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SEMI-INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLD WITH GENERALIZED ALMOST r-CONTACT STRUCTURE ADMITTING SEMI- SYMMETRIC SEMI -METRIC CONNECTION

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Abstract: We consider an almost r-contact Kenmotsu manifold admitting a semi-symmetric semi-metric connection and study semi-invariant submanifolds of an almost r-contact Kenmotsu manifold endowed with a semi- symmetric semi-metric connection. We obtain Gauss and Weingarten formula for such connection. We also discuss the integrability conditions of distributions on Kenmotsu manifold.

Keywords: Kenmotsu manifold, semi-invariant submanifolds, almost r-contact structure, semisymmetric semi-metric connection, integrability conditions.

2000 AMS Subject Classification: 53D12, 53C05

1. Introduction

In 1924, A. Friedmann and J. A. Schouten [10] introduced the notion of semi-symmetric linear connection. In 1975, S. Golab studied some properties of a semi-symmetric and quarter symmetric linear connections [11]. A linear connection ∇ is said to be semi-symmetric if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \qquad (1.1)$$

where η is a 1-form and ϕ is a tensor field of type (1, 1).

In [7] B. Barua introduced the idea of semi-symmetric semi-metric connection and investigated some of its properties. A linear connection ∇ is said to be semi-symmetric if it satisfies (1.1).

$$(\nabla_X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y).$$

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On the otherhand in 1981, Bejancu [8] introduced the idea of semi-invariant or contact CR-submanifolds, as a generalization of invariant and anti-invariant submanifolds of an almost contact-metric manifold and was followed by several geometers (See [1], [2], [3], [4], [5], [9], [13], [14]). Semi-invariant submanifolds of a certain class of almost contact manifolds studied by M. Kobayashi in [12]. Semi-invariant submanifold of Kenmotsu manifold with generalized almost r-contact structure admitting a quarter-symmetric non-metric connection, was studied by first author et. al. in [6].

In this paper, we study semi-invariant submanifolds of Kenmotsu manifold with generalized almost r-contact structure admitting a semi-symmetric semi-metric connection. The paper is organized as follows. In section 2, we give a brief introduction of Kenmotsu manifold with generalized almost r-contact structure. In section 3, we study semi-invariant submanifolds of generalized Kenmotsu manifold with semi-symmetric semi-metric connection. In section 4, we discuss the integrability conditions of distribution D and $D^{\perp}$. In section 5, consequences of parallel horizontal distributions are discussed.

In [12], K. Kenmotsu defined a new class of almost contact manifold called Kenmotsu manifold.

2. Preliminaries

Let $\bar{M}$ be a (2n+r)-dimensional Kenmotsu manifold with generalized almost r-contact structure (ϕ, ξ_p, η_p, g), where ϕ is a tensor field of type (1, 1), ξ_p are r-vector fields, η_p are r 1-forms and g the associated Riemannian metric satisfying

$$(a)\ \phi^2 = a^2 I + \sum_{p=1}^{r} \eta_p \otimes \xi_p,$$

$$(b)\ \eta_p(\xi_q) = \delta_{pq}, \qquad p,q, \in (r) := 1, 2, 3, \ldots\ldots r \qquad (2.1)$$

$$(c)\ \phi(\xi_p) = 0, \qquad p \in (r)$$

$$(d)\ \eta_p(\phi X) = 0$$

and

$$g(\phi X,\phi Y)+a^2 g(X,Y)+\sum_{p=1}^{r}\eta_p(X)\eta_p(Y)=0, \tag{2.2}$$

$$\eta_p(X)=g(X,\xi_p), \tag{2.3}$$

$$(\overline{\overline{\nabla}}_X\phi)Y=-\sum_{p=1}^{r}\eta_p(Y)\phi X-g(X,\phi Y)\sum_{p=1}^{r}\xi_p \quad , \tag{2.4}$$

$$\overline{\overline{\nabla}}_X\xi_p=X-\sum_{p=1}^{r}\eta_p(X)\xi_p, \tag{2.5}$$

where *I* is the identity tensor field and X, Y are vector fields on $\overline{M}$ and $\overline{\overline{\nabla}}$ denotes the Riemannian connection.

3. Semi-invariant submanifolds

An *n*-dimensional Riemannian submanifold *M* of a Kenmotsu manifold with almost r-contact structure $\overline{M}$ is called a semi-invariant submanifold if ξ_p is tangent to *M* and there exists on *M* a pair of orthogonal distributions (D, $D^{\perp}$) such that

(i) $TM=D\oplus D^{\perp}$,

(ii) distribution *D* is invariant under ϕ, that is $\phi D_x = D_x$ for all $x\in$ M,

(iii) distribution $D^{\perp}$ is anti-invariant under ϕ, that is $\phi D_x^{\perp}\subset T_x^{\perp}M$ for all $x\in$ M,

where T_xM and $T_x^{\perp}M$ are respectively the tangent and normal space of *M* at *x*.

The distribution *D* (resp. $D^{\perp}$) can be defined by projection *P* (resp. Q) which satisfies the conditions.

$$P^2=P,\quad Q^2=Q,\quad PQ=QP=0 \tag{3.1}$$

The distribution D (resp. $D^{\perp}$) is called the *horizontal* (resp. *vertical*) distribution.

A semi-invariant submanifold *M* is said to be an invariant (resp. *anti-invariant*) submanifold if we have $D_x^{\perp}=\{0\}$ (resp. $D_x=\{0\}$) for each $x\in M$, we also call *M* is proper if neither D nor $D^{\perp}$ is null. It is easy to check that each hypersurface of *M* which is tangent to ξ_p inherits a structure of the semi-invariant submanifold of $\overline{M}$.

Now we define a semi-symmetric semi-metric connection $\overline{\nabla}$ in a Kenmotsu manifold with generalized almost *r*-contact structure by

$$\overline{\nabla}_X Y=\overline{\overline{\nabla}}_X Y-\sum_{p=1}^{r}\eta_p(X)Y+g(X,Y)\sum_{p=1}^{r}\xi_p \tag{3.2}$$

such that $(\bar{\nabla}_X g)(Y,Z) = 2\eta_p(X)g(Y,Z) - \eta_p(Y)g(Z,X) - \eta_p(Z)g(X,Y)$

for any X and $Y \in TM$, where $\bar{\bar{\nabla}}$ is the induced connection on M.

from (2.4) and (3.2), we have

$$(\bar{\nabla}_X \phi)Y = -\sum_{p=1}^{r} \eta_p(Y)\phi X . \tag{3.3}$$

Let $\bar{\nabla}$ be the semi-symmetric semi-metric connection on $\bar{M}$ and ∇ be the induced connection M with respect to the unit normal N.

Theorem 3.2. The connection induced on semi-invariant submanifolds of a generalized Kenmotsu manifold with semi-symmetric semi-metric connection is also a semi-symmetric semi-metric connection.

Proof. Let ∇ be the induced connection with respect to unit normal N on semi-invariant submanifolds of a generalized Kenmotsu manifold with semi-symmetric semi-metric connection $\bar{\nabla}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + m(X,Y),$$

where m is a tensor field of type (0, 2) on semi-invariant submanifold M. If ∇^* be the induced connection on semi-invariant submanifolds from Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_X Y = \nabla^*_X Y + h(X,Y),$$

where h is second fundamental tensor. Now using (3.2), we have

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) - \sum_{p=1}^{r} \eta_p(X)Y + g(X,Y)\sum_{p=1}^{r} \xi_p.$$

Equating the tangential and normal components from both sides of the above equation, we get

$$h(X,Y) = m(X, Y)$$

and
$$\nabla_X Y = \nabla^*_X Y - \sum_{p=1}^{r} \eta_p(X)Y + g(X,Y)\sum_{p=1}^{r} \xi_p.$$

Thus ∇ is also a semi-symmetric semi-metric connection.

Now, the Gauss formula for semi-invariant submanifold of a generalized Kenmotsu manifold with semi-symmetric semi-metric connection is

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y). \tag{3.4}$$

and the Weingarten formula for *M* is given by

$$\bar{\nabla}_X N = -A_N X - \eta_p(X)N + \nabla^{\perp}_X N \tag{3.5}$$

for $X,Y \in TM$, $N \in T^{\perp}M$, where *h* and *A* are called the second fundamental tensors of *M* in and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X,Y). \tag{3.6}$$

Any vector *X* tangent to *M* is given as

$$X = PX + QX + \eta_p(X)\xi_p, \tag{3.7}$$

where *PX* and *QX* belong to the distribution *D* and $D^{\perp}$ respectively. For any vector field *N* normal to *M*, we have

$$\phi N = BN + CN, \tag{3.8}$$

where *BN* (resp. CN) denotes the tangential (resp. normal) component of ϕN.

4. Integrability of distributions

Lemma 4.1. Let *M* be ξ_p-horizontal semi-invariant submanifold of a generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric semi-metric connection. Then

$$2(\bar{\nabla}_X \phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y,\phi X) - \phi[X,Y] - \eta(X)\phi Y$$

for any $X \in$ D and $Y \in$ D$^{\perp}$.

Proof. Using Weingarten formula, we have

$$\bar{\nabla}_X \phi Y = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \eta(X)\phi Y.$$

Using Gauss formula, we have

$$\bar{\nabla}_Y \phi X = \nabla_Y \phi X + h(Y,\phi X).$$

Subtracting above two equations, we get

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y,\phi X) - \eta(X)\phi Y. \tag{4.1}$$

Also, by covariant differentiation, we obtain

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = \left(\bar{\nabla}_X \phi\right) Y - \left(\bar{\nabla}_Y \phi\right) X + \phi\left(\bar{\nabla}_X Y\right) - \phi \bar{\nabla}_Y X . \quad (4.2)$$

Thus, from (4.1) and (4.2), we get

$$\left(\bar{\nabla}_X \phi\right) Y - \left(\bar{\nabla}_Y \phi\right) X = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \nabla_Y \phi X$$

$$-h\left(Y, \phi X\right) - \phi\left[X, Y\right] - \eta\left(X\right) \phi Y . \quad (4.3)$$

Now, from (3.3), we have

$$\left(\bar{\nabla}_X \phi\right) Y + \left(\bar{\nabla}_Y \phi\right) X = 0 . \quad (4.4)$$

for each $X \in$ D and $Y \in$ D$^{\perp}$.

Adding (4.3) and (4.4), we get the desired result.

Similarly, we can also prove that

Lemma 4.2. Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold with a semi-symmetric semi-metric connection. Then

$$2(\bar{\nabla}_X \phi) Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]$$

for each $X, Y \in D$.

Lemma 4.3. Let M be a semi-invariant submanifold of generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric semi-metric connection. Then

$$P \nabla_X \phi P Y - P A_{\phi Q Y} X = \phi P \nabla_X Y - 2 \sum_{p=1}^{r} \eta_p(Y) \phi P X \ , \quad (4.5)$$

$$Q \nabla_X \phi P Y - Q A_{\phi Q Y} X = B h(X, Y) \ , \quad (4.6)$$

$$h(X, \phi P Y) + \nabla_X^{\perp} \phi Q Y = \phi Q \nabla_X Y$$

$$+ C h(X, Y) + \eta_p\left(X\right) \phi Q Y - \sum_{p=1}^{r} \eta_p\left(Y\right) \phi Q X , \quad (4.7)$$

$$\eta_p(\nabla_X \phi P Y) \xi_p - \eta_p(A_{\phi Q Y} X) \xi_p - \eta_p\left(\nabla_X Y\right) \xi_p = 0 \quad (4.8)$$

for all X and $Y \in TM$.

Proof. We know that $\bar{\nabla}_X \phi Y = (\bar{\nabla}_X \phi) Y + \phi(\bar{\nabla}_X Y)$.

Using (3.4), (3.7), the above equation takes the form

$$(\bar{\nabla}_X \phi) Y = \bar{\nabla}_X \phi P Y + \bar{\nabla}_X \phi Q Y - \phi \nabla_X Y - \phi h(X, Y) . \quad (4.9)$$

Using Gauss and Weingarten formulas and (3.8), we get

$$(\bar{\nabla}_X\phi)Y = P\nabla_X\phi PY + Q\nabla_X\phi PY + \eta_p(\nabla_X\phi PY)\xi_p + h(X,\phi PY)$$
$$-PA_{\phi QY}X - QA_{\phi QY}X - \eta_p(A_{\phi QY}X)\xi_p + \nabla^{\perp}_X\phi QY$$
$$-\eta_p(X)\phi QY - \phi P\nabla_X Y - \phi Q\nabla_X Y - \eta_p(\nabla_X Y)\xi_p$$
$$-Bh(X,Y) - Ch(X,Y). \tag{4.10}$$

Comparing (4.9) and (4.10) and equating horizontal, vertical and normal components, we get (4.5), (4.6), (4.7) and (4.8) respectively.

Definition 4.4. The horizontal distribution *D* is said to be parallel with respect to the connection ∇ on *M*, if $\nabla_X Y \in D$ for all vector fields *X* and $Y \in$ D.

Theorem 4.5. Let *M* be a semi-invariant submanifold of generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric semi-metric connection. If *M* is ξ_p-horizontal, then the distribution *D* is integrable if and only if

$$h(X, \phi Y) = h(\phi X, Y) \tag{4.11}$$

for all *X* and $Y \in D$.

Proof. Let *M* be ξ_p-horizontal, then (4.7) reduces to

$$h(X,\phi Y) = \phi Q \nabla_X Y + Ch(X,Y)$$

Thus, we have

$$h(X,\phi Y) - h(\phi X,Y) = \phi Q[X,Y].$$

If *M* is ξ_p horizontal and

$$h(X,\phi Y) = h(\phi X,Y) \quad \text{for all } X, Y \in D.$$

then $\quad [X,Y] \in D.$

Hence distribution *D* is integrable.

Theorem 4.6. Let *M* be a semi-invariant submanifold of a generalized Kenmotsu manifold $\bar{M}$ with semi-symmetric semi-metric connection. If *M* is ξ_p-vertical then the distribution $D^{\perp}$ is integrable if and only if $A_{\phi X}Y = A_{\phi Y}X$ for each $X,Y \in D^{\perp}$.

Proof. Let *M* be ξ_p-vertical, then from (4.7), we have

$$\nabla^{\perp}_X \phi Y = \phi Q \nabla_X Y + Ch(X,Y) + \sum_{p=1}^{r} \eta_p(X)\phi QY - \sum_{p=1}^{r} \eta_p(Y)\phi QX \tag{4.13}$$

for all $X, Y \in D^{\perp}$.

Using (3.3) and (3.7), we get

$$\bar{\nabla}_X \phi Y = -\sum_{p=1}^{r} \eta_p(Y)\phi X + \phi P \nabla_X Y + \phi Q \nabla_X Y + Bh(X,Y) + Ch(X,Y). \tag{4.14}$$

For ξ_p-vertical manifold M, Weingarten formula is given by

$$\nabla^{\perp}_X \phi Y = \bar{\nabla}_X \phi Y + A_{\phi Y} X + \eta_p(X)\phi Y .$$

Using (4.14), we have

$$\nabla^{\perp}_X \phi Y = -\sum_{p=1}^{r} \eta_p(Y)\phi X + \phi P \nabla_X Y + \phi Q \nabla_X Y$$

$$+ Bh(X,Y) + Ch(X,Y) + A_{\phi Y} X + \sum_{p=1}^{r} \eta_p(X)\phi Y . \tag{4.15}$$

From (4.13) and (4.15), we have

$$\phi P \nabla_X Y = -A_{\phi Y} X - Bh(X,Y).$$

Similarly $\phi P \nabla_Y X = -A_{\phi X} Y - Bh(X,Y),$

which gives

$$\phi P[X,Y] = A_{\phi X} Y - A_{\phi Y} X .$$

Thus if *M* is ξ_p-vertical, then $[X, Y] \in D^{\perp}$, *that is* $P[X, Y] = 0$, if and only if $A_{\phi X}Y = A_{\phi Y}X$.

5. Parallel horizontal distributions

Definition 5.1. A non-zero normal vector field *N* is said to be *D*-parallel normal section if

$$\nabla^{\perp}_X N = 0 \quad \text{for all } X \in D. \tag{5.1}$$

Definition 5.2. *M* is said to be totally r-contact umbilical if there exist a normal vector *H* on *M* such that

$$h(X,Y)=g(\phi X,\phi Y)H+\sum_{p=1}^{r}\eta_p(X)h(Y,\xi_p)+\sum_{p=1}^{r}\eta_p(Y)h(X,\xi_p) \tag{5.2}$$

for all vector fields *X, Y* tangent to *M* .

If *H* = 0, that is the fundamental form is given by

$$h(X,Y)=\sum_{p=1}^{r}\eta_p(X)h(Y,\xi_p) + \sum_{p=1}^{r}\eta_p(Y)h(X,\xi_p)\,, \tag{5.3}$$

then *M* is said to be totally *r*-contact geodesic manifold.

Theorem 5.1. If *M* is totally *r*-contact umbilical semi-invariant submanifold of a generalized Kenmotsu manifold $\bar{M}$ with a semi-symmetric semi-metric connection with parallel horizontal distribution, then *M* is totally *r*-contact geodesic.

Proof. Since *M* is semi-invariant submanifold of a generalized Kenmotsu manifold $\bar{M}$ with semi-symmetric semi-metric connection. From (4.5) and (4.6) we have

$$P\nabla_X\phi PY - PA_{\phi QY}X = \phi P\nabla_X Y - 2\sum_{p=1}^{r}\eta_p(Y)\phi PX\,,$$

$$Q\nabla_X\phi PY - QA_{\phi QY}X = Bh(X,Y)\;.$$

Adding the above equations, we have

$$P\nabla_X\phi PY + Q\nabla_X\phi PY - (PA_{\phi QY}X + QA_{\phi QY}X)$$

$$= \phi P\nabla_X Y + Bh(X,Y) - \sum_{p=1}^{r}\eta_p\left(Y\right)\phi PX\,. \tag{5.4}$$

Interchanging *X* and *Y* in (5.4), we have

$$\nabla_Y\phi PX - A_{\phi QX}Y = \phi P\nabla_Y X + Bh(Y,X) - \sum_{p=1}^{r}\eta_p\left(X\right)\phi PY\,. \tag{5.5}$$

Adding (5.4) and (5.5), we get

$$\nabla_X\phi PY + \nabla_Y\phi PX - A_{\phi QY}X - A_{\phi QX}Y = \phi P\nabla_X Y + \phi P\nabla_Y X + 2Bh(X,Y)$$

$$-\sum_{p=1}^{r}\eta_p\left(Y\right)\phi PX - \sum_{p=1}^{r}\eta_p\left(X\right)\phi PY\,.$$

Taking inner product, we get

$$g(\nabla_X\phi PY + \nabla_Y\phi PX - A_{\phi QY}X - A_{\phi QX}Y,Z) = g(\phi P\nabla_X Y + \phi P\nabla_Y X + 2Bh(X,Y),Z)$$

$$-\sum_{p=1}^{r}\eta_p\left(Y\right)g\left(\phi PX,Z\right) - \sum_{p=1}^{r}\eta_p\left(X\right)g\left(\phi PY,Z\right).$$

Splitting the above equation , we get

$$g(\nabla_X \phi PY,Z)+g(\nabla_Y \phi PX,Z)-g(A_{\phi QY}X,Z)-g(A_{\phi QX}Y,Z)$$

$$=g(\phi P\nabla_X Y,Z)+g(\phi P\nabla_Y X,Z)+g[(2B\{g(\phi X,\phi Y)H+\sum_{p=1}^{r}\eta_p(X)h(Y,\xi_p)$$

$$+\sum_{p=1}^{r}\eta_p(Y)h(X,\xi_p)\},Z)] -\sum_{p=1}^{r}\eta_p(Y)g(\phi PX,Z)-\sum_{p=1}^{r}\eta_p(X)g(\phi PY,Z).$$

$$g(\nabla_X \phi PY,Z)+g(\nabla_Y \phi PX,Z)-g(h(X,Z),\phi QY)-g(h(Y,Z),\phi QX)$$

$$=g(\phi P\nabla_X Y,Z)+g(\phi P\nabla_Y X,Z)+2g(\phi X,\phi Y)g(BH,Z)+2\sum_{p=1}^{r}\eta_p(X)g(Bh(Y,\xi_p),Z)$$

$$+2\sum_{p=1}^{r}\eta_p(Y)g(Bh(X,\xi_p),Z) -\sum_{p=1}^{r}\eta_p(Y)g(\phi PX,Z)-\sum_{p=1}^{r}\eta_p(X)g(\phi PY,Z).$$

$$=g(\phi P\nabla_X Y,Z)+g(\phi P\nabla_Y X,Z)-2a^2g(X,Y)g(BH,Z)-2\sum_{p=1}^{r}\eta_p(X)\eta_p(Y)g(BH,Z)$$

$$+2\sum_{p=1}^{r}\eta_p(X)g(h(Y,\xi_p),\phi Z) +2\sum_{p=1}^{r}\eta_p(Y)g(h(X,\xi_p),\phi Z)$$

$$-\sum_{p=1}^{r}\eta_p(Y)g(\phi PX,Z) -\sum_{p=1}^{r}\eta_p(X)g(\phi PY,Z).$$

Replacing *Z* by *X* and Y by *BH* and using (5.2), we get

$$g(\nabla_X \phi PBH,X)+g(\nabla_{BH}\phi PX,X)-g(X,X)g(H,\phi QBH)$$
$$-g(BH,X)g(H,\phi QX)$$

$$=g(\phi P\nabla_X BH,X)+g(\phi P\nabla_{BH}X,X)-2a^2g(X,BH)g(BH,X)$$
$$+2\sum_{p=1}^{r}\eta_p(BH)g\left(h\left(X,\xi_p\right),\phi X\right)-\sum_{p=1}^{r}\eta_p(BH)g(\phi PX,X)$$
$$-\sum_{p=1}^{r}\eta_p(X)g(\phi PBH,X). \tag{5.6}$$

Thus, we have

$$g(X,BH) = g(\phi X,BH) = 0 \text{ for any } X\in D .$$

Differentiating above equation covariantly with respect to *X*, we get

$$g(\nabla_X \phi X, BH) + g(\phi X, \nabla_X BH) = 0.$$

For parallel horizontal distribution *D*, we have

$$g(\phi X, \nabla_X BH) = 0. \tag{5.7}$$

Using (5.7) in (5.6), we get

$$g(\nabla_{BH}\phi PX, X) - g(H, \phi QBH) = g(\phi P\nabla_{BH}X, X) - \sum_{p=1}^{r}\eta_p(BH)g(\phi X, X).$$

$$g((\nabla_{BH}\phi P)X, X) + g(\phi P\nabla_{BH}X, X) - g(H, \phi QBH)$$

$$= g(\phi P\nabla_{BH}X, X) - \sum_{p=1}^{r}\eta_p(BH)g(\phi X, X).$$

for a unit vector $X \in D$.

$$g((\nabla_{BH}\phi P)X, X)) = \sum_{p=1}^{r}\eta_p(BH)g(\phi X, X). \tag{5.8}$$

From (5.8), we have

$$g(BH, QBH) + \sum_{p=1}^{r}\eta_p(BH)g(\phi X, X) = 0$$

which implies that BH=0.

Since $\phi H \in D^{\perp}$, we have $CH = 0$, hence $\phi H = 0$, thus $H = 0$.

Hence *M* is totally *r*-contact geodesic.

Remark. For a generalized Kenmotsu manifold with semi-symmetric semi-metric connection, we have

$$\bar{\nabla}_X \xi_p = \bar{\bar{\nabla}}_X \xi_p - \sum_{p=1}^{r}\eta_p(X)\xi_p + g(X, \xi_p)\sum_{p=1}^{r}\xi_p \tag{5.9}$$

$$= PX + QX.$$

Equating the tangential and normal components, we have

$$\nabla_X \xi_P + h(X, \xi_p) = PX + QX, \tag{5.10}$$

$$\nabla_X \xi_p = PX, \tag{5.11}$$

$$h(X, \xi_p) = QX. \tag{5.12}$$

From (5.10) and (5.11), we obtain

$$\nabla_X \xi_p = 0 \text{ for } X \in D^{\perp},$$

$$h\left(X, \xi_p\right) = 0 \text{ for } X \in D.$$

Theorem 5.2. Let M be D-umbilic (resp. $D^{\perp}$-umbilic) semi-invariant submanifold of a generalized Kenmotsu manifold M with semi-symmetric semi-metric connection. If ξ_p-horizontal (resp. ξ_p-vertical) then M is totally geodesic (resp. $D^{\perp}$ - totally geodesic).

Proof. If M is D -umbilic semi-invariant submanifold of a generalized Kenmotsu manifold with semi-symmetric semi-metric connection with ξ_p-horizontal then, we have

$$h(X, \xi_p) = g(X, \xi_p)L,$$

which means that $L=0$ from which we get $h(X,\xi_p) = 0$.

Hence M is D-totally geodesic.

Similarly, we can prove that if M is a $D^{\perp}$-umbilic semi-invariant submanifold with ξ_p-vertical, then M is $D^{\perp}$-totally geodesic.

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