Available online at http://scik.org
J. Math. Comput. Sci. 2023, 13:10
https://doi.org/10.28919/jmcs/8043
ISSN: 1927-5307

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO NONLINEAR IMPULSIVE HYBRID DIFFERENTIAL EQUATIONS WITH LINEAR AND NONLINEAR PERTURBATIONS

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Abstract. In this paper, we establish the existence and uniqueness of solutions to impulsive nonlinear hybrid fractional differential equations, considering both linear and nonlinear perturbations. Our approach relies on the nonlinear alternative of Leray-Schauder type, in conjunction with Banach's fixed-point theorem. Additionally, we provide an illustrative example to showcase the applicability of our results.

Keywords: impulsive hybrid fractional differential equations; fixed point theorems; linear perturbation.

2020 AMS Subject Classification: 47H10.

1. Introduction

During the last three decades fractional calculus and its applications become diversified more and has materialize as a significant tool for the comprehensive applications in mathematical modeling of nonlinear systems. The nonlocal nature of fractional order operators accounts the hereditary properties involved in various systems in terms of fractional differential operator. For

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Received June 06, 2023

further reference see [16, 17, 19, 20]. and the references cited therein.

The definitions like Riemann-Liouville (1832), Grunwald-Letnikov(1867), Hadamard (1891,[21]) and Caputo(1997) are used to model problems in engineering and applied sciences and the formulations are used to model the physical systems and has given more accurate results. The study of impulsive differential equations has served as a cornerstone in the micro world of biology, leading to a reevaluation of our understanding of nature. Furthermore, the implications of these equations are of significant importance in a range of fields, including bioinformatics and biotechnology. Through the use of impulsive differential equations, we can gain a deeper insight into the behavior of biological systems and develop more effective tools for practical applications in the field of biotechnology [6, 18].

these as examples of impulsive systems. The study of impulsive differential equations has significant implications across a range of fields, including physics, engineering, economics, and biology. By incorporating the effects of sudden changes or impulses into mathematical models, we can better understand the behavior of these systems and make more accurate predictions. In particular, the study of impulsive differential equations has provided invaluable insights into the functioning of the heart and other biological systems. Therefore, impulsive differential equations are a powerful tool for understanding and modeling complex systems that experience abrupt changes we refer [4] for an introduction to the theory of impulsive differential equations. It is well known that in the evolution processes the impulsive phenomena can be found in many situations. For example, disturbances in cellular neural networks [8], operation of a damper subjected to the percussive effects [5], change of the valve shutter speed in its transition from open to closed state [7], fluctuations of pendulum systems in the case of external impulsive effects [9],percussive systems with vibrations [1], relaxational oscillations of the electromechanical systems [2], dynamic of system with automatic regulation [3], control of the satellite orbit, using the radial acceleration [3] and so on.

Hybrid differential equations is a rich field of differential equations. It is quadratic perturbations of non linear differential equations. It has lately years been an object of increasing interest because of its vast applicability in several fields. For more details about hybrid differential equations, we refer to [22], [24], [26], [27], [28].

The authors of [12], S. Melliani, A. El Allaoui, and L. S. Chadli, examined a boundary value problem involving nonlinear hybrid differential equations with both linear and nonlinear perturbations.

$$(1)\quad \begin{cases} \frac{d}{dt}\Big(\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\xi(\hat{\varkappa},\vartheta(\hat{\varkappa})), \hat{\varkappa}\in I=[0,a], a>0 \\ \vartheta(0)\eta(0,\vartheta(0))+\hat{v}\vartheta(a)\eta(a,\vartheta(a))=\vartheta(0)\chi(0,\vartheta(0))+\hat{v}\chi(a,\vartheta(a))+\beta, \end{cases}$$

where $\eta\in C(I\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $\chi,\xi\in C(I\times\mathbb{R},\mathbb{R})$ are given functions and $\hat{v}$, $\beta\in\mathbb{R}$ such that $\hat{v}\neq-1$.

Motivated by the good effect of model (1), we consider the following problem of impulsive hybrid fractional differential equation:

$$(2)\quad \begin{cases} D^{\hat{v}}\Big(\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\xi(\hat{\varkappa},\vartheta(\hat{\varkappa})), \hat{\varkappa}\in\hat{\mathfrak{J}}=[0,1], \hat{\varkappa}\neq\hat{\varkappa}_i, \\ i=1,2,\ldots,n, 0<\hat{v}<1, \\ \quad \vartheta(\hat{\varkappa}_i^+)=\vartheta(\hat{\varkappa}_i^-)+I_i(\vartheta(\hat{\varkappa}_i^-)), \hat{\varkappa}_i\in(0,1), i=1,2,\ldots,n, \\ \vartheta(0)\eta(0,\vartheta(0))-\chi(0,\vartheta(0))=\phi(\vartheta), \end{cases}$$

where $D^{\hat{v}}$, denote the Caputo fractional derivative of order $\hat{v}$, $\eta\in\mathscr{C}(\hat{\mathfrak{J}}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $\chi\in\mathscr{C}(\hat{\mathfrak{J}}\times\mathbb{R},\mathbb{R})$, and $\xi\in\mathscr{C}(\hat{\mathfrak{J}}\times\mathbb{R},\mathbb{R})$, and $\phi:\mathscr{C}(\hat{\mathfrak{J}},\mathbb{R})\longrightarrow\mathbb{R}$ are continuous functions defined by $\phi(\vartheta)=\sum_{i=1}^{n}\lambda_i\vartheta(\xi_i)$, where $\xi_i\in(0,1)$ for $i=1,2,...,n$, and $I_i:\mathbb{R}\longrightarrow\mathbb{R}$ and $\vartheta(\hat{\varkappa}_i^+)=\lim\limits_{\varepsilon\to0^+}\vartheta(\hat{\varkappa}_i+\varepsilon)$ and $\vartheta(\hat{\varkappa}_i^-)=\lim\limits_{\varepsilon\to0^-}\vartheta(\hat{\varkappa}_i+\varepsilon)$ represent the right and left limits of $\vartheta(\hat{\varkappa})$ at $\hat{\varkappa}=\hat{\varkappa}_i$.

The paper is organized as follows. Section 2 provides a concise overview of fundamental concepts, fractional calculation laws, and introduces preliminary results. In Section 3, we examine the existence and uniqueness of solutions to the initial value problem (2), employing both the Banach contraction mapping principle (BCMP) and Leray-Schauder fixed point theorem. In Section 4, we present an example that serves to illustrate the findings of our study. Lastly, Section 5 contains concluding remarks and proposes potential avenues for future research.

2. Preliminaries

In this section, we provide a concise review of the fundamental concepts and properties of fractional calculus theory. Additionally, we introduce several preparatory results that will be used in our subsequent analysis. Throughout this paper denotes $\hat{\mathfrak{J}}_0 = [0, \hat{\varkappa}_1]$, $\hat{\mathfrak{J}}_1 = (\hat{\varkappa}_1, \hat{\varkappa}_2], \ldots,$ $\hat{\mathfrak{J}}_{n-1} = (\hat{\varkappa}_{n-1}, \hat{\varkappa}_n]$, $\hat{\mathfrak{J}}_n = (\hat{\varkappa}_n, 1], n \in \mathbb{N}, n > 1$.

For $\hat{\varkappa}_i \in (0,1)$ such that $\hat{\varkappa}_1 < \hat{\varkappa}_2 < \ldots < \hat{\varkappa}_n$, we define the following spaces:

$I' = I \setminus \{\hat{\varkappa}_1, \hat{\varkappa}_2, ..., \hat{\varkappa}_n\}$,

$$X = \{\vartheta \in \mathscr{C}(\hat{\mathfrak{J}}, \mathbb{R}) : \vartheta \in C(I') \text{ and left } \vartheta(\hat{\varkappa}_i^+) \text{ and right limit } \vartheta(\hat{\varkappa}_i^-)) \text{ exist and}$$
$$\vartheta(\hat{\varkappa}_i^-) = \vartheta(t_i), 1 \le i \le n\}.$$

Then, clearly $(X, \|.\|)$ is a Banach space under the norm $\|\vartheta\| = \max_{\hat{\varkappa} \in [0,1]} |\vartheta(\hat{\varkappa})|$.

Definition 2.1. [10] *The fractional integral of the function $\xi \in L^1([a,b], \mathbb{R}^+)$ of order $\hat{\nu} \in \mathbb{R}^+$ is defined by*

$$I_a^{\hat{\nu}} \xi(\hat{\varkappa}) = \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \xi(s) ds,$$

where Γ is the gamma function.

Definition 2.2. [10] *For a function ξ defined on the interval $[a,b]$, the Riemann-Liouville fractional-order derivative of h, is defined by*

$$({}^R D_{a^+}^{\hat{\nu}} \xi)(\hat{\varkappa}) = \frac{1}{\Gamma(n - \hat{\nu})} \left(\frac{d}{dt}\right)^n \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{n-\hat{\nu}-1}}{\Gamma(\hat{\nu})} \xi(s) ds,$$

where $n = [\hat{\nu}] + 1$ and $[\hat{\nu}]$ denotes the integer part of $\hat{\nu}$.

Definition 2.3. [10] *For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of ξ, is defined by*

$$({}^c D_{a^+}^{\hat{\nu}} \xi)(\hat{\varkappa}) = \frac{1}{\Gamma(n - \hat{\nu})} \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{n-\hat{\nu}-1}}{\Gamma(\hat{\nu})} \xi^{(n)}(s) ds,$$

where $n = [\hat{\nu}] + 1$ and $[\hat{\nu}]$ denotes the integer part of $\hat{\nu}$.

In this section, we introduce the notation, definitions, and lemmas that will be utilized in our proofs later.

Lemma 2.1. [11] *Let $n \in \mathbb{N}$ and $n - 1 < \hat{\nu} < n$. If η is a continuous function, then we have*

$$I^{\hat{\nu}} \; {}^c D^{\hat{\nu}} \eta(\hat{\varkappa}) = \eta(\hat{\varkappa}) + a_0 + a_1 \hat{\varkappa} + a_2 \hat{\varkappa}^2 + ... + a_{n-1} \hat{\varkappa}^{n-1}.$$

Lemma 2.2. *(Leray-Schauder alternative see* [15]*). Let* $\hat{\mathfrak{F}} : \hat{\mathfrak{G}} \longrightarrow \hat{\mathfrak{G}}$ *be a completely continuous operator (i.e., a map that is restricted to any bounded set in* $\hat{\mathfrak{G}}$ *is compact). Let* $\hat{\mathfrak{P}}(\hat{\mathfrak{F}}) = \{\vartheta \in \hat{\mathfrak{G}} : \vartheta = \lambda \hat{\mathfrak{F}}\vartheta \quad for \quad some \quad 0 < \lambda < 1\}$*. Then either the set* $\hat{\mathfrak{P}}(\hat{\mathfrak{F}})$ *is unbounded or* $\hat{\mathfrak{F}}$ *has at least one fxed point.*

3. Main Results

In this section, we will prove the existence of a mild solution for problem (2) .
To obtain the existence of a mild solution, we will need the following assumptions:

(H_1) The function $\vartheta \longrightarrow \vartheta\eta(\hat{\varkappa}, \vartheta)$ is increasing in $\mathbb{R}$ for every $\hat{\varkappa} \in [0,1]$.

(H_2) i) The functions η and χ are continuous and bounded, that is, there exist positive numbers $\nu_\eta > 0$ and $\mu_\chi > 0$, such that
$|\eta(\hat{\varkappa}, \vartheta)| \geq \nu_\eta$ and $|\chi(\hat{\varkappa}, \vartheta)| \leq \mu_\chi$ for all $(\hat{\varkappa}, \vartheta) \in [0,1] \times \mathbb{R}$.
ii) There exist positive numbers $M_\eta > 0$ and $M_\chi > 0$, such that

$$|\eta(\hat{\varkappa}, \vartheta) - \eta(\hat{\varkappa}, \bar{\vartheta})| \leq M_\eta |\vartheta - \bar{\vartheta}|,$$

and

$$|\chi(\hat{\varkappa}, \vartheta) - \chi(\hat{\varkappa}, \bar{\vartheta})| \leq M_\chi |u - \bar{\vartheta}|.$$

for all $\vartheta, \bar{\vartheta} \in \mathbb{R}$ and $\hat{\varkappa} \in [0,1]$.

(H_3) There exist positive number $M_\xi > 0$, such that

$$|\xi(\hat{\varkappa}, \vartheta) - \xi(\hat{\varkappa}, \bar{\vartheta})| \leq M_\xi |\vartheta - \bar{\vartheta}|.$$

for all $\vartheta, \bar{\vartheta} \in \mathbb{R}$ and $\hat{\varkappa} \in [0,1]$.

(H_4) There exists constant $A > 0$, such that for all

$$|I_i(\vartheta) - I_i(\bar{\vartheta})| \leq A|\vartheta - \bar{\vartheta}|, \qquad i = 1,2,...,n, \forall \vartheta, \bar{\vartheta} \in \mathbb{R}.$$

(H_5) There exist constant K_ϕ, such that
$|\phi(\vartheta)| \leq K_\phi \|\vartheta\|, \quad$ for all $\vartheta \in C([0,1], \mathbb{R})$,

(H_6) There exist constant $M_\phi, N_\vartheta > 0$, such that
$|\phi(\vartheta)| \leq M_\phi \|\vartheta\|, \quad$ for all $\vartheta \in C([0,1], \mathbb{R})$,
$|I_i(\vartheta)| \leq N_\vartheta \|\vartheta\|, \qquad i = 1,2,...,n, \quad$ for all $\vartheta \in \mathbb{R}$,

(H_7) There exists constant $C > 0$, such that

$|I_i(\vartheta)| \leq C, \quad i = 1, 2, \ldots, n$, for all $\vartheta \in \mathbb{R}$.

(H_8) There exists constant $\rho > 0$, such that

$|\phi(\vartheta)| \leq \rho, \quad \forall \vartheta \in C([0,1], \mathbb{R})$.

(H_9) There exist constants $\rho_0, \rho_1 > 0$, such that

$|h(\hat{\varkappa}, \vartheta)| \leq \rho_0 + \rho_1 \|\vartheta\|, \quad$ for all $\vartheta \in X$ and $\hat{\varkappa} \in [0,1]$.

For brevity, let us set

$$\pi \;=\; \frac{1}{\nu_\eta}\Big(M_\chi + K_\phi + M_\eta + nA + \frac{M_\xi}{\Gamma(\alpha+1)}\Big). \tag{3}$$

Lemma 3.1. *: Let $\hat{\nu} \in (0,1)$ and $\xi : [0,a] \longrightarrow \mathbb{R}$ be continuous. A function $\vartheta \in \mathscr{C}([0,a], \mathbb{R})$ is a solution to the fractional integral equation*

$$\vartheta(\hat{\varkappa}) = \vartheta_0 - \int_0^a \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\alpha)}\xi(s)ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big),$$

if and only if ϑ is a solution to the following fractional Cauchy problem:

$$\begin{cases} D^{\hat{\nu}}\vartheta(\hat{\varkappa}) = \xi(\hat{\varkappa}), \hat{\varkappa} \in [0,a] \\ \vartheta(a) = \vartheta_0, \quad a > 0, \end{cases} \tag{4}$$

Lemma 3.2. *Let's assume that hypotheses (H_1) and (H_3) hold. Let $\hat{\nu} \in (0,1)$ and $\xi : J \longrightarrow \mathbb{R}$ be continuous. A function ϑ is a solution to the fractional integral equation*

$$\begin{aligned} \vartheta(\hat{\varkappa}) = \frac{1}{\eta(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}\Big(& \phi(\vartheta) + \chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) + \theta(\hat{\varkappa})\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) \\ & + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big), \hat{\varkappa} \in [\hat{\varkappa}_i, \hat{\varkappa}_{i+1}] \end{aligned} \tag{5}$$

where

$$\theta(\hat{\varkappa}) = \begin{cases} 0, & \hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1] \\ 1, & \hat{\varkappa} \notin [\hat{\varkappa}_0, \hat{\varkappa}_1[\end{cases}$$

if and only if ϑ is a solution of the following impulsive problem:

(6)
$$\begin{cases} D^{\hat{\nu}}\Big(\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\xi(\hat{\varkappa}), \hat{\varkappa}\in J=[0,1], \hat{\varkappa}\neq\hat{\varkappa}_i, i=1,2,\ldots,n, \quad 0<\hat{\nu}<1 \\ \vartheta(\hat{\varkappa}_i^+)=\vartheta(\hat{\varkappa}_i^-)+I_i(\vartheta(\hat{\varkappa}_i^-)), \quad \hat{\varkappa}_i\in(0,1), i=1,2,\ldots,n \\ \vartheta(0)\eta(0,\vartheta(0))-\chi(0,\vartheta(0)))=\phi(\vartheta), \end{cases}$$

Proof. Assume that u satisfies (6). If $\hat{\varkappa}\in[\hat{\varkappa}_0,\hat{\varkappa}_1[$, then

$$D^{\hat{\nu}}\Big(\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\xi(\hat{\varkappa}), \hat{\varkappa}\in[\hat{\varkappa}_0,\hat{\varkappa}_1[\tag{7}$$

$$\vartheta(0)\eta(0,\vartheta(0))-\chi(0,\vartheta(0))=\phi(\vartheta), \tag{8}$$

Applying $I^{\hat{\nu}}$ on both sides of (7), we obtain

$$\begin{aligned} \vartheta(\hat{\varkappa})f(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) &= \vartheta(0)\zeta(0,\vartheta(0))-\chi(0,\vartheta(0))+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds \\ &= \phi(\vartheta)+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds, \end{aligned}$$

Then we get

$$\vartheta(\hat{\varkappa}) = \tfrac{1}{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\int_0^{\hat{\varkappa}}\tfrac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big).$$

If $\hat{\varkappa}\in[\hat{\varkappa}_1,\hat{\varkappa}_2[$, then

$$D^{\hat{\nu}}\Big(\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\xi(\hat{\varkappa}), \quad \hat{\varkappa}\in[\hat{\varkappa}_1,\hat{\varkappa}_2[\tag{9}$$

$$\vartheta(\hat{\varkappa}_1^+)=\vartheta(\hat{\varkappa}_1^-)+I_1(\vartheta(\hat{\varkappa}_1^-)), \tag{10}$$

According to Lemma 3.1 and the continuity of $\hat{\varkappa}\longrightarrow\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))$, we have

$$\begin{aligned} \vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) &= \vartheta(\hat{\varkappa}_1^+)\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))-\chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))-\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds \\ &\quad+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds \\ &= (\vartheta(\hat{\varkappa}_1^-)+I_1(\vartheta(\hat{\varkappa}_1^-)))\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))-\int_0^{t_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\nu}-1}}{\Gamma(\alpha)}\xi(s)ds \\ &\quad+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds. \end{aligned}$$

Since

$$\vartheta(\hat{\varkappa}_1^-) = \frac{1}{\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))}\Big(\chi(\hat{\varkappa}_1,\vartheta(t_1))+\phi(\vartheta)+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big),$$

then we get

$$\begin{aligned}
&\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\\
&=\Big[\frac{1}{\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))}\Big(\chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1)))+\phi(\vartheta)+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big)\\
&+I_1(\vartheta(\hat{\varkappa}_1^-))\Big]\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))-\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\\
&=\phi(\vartheta)+I_1(u(\hat{\varkappa}_1^-))\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds,
\end{aligned}$$

So, one has

$$\vartheta(\hat{\varkappa}) = \frac{1}{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\Big(\phi(\vartheta)+\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+I_1(\vartheta(\hat{\varkappa}_1^-))\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big).$$

If $\hat{\varkappa}\in[\hat{\varkappa}_2,\hat{\varkappa}_3[$, we have

$$\begin{aligned}
&\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\\
&=\vartheta(\hat{\varkappa}_2^+)\eta(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))-\chi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))-\int_0^{\hat{\varkappa}_2}\frac{(t_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\\
&+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\\
&=(\vartheta(\hat{\varkappa}_2^-)+I_2(\vartheta(\hat{\varkappa}_2^-)))\eta_1(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))-\int_0^{\hat{\varkappa}_2}\frac{(t_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\\
&+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds,
\end{aligned}$$

and

$$\vartheta(\hat{\varkappa}_2^-) = \frac{1}{\eta(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))}\Big(\phi(\vartheta)+\chi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))+I_1(\vartheta(\hat{\varkappa}_1^-))\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big).$$

Therefore, we obtain

$$\begin{aligned}
\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))=&\Big[\frac{1}{\eta(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))}\Big(\phi(\vartheta)+\chi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))\\
&+I_1(\vartheta(\hat{\varkappa}_1^-))\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}-s)^{\alpha-1}}{\Gamma(\alpha)}\xi(s)ds\Big)\Big]\eta(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))\\
&+I_2(\vartheta(\hat{\varkappa}_2^-)))\eta(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))
\end{aligned}$$

$$-\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds,$$

$$=\phi(\vartheta)+I_1(\vartheta(\hat{\varkappa}_1^-))\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+I_2(\theta(\hat{\varkappa}_2^-))\eta(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))$$

$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds,$$

Consequently, we get

$$\vartheta(\hat{\varkappa}) = \frac{1}{\eta(\hat{\varkappa},u(\hat{\varkappa}))}\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\textstyle\sum_{i=1}^{2}I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i))+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big).$$

If $\hat{\varkappa}\in[\hat{\varkappa}_i,\hat{\varkappa}_{i+1}[(i=3,4,...,n)$, using the same method, one has

$$\vartheta(\hat{\varkappa}) = \frac{1}{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\textstyle\sum_{i=1}^{n}I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i))+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big).$$

Conversely, assume that u satisfies (5). If $\hat{\varkappa}\in[\hat{\varkappa}_0,\hat{\varkappa}_1[$, we have

$$\vartheta(\hat{\varkappa}) = \frac{1}{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\textstyle\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big). \tag{11}$$

Then, we multiplied by $\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))$ and applying $D^{\hat{\nu}}$ on both sides of (11), we get equation (7). Again, substituting $\hat{\varkappa}=0$ in (11), we obtain $\vartheta(0)\zeta(0,\vartheta(0))-\chi(0,\vartheta(0))=\phi(\vartheta)$. Since $\vartheta\longrightarrow\vartheta\eta(\hat{\varkappa},\vartheta)$ is increasing in $\mathbb{R}$ for $\hat{\varkappa}\in[\hat{\varkappa}_0,\hat{\varkappa}_1[$, the map $\vartheta\longrightarrow\vartheta\eta(\hat{\varkappa},\vartheta)$ is injective in $\mathbb{R}$. Then we get (8).

Similarly, for $\hat{\varkappa}\in[\hat{\varkappa}_1,\hat{\varkappa}_2[$, we get

$$\vartheta(\hat{\varkappa}) = \frac{1}{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\Big(\phi(\vartheta)+\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+I_1(\vartheta(\hat{\varkappa}_1^-))\eta(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+\textstyle\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big). \tag{12}$$

Then, we multiplied by $\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))$ and applying $D^{\hat{\nu}}$ on both sides of (12), we get equation (13) . Again by (H_3), substituting $\hat{\varkappa}=\hat{\varkappa}_1$ in (11) and taking the limit of (12), then (12) minus (11) gives (14).

If $\hat{\varkappa}\in[\hat{\varkappa}_i,\hat{\varkappa}_{i+1}[(i=2,3,...,n)$, similarly we get

$$D^{\hat{\nu}}\Big(\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\xi(\hat{\varkappa}),\hat{\varkappa}\in[\hat{\varkappa}_k,\hat{\varkappa}_{k+1}[\tag{13}$$

$$\vartheta(\hat{\varkappa}_1^+)=\vartheta(\hat{\varkappa}_1^-)+I_1(\vartheta(\hat{\varkappa}_1^-)), \tag{14}$$

□

This completes the proof.

Lemma 3.3. *Let ξ be continuous, then $\vartheta \in X$ is a solution of (2) if and only if ϑ is the solution of the integral equations*

$$\vartheta(\hat{\varkappa}) = \frac{1}{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) + \phi(\vartheta) + \theta(\hat{\varkappa})\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i)) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big), \hat{\varkappa} \in [\hat{\varkappa}_i, \hat{\varkappa}_{i+1}]$$

where

$$\theta(\hat{\varkappa}) = \begin{cases} 0, & \hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1] \\ 1, & \hat{\varkappa} \notin [\hat{\varkappa}_0, \hat{\varkappa}_1[\end{cases}$$

We define an operator $\Theta : X \longrightarrow X$ by

$$\Theta(\vartheta)(\hat{\varkappa}) = \frac{1}{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\Big(\eta(\hat{\varkappa},\vartheta(\hat{\varkappa})) + \phi(\vartheta) + \theta(\hat{\varkappa})\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i)) \tag{15}$$

$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big). \tag{16}$$

FIRST RESULT

Now, we are ready to present our first result, which addresses the existence and uniqueness of solutions for problem (2). This result is established using Banach's contraction mapping principle.

Theorem 3.1. *Assume that conditions $(H_1)-(H_7)$ holds and the function $h : [0,1] \times \mathbb{R}^{\not\leq} \longrightarrow \mathbb{R}$ is continuous functions. Then problem (2) has an unique solution provided that $\pi < 1$, π is the constant given in equation* (3).

Proof. Let us set $\sup_{\hat{\varkappa} \in J} \xi(\hat{\varkappa}, 0) = \kappa < \infty$, and define a closed ball $\bar{B}$ as follows

$$\bar{B} = \{\vartheta \in X : \|\vartheta\| \leq r\},$$

where

$$r \geq \frac{\mu_\chi + \frac{\kappa}{\Gamma(\hat{\nu}+1)}}{\nu_\eta - (M_\phi + nN_u + \frac{1}{\Gamma(\hat{\nu}+1)}M_\xi)}. \tag{17}$$

We show that $\Theta\bar{B} \subset \bar{B}$. For $u \in \bar{B}$, we obtain

$$
\begin{aligned}
|\Theta(\vartheta)(\hat{\varkappa})| &\leq \frac{1}{|\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\Big|\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\theta(\hat{\varkappa})\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i)) \\
&\quad + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big)\Big| \\
&\leq \frac{1}{\nu_\eta}\Big(\mu_\chi + M_\phi\|\vartheta\| + nN_\vartheta\|\vartheta\| + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}(|\xi(s,\vartheta(s)))-\xi(s,0)|+|\xi(s,0)|)ds \\
&\leq \frac{1}{\nu_\eta}\Big(\mu_\chi + M_\phi\|\vartheta\| + nN_\vartheta\|\vartheta\| + \frac{1}{\Gamma(\alpha+1)}(\lambda\|\vartheta\|+\kappa)\Big) \\
&\leq \frac{1}{\nu_\eta}\Big(\mu_\chi + (M_\phi + nN_\vartheta)r + \frac{1}{\Gamma(\hat{\nu}+1)}(\lambda r+\kappa)\Big),
\end{aligned}
$$

Hence, we get

$$
\|\Theta(\vartheta)\| \leq \frac{1}{\nu_\eta}\Big(\mu_\chi + (M_\phi + nN_\vartheta)r + \frac{1}{\Gamma(\hat{\nu}+1)}\big(\lambda r+\kappa\big)\Big). \tag{18}
$$

From (18), it follows that $\|\Theta(\vartheta)\| \leq r$.

Next, for $(\vartheta,\bar{\vartheta}) \in X^2$and for any $\hat{\varkappa} \in [0,1]$, we have

$$
\begin{aligned}
|\Theta(\vartheta)(\hat{\varkappa}) - \Theta(\bar{\vartheta})(\hat{\varkappa})| &= \Big|\frac{1}{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\theta(\hat{\varkappa})\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i),)) \\
&\quad + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big) \\
&\quad - \frac{1}{\eta(\hat{\varkappa},\bar{\vartheta}(\hat{\varkappa}))}\Big(\chi(\hat{\varkappa},\bar{\vartheta}(\hat{\varkappa}))+\phi(\vartheta)+\theta(\hat{\varkappa})\sum_{i=1}^{n} I_i(\bar{\vartheta}(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i,\bar{\vartheta}(\hat{\varkappa}_i)) \\
&\quad + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s)ds\Big)\Big| \\
&\leq \frac{1}{\nu_\eta}\Big(M_\chi|\vartheta-\bar{\vartheta}| + K_\phi|\vartheta-\bar{\vartheta}| + M_\eta|\vartheta-\bar{\vartheta}| + nA|\vartheta-\bar{\vartheta}| \\
&\quad + \frac{M_h}{\Gamma(\hat{\nu}+1)}|\vartheta-\bar{\vartheta}|\Big),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\Theta(\vartheta) - \Theta(\bar{\vartheta})\| &\leq \frac{1}{\nu_\eta}\Big(M_\chi + K_\phi + M_\eta + nA + \frac{M_\xi}{\Gamma(\alpha+1)}\Big)(\|\vartheta-\bar{\vartheta}\|) \\
&= \pi\|\vartheta-\bar{\vartheta}\|. \qquad (19)
\end{aligned}
$$

From (19), we deduce that

$$
\|\Theta(\vartheta) - \Theta(\bar{\vartheta})\| \leq \pi\|\vartheta-\bar{\vartheta}\|
$$

□

Due to the condition $\pi < 1$, we can conclude that Θ is a contraction operator. As a result, Banach's fixed point theorem is applicable, ensuring that the operator Θ possesses a unique fixed point. This unique fixed point serves as the unique solution to the Cauchy problem (2). This completes the proof.

SECOND RESULT

Our second result focuses on establishing the existence of solutions for the problem (2) using the Leray-Schauder alternative. For brevity, let us set

$$\Lambda_1 = \frac{1}{\nu_\eta \Gamma(\hat{\nu}+1)}, \tag{20}$$

$$\Lambda_0 = 1 - \Lambda_1 \rho_1. \tag{21}$$

Theorem 3.2. *Assume that conditions $(H_1)-(H_3)$ and $(H_8)-(H_9)$ hold. Furthermore, it is assumed that $\Lambda_1\rho_1 < 1$, where Λ_1 isgiven by (20). Then the boundary value problem (2) has at least one solution.*

Proof. We will show that the operator $\Pi : X \longrightarrow X$ satisfes all the assumptions of Lemma 2.2.
Step 1: We will prove that the operator Π is completely continuous.
Clearly, it follows by the continuity of functions η, χ, ξ that the operator Π is continuous.
Let $S \subset X$ be bounded. Then we can find positive constant Ω such that:
$|\xi(\hat{\varkappa}, \vartheta)| \leq \Omega, \quad \forall \vartheta \in S.$
Thus, for any $\vartheta \in S$, we can get

$$\begin{aligned}|\Pi(\vartheta)(\hat{\varkappa})| &\leq \frac{1}{\nu_\eta}\Big(\mu_\chi + \rho + \sum_{i=1}^{n} C + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\Omega ds\Big) \\ &\leq \frac{1}{\nu_\eta}\big(\mu_\chi + \rho + nC + \frac{\Omega}{\Gamma(\alpha+1)}\big).\end{aligned}$$

which yields

$$\|\Pi(\vartheta)\| \;\leq\; \frac{1}{\nu_\eta}\big(\mu_\chi + \rho + nC + \frac{\Omega}{\Gamma(\hat{\nu}+1)}\big). \tag{22}$$

From the inequalities (22), we deduce that the operator Π is uniformly bounded.

Setep 2: Now we show that the operator Π is equicontinuous.

We take $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ we obtain

$$
\begin{aligned}
&|\Pi(\vartheta(\tau_2)) - \Pi(\vartheta(\tau_1))| \\
&\quad \leq \Big|\frac{1}{\eta(\tau_2, \vartheta(\tau_2))}\Big(\phi(\vartheta) + \chi(\tau_2, \vartheta(\tau_2)) + \theta(\tau_2)\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) \\
&\quad + \Omega\int_0^{\tau_2} \frac{(\tau_2 - s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} ds\Big) \\
&\quad - \frac{1}{\eta(\tau_1, \vartheta(\tau_1))}\Big(\phi(\vartheta) + \chi(\tau_1, \vartheta(\tau_1)) + \theta(\tau_1)\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) \\
&\quad + \Omega\int_0^{\tau_1} \frac{(\tau_1 - s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} ds\Big)\Big| \\
&\quad \leq \frac{1}{\nu_\eta}\Big(\Big|(\chi(\tau_2, \vartheta(\tau_2)) - \chi(\tau_1, \vartheta(\tau_1)) + (\theta(\tau_2) - \theta(\tau_1))\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))\Big| \\
&\quad + \Omega\Big|\int_0^{\tau_2} \frac{(\tau_2 - s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} ds - \int_0^{\tau_1} \frac{(\tau_1 - s)^{\hat{\nu}-1}}{\Gamma(\hat{\mu})} ds\Big|\Big) \\
&\quad \leq \frac{1}{\nu_\eta}\Big(\Big|(\chi(\tau_2, \vartheta(\tau_2)) - \chi(\tau_1, \vartheta(\tau_1)) + (\theta(\tau_2) - \theta(\tau_1))\sum_{i=1}^{n} I_i(\vartheta(\hat{\varkappa}_i^-))\eta(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))\Big| \\
&\quad + \Omega\Big|\int_0^{\tau_1} \frac{(\tau_1 - s)^{\hat{\nu}-1} - (\tau_2 - s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} ds - \int_{\tau_2}^{\tau_2} \frac{(\tau_2 - s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} ds\Big|\Big).
\end{aligned}
$$

Which tend to 0 independently of ϑ. This implies that the operator $\Pi(\vartheta)$ is equicontinuous. Thus, by the above fndings, the operator $\Pi(\vartheta)$ is completely continuous.

In the next step, it will be established that the set $P = \{\vartheta \in X/\vartheta = \lambda\Pi(\vartheta), 0 < \lambda < 1\}$ is bounded.

Let $\vartheta \in \mathscr{P}$. We have $\vartheta = \lambda\Pi(\vartheta)$. Thus, for any $\hat{\varkappa} \in [0,1]$, we can write

$$
\vartheta(\hat{\varkappa}) = \lambda\Pi(\hat{\varkappa})(\hat{\varkappa}),
$$

Hence, we get

$$
\begin{aligned}
\|\vartheta\| &\leq \frac{1}{\nu_\eta}\Big(\mu_\chi + \rho + nC + \frac{1}{\Gamma(\hat{\nu}+1)}(\rho_0 + \rho_1\|\vartheta\|)\Big) \\
&\leq \frac{1}{\nu_\eta}(\mu_g + \rho + nC) + \Lambda_1(\rho_0 + \rho_1\|\vartheta\|),
\end{aligned}
$$

which, in view of (21), can be expressed as

$$\|\vartheta\| \leq \frac{\frac{1}{v_\eta}\left(\mu_\chi+\rho+nC\right)+\Lambda_1\rho_0}{\Lambda_0}.$$

This demonstrates that the set $\mathscr{P}$ is bounded. As a result, all the conditions of Lemma 2.2 are satisfied. Therefore, the operator Π has at least one fixed point, which corresponds to a solution of problem (2). This completes the proof. □

4. Example

Consider the following impulsive hybrid fractional differential equation::

$$\begin{cases} D^{\frac{1}{2}}\Big(\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\xi(\hat{\varkappa},\vartheta(\hat{\varkappa})), \hat{\varkappa}\in[0,1]\setminus\{\hat{\varkappa}_1\}, \\ \vartheta(\hat{\varkappa}_1^+)=\vartheta(\hat{\varkappa}_1^-)+(-2u(\hat{\varkappa}_1^-)), \ \ \hat{\varkappa}_1\neq 0,1 \\ \vartheta(0)\eta(0,\vartheta(0)))-\chi(0,\vartheta(0))=\sum_{i=1}^n c_i\vartheta(\hat{\varkappa}_i), \end{cases} \tag{23}$$

Here, we have

$$\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))=\frac{\arctan\hat{\varkappa}}{3}|\vartheta(\hat{\varkappa})|,$$

$$\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))=\frac{1}{7}+\frac{1}{9}\vartheta(\hat{\varkappa}),$$

$$\xi(\hat{\varkappa},\vartheta(\hat{\varkappa}))=\frac{1}{4\hat{\varkappa}^2}(\vartheta(\hat{\varkappa})+\sqrt{2}),$$

Note that

$$|\chi(\hat{\varkappa},\vartheta_1)-\chi(\hat{\varkappa},\vartheta_2)|\leq\frac{1}{9}\mid\vartheta_2-\vartheta_1\mid,$$
$$\hat{\varkappa}\in[0.1],\vartheta_1,\vartheta_2\in\mathbb{R}.$$

and

$$|\xi(\hat{\varkappa},\vartheta_1)-\xi(\hat{\varkappa},\vartheta_2)|\leq\frac{1}{4}\mid\vartheta_2-\vartheta_1\mid,\hat{\varkappa}\in[0.1],\vartheta_1,\vartheta_2\in\mathbb{R}.$$

$$\pi = \frac{1}{v_\eta}\Big(M_\chi+K_\phi+M_\eta+nA+\frac{M_\xi}{\Gamma(\hat{v}+1)}\Big)=0.12345678<1,$$

As all of the assumptions in Theorem 3.2 are satisfied, our results can be directly applied to the problem (23).

5. Conclusion

The main focus of this paper is to explore the existence of solutions for impulsive nonlinear hybrid fractional differential equations involving both linear and nonlinear perturbations. Our results not only improve upon existing findings in this research area but also provide a more generalized perspective. Furthermore, we anticipate that the theory we have developed can be extended to address broader problems related to impulsive fractional differential equations featuring both linear and nonlinear perturbations. The fixed-point theorems employed in our analysis can also be applied to investigate the existence of solutions for other types of impulsive fractional differential equations, including those involving alternative forms of fractional derivatives such as Hilfer's and Hadamard's derivatives. By contributing to the advancement of more comprehensive and efficient tools for studying these problems, we aim to enhance our understanding of the dynamics of complex systems and their behavior in impulsive conditions.

Conflict of Interests

The authors declare that there is no conflict of interests.

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