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AN ORTHOGONAL PROJECTION ALGORITHM FOR SOLVING QUADRATIC PROGRAMMING PROBLEMS

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Abstract: In this paper, we combine the classical method of unconstrained optimization, the methods for solving linear programming problems and the orthogonal projection technique to evolve a new algorithm for solving quadratic programming problems. We solve the quadratic programming problem ignoring the constraints using the classical method to obtain the unconstrained optimum. The unconstrained optimum is then tested for feasibility in the original quadratic programming problem. Feasibility of the unconstrained optimum implies that it is optimal solution to the original problem. If the unconstrained optimum is infeasible we then make moves to search for the feasible optimal solution by projecting it orthogonally on the hyper-plane of each of the violated constraints. Feasibility of any of the projected points indicates optimal solution, while infeasibility indicates that the optimal solution is at extreme point of the feasible region and is obtained by solving linear approximation of the quadratic programming problem. From the computational results, our proposed algorithm performed well to solve quadratic programming problems.

Key words: Quadratic programming, unconstrained optimum, feasible region, boundary point, extreme point, orthogonal projection, optimal solution, linear approximation

1. Introduction

A Quadratic Programming Problem (QPP) is a mathematical program in which a quadratic objective function is being optimized subject to a set of linear constraints. It is a Non-Linear Programming Problem which differs from the linear programming problem (LPP) with a non-linear objective function (consisting of terms involving the square of a variable or product of two

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variables) and a linear constraint(s). Quadratic programming, especially convex quadratic programming, plays important role in optimization. In one sense it is a continuous optimization problem and a fundamental subroutine frequently employed by general nonlinear programming. The optimality conditions of quadratic programming are linear equalities and inequalities and this make it one of the challenging combinatorial problems.

QPP arises naturally in a variety of applications such as Portfolio selection, Martin et al (1955); structural analysis, Atkociunas (1996); and many Economic and Business problems like Elasticity of demand, manufacturing companies and Spatial Equilibrium Analysis, Sharma (2008).

Several solution techniques have been developed to solve quadratic programming problems, see Wolf (1959), Fiacco and Cormick (1968), Fletcher (1981), Griffith and Stewart (1961), Nesterov (1999), Cottle et al (2009), Zontendijk (1960), Rosen (1960), Kozlou, et al (1970), Huynh (2008).

The current method of solving quadratic programming problem is the interior point method. The interior point method starts from an initial feasible point and make progress through the feasible region toward the optimal solution in a polynomial time, Karmarkar (1984). This method which is considered to be the most efficient method for solving both linear and quadratic programming problems has some pitfalls. It is computationally demanding and takes too much iteration to converge at the approximate optimal solution.

We are motivated by the fact that it is easier to solve linear programming problems and the unconstrained convex optimization problems than to solve constrained convex quadratic programming problems. We are further motivated by the fact that the unconstrained optimum is the centre of the concentric parabolic curves described by the quadratic objective function, and that the objective function increases in values as we move away from the centre towards the feasible region. The first point of contact of the curve with the feasible region yields the optimal solution to problem. This point could be a boundary point or an extreme point, and could be determined easily either by solving a linear programming approximation of the QPP or by the orthogonal projection technique.

In this paper, we combine the classical method of unconstrained optimization, the methods of solving linear programming problems and the orthogonal projection technique to evolve a new algorithm for solving quadratic programming problems.

2. Preliminaries

2.1 General Quadratic Programming Problems

The generic quadratic programming problem (QPP) as given in Sharma (2008) is

$$\text{Minimize } Z = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i x_j \qquad 1(a)$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \text{ ; } i = 1,2,\ldots,m \qquad 1(b)$$

$$x_j \geq 0; \ \ j = 1,2,\ldots,n. \qquad 1(c)$$

It is also written in matrix notation as :

$$\text{Minimize } Z = Cx + \tfrac{1}{2} x^T Q x \qquad 2(a)$$

Subject to:

$$Ax \le b \qquad 2(b)$$

$$x \geq 0 \qquad 2(c)$$

Where $x = (x_1, x_2, \ldots, x_n)^T$; $C = (C_1, C_2, \ldots, C_n)$;

$b = (b_1, b_2, \ldots, b_n)^T$; $Q = n \times n$ symmetric matrix and $A = m \times n$ matrix.

2.2. The classical theory of unconstrained optimization

Given a function of n – variables, $f(x_1, x_2, \ldots, x_n)$ which is convex and has continuous partial derivatives, then the minimum point is the solution of the following system of equations

$$\frac{\partial f}{\partial x_1} = 0,\ \frac{\partial f}{\partial x_2} = 0, \cdots, \frac{\partial f}{\partial x_n} = 0 \qquad (3)$$

2.3. Linear Programming approximation of Quadratic Programming

Let y be any feasible solution of a quadratic programming problem. Here y might be determined by phase 1 of the simplex method or the modified simplex splitting algorithm, see Effanga, et al (2012).

Given the quadratic objective function, 1(a) or 2(a), then its linear approximation at the feasible point y is obtained through

$$Z' = Z^0 + m\,(x - y) \qquad (4)$$

Where Z^0 is the value of Z evaluated at the point y, m is the slope, or partial derivative, of Z with respect to x, evaluated at the point y, Magnanti, et al (1977)

That is,

$$m = c + y^T Q \qquad (5)$$

Now, since Z^0, y and m are fixed, minimizing Z' in (4) is equivalent to minimizing

$$Z'' = mx \qquad (6)$$

So the linear programming approximation of the quadratic programming is

$$\text{Min } Z'' = (c + y^T Q)x \qquad (7a)$$

Subject to:

$$Ax \le b \qquad (7b)$$

$$x \ge 0 \qquad (7c)$$

2.4. The orthogonal projection

The perpendicular distance d of the point x^0 from the hyper-plane $A_i x = b_i$ is given by

$$d_i = \frac{A_i x^0 - b_i}{\sqrt{A_i A_i^T}} \qquad (8)$$

Where $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ is a vector normal to the i^{th} hyper - plane

The orthogonal projection of x^0 onto the hyper-plane $A_i x = b_i$ is therefore define through

$$x^i = x^0 - d_i \hat{A}_i^T \qquad (9)$$

Where $\hat{A}_i$ is a unit vector normal to the i^{th} hyper – plane.

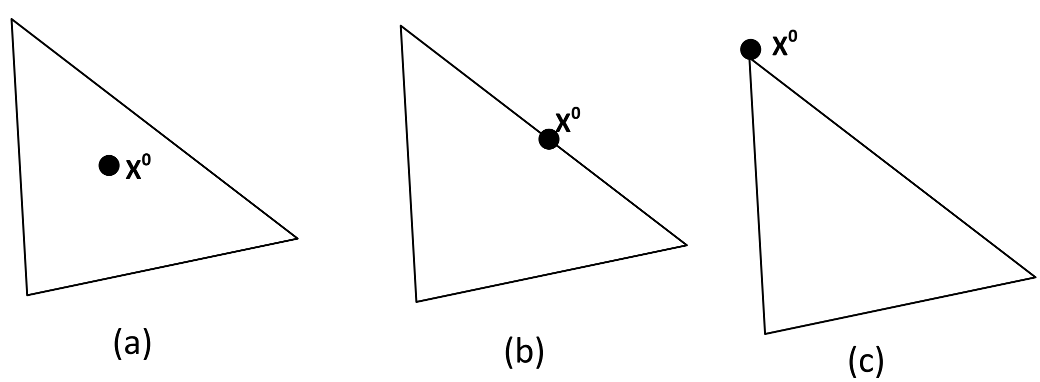
2.5. Geometrical consideration of solution for QPP

The optimal solution of a quadratic programming problem can occur at any of the following points of the feasible region

(a) Interior of the feasible region;

(b) On the binding hyper plane;

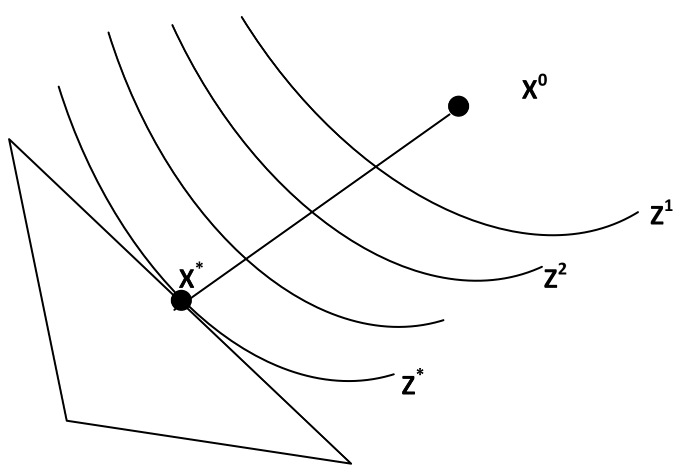
(c) At the extreme point of the feasible region.



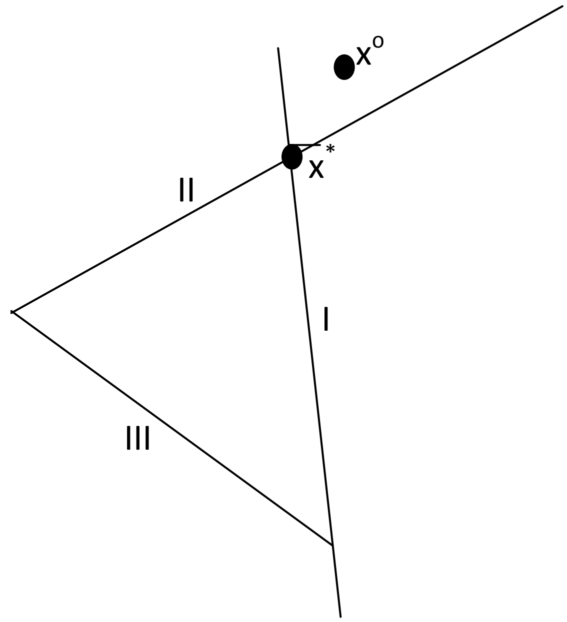
If the constraints of a QPP are ignored, then the unconstrained problem can easily be solved by the classical unconstrained optimization method. If the unconstrained optimum coincides with any of the above points, then it is optimal for the original quadratic programming problem, otherwise it is not optimal and so we have to search for the optimal solution.

The unconstrained optimum x^0 is the centre of the concentric parabolic curve described by the quadratic objective function. As we move away from this centre the objective function increases in value. Thus if the unconstrained optimum is infeasible in the original QP the optimum solution will be located at the first point the objective function curve come in contact with the feasible region. See Eiselt, et al (1987).

If the unconstrained optimum x^0 violate any of the constraints and the optimum solution to the original problem lies on the binding hyper-plane, then the orthogonal projection of x^0 on the violated hyper-plane yields the optimum solution. But, if the optimum solution to the original problem is at the corner of the feasible region, then a linear approximation of the QPP at the unconstrained optimum yields the optimum solution to QPP. See figure below.



Clearly, x^* is an orthogonal projection of x^0 on the violated hyper plane



In the above figure, x^0 violates two constraints (I and II). The orthogonal projections of x^0 on I and II, respectively, are infeasible, so the optimum solution x^* is located at the extreme point of the feasible region.

From our geometrical considerations the following observations are made.

Observation 1:

If the unconstrained minimum x^0 of QP violates the i^{th} constraint, i.e. $A_i x^0 > b_i$, $i = 1, 2, \ldots, p$, then the set S of all projections x^i of x^0 on the hyper-plane of the violated constraints such that $A_i x^i \leq b_i$ is either empty or singleton. This is true because the feasible region is a convex set.

Observation 2:

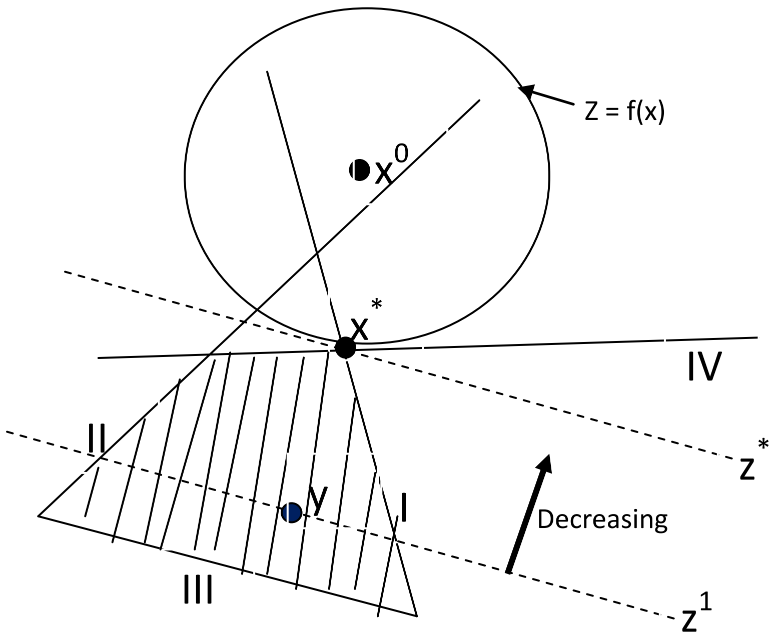
If S is empty, then the optimal solution of QP is an extreme point solution.

Observation 3:

If S is non-empty, then the optimal solution of QP is on the binding hyper-plane.

Observation 4:

Linear approximation of the quadratic objective function at any feasible point y, is parallel to the tangent to the parabolic curve described by the quadratic function at the optimal solution of the quadratic programming problem.



In the figure above, the feasible region is bounded by four hyper-planes, I, II, III and IV. The dotted line through the point y is a linear approximation of the curve with centre at x^0 and is parallel to the tangent to the curve at x^*

Observation 5:

If the optimal solution of a quadratic programming problem is an extreme point solution, then it is equal to the optimal solution of its linear programming approximation.

That is, the optimal solution of the linear programming problem (7) and the optimal solution of the quadratic programming problem (1) or (2) are the same provided it is an extreme point solution.

3. Our Proposed algorithm

Step 0. Ignoring the constraints, find the unconstrained minimum x^0 of the quadratic programming problem.

Step 1. Testing x^0 for feasibility in the original QPP.

Is $Ax^0 \leq b$?

Yes: x^0 is the optimal solution to QPP. Go to step 7

No: Go to step 2.

Step 2. Computing distances of x^0 from the violated hyper-planes.

Compute d_i for all $A_i x^0 > b_i$, i = 1, 2, . . . , p through

$$d_i = \frac{A_i x^0 - b_i}{\sqrt{A_i A_i^T}}$$

Step 3. Projecting x^0 on each of the violated hyper-plane

Project x^0 on the violated hyper-plane through

$$x^i = x^0 - d_i \hat{A}_i^T$$

Step 4. Testing the projected points for feasibility.

Form the set $S = \{x^i : Ax^i \leq b, i = 1, 2, \ldots, s\}$

Step 5. Is S empty?

Yes: Go to step 6

No: set $x^* = x^i \varepsilon S$. Go to step 7.

Step 6. Using any existing method, solve the LP problem

$\text{Min} Z = (c + y^T Q)x$

Subject to:

$$Ax \leq b$$

$$x \geq 0$$

Go to step 7.

Step 7. Stop

4. Illustrations of the algorithm

To examine the workability of our proposed algorithm, we picked four quadratic programming problems and solved. We code the algorithm in MATLAB 6.5 and all the problems solved on a Pentium IV 600 CPU with 320 MB RAM Windows XP computer.

In order to assess the quality of the optimal solution obtained by our algorithm, we solved the same problems using an OPTIMIZER, existing software for solution of both linear and quadratic programming problems.

Example 1

$$\text{Min } Z = -26x_1 - 8x_2 + 3x_1^2 - 2x_1x_2 + 2x_2^2$$

Subject to:

$$x_1 + 2x_2 \leq 6$$

$$x_1 - x_2 \geq 1$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Differentiating Z with respect to x_1 and x_2, respectively, and equating to zero, we have

$$3x_1 - x_2 = 13$$

$$-x_1 + 2x_2 = 4$$

Solving simultaneously we have $x_1 = 6$ and $x_2 = 5$. So the unconstrained minimum is $x^o = (6, 5)$.

This solution is infeasible and does not satisfy the first constraint. So we project it on the hyper plane.

$$d_1^0 = \frac{(1,2)\begin{pmatrix}6\\5\end{pmatrix} - 6}{\sqrt{1^2+2^2}} = \frac{6+10-6}{\sqrt{5}} = 2\sqrt{5}$$

$$x^1 = (6,5) - 2\sqrt{5}\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = (4,1)$$

x^1 is feasible, so $x^* = (4, 1)$ is optimal solution for the constrained problem with the objective function value of $Z^* = -70$

Example 2

$$\text{Min } Z = -16x_1 - 10x_2 + x_1^2 + x_2^2$$

Subject to :

$$-2x_1 + 4x_2 \leq 8$$

$$x_1 + x_2 \leq 6$$

$$X_2 \leq 3$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Differentiating Z with respect to x_1 and x_2, respectively, and equating to zero, we have

$$2x_1 - 16 = 0$$

$$2x_2 - 10 = 0$$

Solving simultaneously we have $x_1 = 8$ and $x_2 = 5$. So the unconstrained minimum is $x^o = (8, 5)$.

This solution is infeasible and does not satisfy the second and the third constraints. So we project it on the two hyper planes.

$$d_1^0 = \frac{(1,1)\begin{pmatrix}8\\5\end{pmatrix} - 6}{\sqrt{1^2 + 1^2}} = \frac{8+5-6}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

$$d_2^0 = \frac{(0,1)\begin{pmatrix}8\\5\end{pmatrix} - 3}{\sqrt{0^2 + 1^2}} = \frac{0+5-3}{1} = 2$$

$$x^1 = (8,5) - \frac{7}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (4.5, 1.5)$$

$$x^2 = (8,5) - 2(0,1) = (8,3)$$

x^1 is feasible and x^2 is infeasible, so $x^* = (4.5, 1.5)$ is optimal solution for the constrained problem with the objective function value of $Z^* = -64.5$

Example 3

$$\text{Min } Z = -x_1 - x_2 + 0.5x_1^2 - x_1x_2 + x_2^2$$

Subject to :

$$x_1 + x_2 \leq 3$$

$$x_1 + 3x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Differentiating Z with respect to x_1 and x_2, respectively, and equating to zero, we have

$x_1 - x_2 = 1$

$x_1 + 2x_2 = 1$

Solving simultaneously we have $x_1 = 3$ and $x_2 = 2$. So the unconstrained minimum is $x^o = (3, 2)$.

This solution is infeasible and does not satisfy the first constraint. So we project it on the hyper plane.

$$d_1^0 = \frac{(1,1)\begin{pmatrix}3\\2\end{pmatrix} - 3}{\sqrt{1^2+1^2}} = \frac{3+2-3}{\sqrt{2}} = \sqrt{2}$$

$$x^1 = (3,2) - \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (2,1)$$

x^1 is infeasible.

The set S is empty, so the optimal solution is an extreme point solution.

We now linearize the objective function at the point y = (1, 2) as

$$Z = -2x_1 + 4x_2$$

The LPP approximation of the QPP is therefore the following problem.

Min $Z = -2x_1 + 4x_2$

Subject to:

$$x_1 + x_2 \leq 3$$

$$x_1 + 3x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

The optimal solution to this problem is $x_1 = 1.5$, $x_2 = 1.5$ with Z = 1.5

Hence the optimal solution to the quadratic programming problem is $x_1 = 1.5$, $x_2 = 1.5$ with Z = -1.875

Example 4

$$\text{Max } Z = 10x_1 + 4x_2 - x_1^2 + 4x_1x_2 - 5x_2^2$$

Subject to:

$x_1 + x_2 \leq 6$

$4x_1 + x_2 \leq 18$

$x_1 \geq 0,\ x_2 \geq 0$

Changing the sense of optimization, we solve the following problem

$$\text{Min}(-Z) = -10x_1 - 4x_2 + x_1^2 - 4x_1x_2 + 5x_2^2$$

Subject to:

$x_1 + x_2 \leq 6$

$4x_1 + x_2 \leq 18$

$x_1 \geq 0,\ x_2 \geq 0$

Differentiating Z with respect to x_1 and x_2, respectively, and equating to zero, we have

$2x_1 - 4x_2 = 10$

$4x_1 - 10x_2 = -4$

Solving simultaneously we have $x_1 = 29$ and $x_2 = 12$. So the unconstrained minimum is $x^o = (29, 12)$.

This solution is infeasible and does not satisfy both constraints. So we project it on the two hyper planes.

$$d_1^0 = \frac{(1,1)\begin{pmatrix}29\\12\end{pmatrix} - 6}{\sqrt{1^2 + 1^2}} = \frac{29 + 12 - 6}{\sqrt{2}} = \frac{35}{\sqrt{2}}$$

$$d_2^0 = \frac{(4,1)\begin{pmatrix}29\\12\end{pmatrix} - 18}{\sqrt{4^2 + 1^2}} = \frac{116 + 12 - 18}{\sqrt{17}} = \frac{110}{\sqrt{17}}$$

$$x^1 = (29, 12) - \frac{35}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{23}{2}, -\frac{11}{2}\right)$$

$$x^2 = (29, 12) - \frac{110}{\sqrt{17}}\left(\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right) = \left(\frac{53}{17}, \frac{94}{17}\right)$$

x^1 and x^2 are infeasible.

The set S is empty, so the optimal solution is an extreme point solution.

We therefore linearize the objective function at the point (2, 2) as

$c_1 = 2x_1 - 4x_2 - 10 = -14$

$c_2 = -4x_1 + 10x_2 - 4 = 8$

$Z = -14x_1 + 8x_2$

The LPP approximation of the QPP is therefore the following problem.

$\text{Min}Z = -14x_1 + 8x_2$

Subject to:

$x_1 + x_2 \leq 6$

$4x_1 + x_2 \leq 18$

$x_1 \geq 0, x_2 \geq 0$

The optimal solution to this problem is $x_1 = 4.5$, $x_2 = 0$ with $Z = -63$

Hence the optimal solution to the quadratic programming problem is $x_1 = 4.5$, $x_2 = 0$ with $Z = 24.75$

9. Computational Results Using Different methods

Problem No.	Optimizer	Propose
1	(4, 1), Z = -70	(4, 1) Z = -70
2	(4.5, 1.5) Z = -64.5	(4.5, 1.5) Z = -64.5
3	(1.5, 1.5) Z = -1.875)	(1.5, 1.5) Z = -1.875)
4	(4.47, 0.12) Z = 25.04	(4.5, 0) Z = 24.75

5. Conclusions

This paper proposes a new algorithm for solving quadratic programming problems. We use classical technique of unconstrained optimization to first determine the unconstrained optimum which is then tested for feasibility in the constrained problem. When the unconstrained optimum is infeasible in QPP we use the method of orthogonal projection to project it on the hyper-plane of each of the violated constraints. Where one of the projected points is feasible yields the optimal solution to the QPP. In the case where all the projected points are infeasible we adopt the technique of linear programming approximation to obtain the optimum solution. The proposed algorithm in our test problems performs excellently well and competes favorably with the available techniques.

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