_M(\nabla_U\phi\psi V,X)$$

Considering Theorem 3.6 and decomposition (15), we obtained

$$\sin^2\theta_1\ g_M(\nabla_U V,X) = (\cos^2\theta_2-\cos^2\theta_1)g_M(\nabla_U P_2V,X) + g_M(\nabla_U\omega\psi V,X)+g_M(\nabla_U\omega V,\phi X). \tag{41}$$

From equation (41) and (40), we get

$$\sin^2\theta_1\ \lambda^{-2}g_N((\nabla\varphi_*)(U,V),\varphi_*X) = (\cos^2\theta_2-\cos^2\theta_1)g_M(\nabla_U P_2V,X) + g_M(\nabla_U\omega\psi V,X)+g_M(\nabla_U\omega V,\phi X).$$

On using equations (8) and (9), we have

$$\begin{aligned}\sin^2\theta_1\ \lambda^{-2}g_N((\nabla\varphi_*)(U,V),\varphi_*X) &= (\cos^2\theta_2-\cos^2\theta_1)g_M(\mathscr{T}_U P_2V,X)\\ &\quad + g_M(\mathscr{H}\nabla_U\omega\psi V,X)+g_M(\mathscr{T}_U\omega V,tX)\\ &\quad + g_M(\mathscr{H}\nabla_U\omega V,fX).\end{aligned}$$

From equation (14) and Lemma 2.4, we get

$$\begin{aligned}\sin^2\theta_1\ \lambda^{-2}g_N((\nabla\varphi_*)(U,V),\varphi_*X) &= \lambda^{-2}g_N(U(\ln\lambda)\varphi_*\omega\psi V,\varphi_*X)\\ &\quad + \lambda^{-2}g_N(\nabla^{\varphi}_U\varphi_*\omega\psi V,\varphi_*X)\\ &\quad + \lambda^{-2}g_N(U(\ln\lambda)\varphi_*\omega V,\varphi_*fX)\\ &\quad + \lambda^{-2}g_N(\nabla^{\varphi}_U\varphi_*\omega V,\varphi_*fX).\end{aligned}$$

On the other hand, for $U\in\Gamma(ker\varphi_*)$ and $X,Y\in\Gamma((ker\varphi_*)^{\perp})$, we get

$$\lambda^{-2}g_N((\nabla\varphi_*)(X,U),\varphi_*Y) = g_M(\nabla_X U,Y). \tag{42}$$

On using (2), (3) and (16), we have

$$g_M(\nabla_X U,Y) = -g_M(\nabla_X\phi\psi U,Y)+g_M(\nabla_X\omega U,\phi Y)$$

With the help of decomposition (34) and Theorem 3.6, we obtain

$$\sin^2\theta_1\ g_M(\nabla_X U,Y) = (\cos^2\theta_2-\cos^2\theta_1)g_M(\nabla_X P_1U,Y) + g_M(\nabla_X\omega U,\phi Y)-g_M(\nabla_X\omega\psi U,Y) \tag{43}$$

From (42) and (28), we can write

$$\sin^2\theta_1\ \lambda^{-2}g_N((\nabla\varphi_*)(X,U),\varphi_*Y) = (\cos^2\theta_2-\cos^2\theta_1)g_M(\nabla_X P_1U,Y)$$
$$+g_M(\nabla_X\omega U,\phi Y)-g_M(\nabla_X\omega\psi U,Y).$$

From equation (10) and (11), we have

$$\sin^2\theta_1\ \lambda^{-2}g_N((\nabla\varphi_*)(X,U),\varphi_*Y) = (\cos^2\theta_2-\cos^2\theta_1)g_M(\mathscr{A}_X P_1U,Y)$$
$$+g_M(\mathscr{A}_X\omega U,tY)-g_M(\mathscr{H}\nabla_X\omega\psi U,Y)$$
$$+g_M(\mathscr{H}\nabla_X\omega U,Y).$$

Considering Lemma 2.4 with equation (14), we yields

$$\begin{aligned}\sin^2\theta_1\lambda^{-2}g_N((\nabla\varphi_*)(X,U),\varphi_*Y) = {} & (\cos^2\theta_2-\cos^2\theta_1)g_M(\mathscr{A}_X P_1U,Y)\\ & -g_M(grad\ln\lambda,X)g_M(\omega\psi U,Y)\\ & -g_M(grad\ln\lambda,\omega\psi U)g_M(X,Y)\\ & +g_M(grad\ln\lambda,Y)g_M(X,\omega\psi U)\\ & +g_M(grad\ln\lambda,X)g_M(\omega U,fY)\\ & +g_M(grad\ln\lambda,\omega U)g_M(X,fY)\\ & -g_M(grad\ln\lambda,fY)g_M(X,\omega U)\\ & +\lambda^{-2}g_N(\nabla^{\varphi}_X\varphi_*\omega\psi U,Y)+g_M(\mathscr{A}_X\omega U,tY)\\ & -\lambda^{-2}g_N(\nabla^{\varphi}_X\varphi_*\omega U,\varphi_*fY).\end{aligned}$$

Finally we show that λ is constant on $\Gamma(D_1)$. For $U_1,U_2\in\Gamma(D_1)$ and from Lemma 2.4. we obtain

$$(\nabla\varphi_*)(\omega U_1,\omega U_2) = \omega U_1(\ln\lambda)\varphi_*\omega U_2+\omega U_2(\ln\lambda)\varphi_*\omega U_1$$
$$-g_M(\omega U_1,\omega U_2)\varphi_*(grad\ln\lambda).$$

Replacing U_2 by U_1 in above equation, we get

$$(\nabla\varphi_*)(\omega U_1,\omega U_1) = 2\omega U_1(\ln\lambda)\varphi_*\omega U_1 - g_M(\omega U_1,\omega U_1)\varphi_*(grad\ln\lambda). \tag{44}$$

Taking inner product with $\varphi_*\omega U_1$ in (44), we can write

$$2g_M(grad\ln\lambda,\omega U_1)g_N(\varphi_*\omega U_1,\varphi_*\omega U_1)$$

$$-g_M(\omega U_1,\omega U_1)g_N(\varphi_* grad\ln\lambda,\varphi_\omega U_1)=0,$$

which shows that λ is constant on $\Gamma(D^{\theta_1})$. Similarly, we can show that λ is constant on $\Gamma(D^{\theta_2})$ and $\Gamma(\mu)$. This completes the proof of the Theorem. □

5. Decomposition Theorems

In this section, we give some decomposition theorems with the help of previous results. We start as follows:

Theorem 5.1. *Let φ be the CBSS from the cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with slant angles θ_1 and θ_2. Then vertical distribution $(ker\varphi_*)$ is locally Riemannian product $M_{D^{\theta_1}}\times M_{D^{\theta_2}}$ if and only if equation (33)-(36) holds where $M_{D^{\theta_1}}$ and $M_{D^{\theta_2}}$ are integral manifolds of distributions D^{θ_1} and D^{θ_2} respectively.*

Theorem 5.2. *Let φ be the CBSS from the cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with slant angles θ_1 and θ_2. Then the total space $M_{D^{\theta_1}}\times M_{D^{\theta_2}}\times M_{((ker\varphi_*)^\perp)}$ is locally product if and only if equation (33)-(38) are holds, where $M_{D^{\theta_1}}, M_{D^{\theta_2}}$ and $M_{((ker\varphi_*)^\perp)}$ are integral manifolds of the distributions D^{θ_1}, D^{θ_2} and $((ker\varphi_*)^\perp)$ respectively.*

Theorem 5.3. *Let φ be the CBSS from the cosymplectic manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with slant angles θ_1 and θ_2. Then the total space $M_{ker\varphi_*}\times M_{((ker\varphi_*)^\perp)}$ is locally product if and only if equation (37)-(39) are holds where $M_{ker\varphi_*}$ and $M_{((ker\varphi_*)^\perp)}$ are integral manifolds of the distributions $(ker\varphi_*)$ and $((ker\varphi_*)^\perp)$ respectively.*

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