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RICCI SOLITONS ON ε-SASAKIAN MANIFOLD UNDER D_a-HOMOTHETIC DEFORMATION

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Abstract. The objective of the present paper is to study D_a-homothetic deformation of ε-sasakian manifold. We have analyzed ε-sasakian metric as Ricci soliton based on the associated vector field, by taking the vector field as orthogonal to Reeb vector field which results in constant scalar curvature. We have also shown that the vector field V transforms (1,1) tensor field ϕ, 1- form η and Reeb vector field ξ. Further we have shown that η- Einstein Ricci soliton and η-Einstein ρ Ricci soliton remains η-Einstein Ricci soliton with constant scalar curvature.

Keywords: D_a-homothetic; η-Einstein manifold; concurrent vector field.

2020 AMS Subject Classification: 53C15, 53C25.

1. INTRODUCTION

In modern mathematics of contact manifolds, geometric structures in Riemannian geometry are analysed with the significant role in geometric flows. One such flow is Ricci flow, which was introduced by Hamilton[5] in the year 1982. The concept of Hamilton's Ricci flow describes certain partial differential equation for a Riemannian metric which is as follows,

$$\frac{\partial}{\partial t} g(t) = -2S(t), t \geq 0, g(0) = g, \tag{1}$$

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where g is the Riemannian metric and S denotes (0,2)-symmetric Ricci tensor. This flow symbolises dynamical system on the space of Riemannian metrics modulo diffeomorphisms, scalings and also appear as self similar solutions.

Diligently, Ricci Soliton on pseudo- Riemannian manifold (M,g) is a triple (g,V,λ) which is defined as [7],

$$L_V g + 2S + 2\lambda g = 0, \tag{2}$$

where L_V denotes the Lie derivative along a vector field V and λ is a dilation constant. The equation of Ricci soliton represents a special case in the extension of Einstein field equation. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively. Therefore, up to diffeomorphism and depending on the sign of λ , a Ricci soliton homothetically shrinks, remains steady or expands under Ricci flow. The concept of Ricci solitons were initially studied by Sharma[12]. Further D- homothetic deformation of almost contact structures admitting Ricci solitons were extensively studied by several authors such as Nagaraja and Premalatha [10][11][13], De and Ghosh [3][4] [15].

In 1979, Bourguignon introduced the notion of Ricci-Bourguignon flow as

$$\frac{dg}{dt} = -2(S - \rho g), \tag{3}$$

where ρ is a non-zero constant. Later Catino and Mazzieri [4] gave the defnition of gradient ρ-Einstein soliton as

$$\nabla^2 f + S = \lambda g + \rho r g \tag{4}$$

The function f is called Einstein potential. The gradient ρ-Einstein soliton is called expanding, steady or shrinking according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

In motivation from the above study, the outline of the paper is structured as follows: after the introduction, we give brief review of ε-Sasakian manifold. We consider D_a-homothetically deformed ε-Sasakian metrics as Ricci solitons.

2. PRELIMINARIES

We give some definitions and basic formulas which are necessary for proving the results. An almost contact smooth manifold (M,g) of dimension $(2n+1)$ is said to be almost contact metric manifold if it is a triple (ϕ,ξ,η),where ϕ is a $(1,1)$ tensor field, ξ is a characteristic vector field(Reeb vector field), η is a global 1-form, g is an associated metric satisfying the following,

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \tag{5}$$

where I denotes the identity endomorphism.

We obtain a Riemannian metric g and a (1,1)-tensor field ϕ of contact metric on doing polarization of $d\eta$ on the contact subbundle $\eta = 0$ such that

$$d\eta(X,Y) = g(X,\phi Y), \tag{6}$$

where X,Y denote arbitrary vector fields on M. A semi-Riemannian metric g on M is said to be compatible with almost contact structure if it satisfies,

$$g(\phi X,\phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \quad g(X,\xi) = \varepsilon\eta(X), \tag{7}$$

$\forall$ X,Y$\in$ $\chi(M)$, where $\chi(M)$ is a lie algebra of smooth vector fields and $\varepsilon = \pm 1$. An ε-almost contact manifold is said to be an ε- sasakian manifold if the following relations hold [14],

$$\nabla_X \xi = -\varepsilon\phi X \tag{8}$$

$$(\nabla_X \eta)Y = -g(Y,\phi X), \tag{9}$$

where ∇ is the Levi-Civita connection with respect to g.

In an ε-Sasakian manifold the following relations hold [14]

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{10}$$

$$R(\xi,X)Y = \varepsilon g(X,Y)\xi - \eta(Y)X \tag{11}$$

$$R(X,\xi)Y = -\varepsilon g(X,Y)\xi + \eta(Y)X \tag{12}$$

$$R(\xi,X)\xi=\varepsilon(\eta(X)\xi-X) \tag{13}$$

$$S(X,\xi)=\varepsilon(n-1)\eta(X), \tag{14}$$

where X,Y are vector fields on M, R and S denote curvature tensor and Ricci tensor of g respectively.

3. D_a-Homothetic Deformation of ε-Sasakian Manifold

Let (M,g,ϕ,ξ,η) be an ε-Sasakian manifold, where g is a Ricci soliton. The D_a-homothetic deformation on M is given by,

$$\phi^*=\phi, \quad \xi^*=\frac{1}{a}\xi, \quad \eta^*=a\eta, \quad g^*=ag+a(a-1)\eta\otimes\eta, \tag{15}$$

where 'a' is a positive constant. If (M,ϕ,ξ,η,g) is an almost contact metric structure with contact form η, then $(M,\phi^*,\xi^*,\eta^*,g^*)$ is also an almost contact metric structure.
The relation between the connections ∇ and ∇^*[1] is given by,

$$\nabla^*_X Y=\nabla_X Y+(1-a)(\eta(Y)\phi X+\eta(X)\phi Y), \tag{16}$$

for any vector fields X, Y on M.
We now calculate the Riemann curvature tensor R^* from (16) of $(M,\phi^*,\xi^*,\eta^*,g^*)$ which is as follows:

$$\begin{aligned} R^*(X,Y)Z &= R(X,Y)Z+[\varepsilon-(1-a)][(1-a)g(Z,\phi Y)\phi X-(1-a)g(Z,\phi X) \\ &\phi Y]+2(1-a)(\varepsilon+a-1)g(X,\phi Y)\phi Z+(1-a)^2\eta(Z)[\eta(Y)X-\eta(X)Y], \end{aligned} \tag{17}$$

where X, Y and Z denote the vector fields.
Now, we obtain the Ricci tensor S^* of D_a-homothetically deformed ε-Sasakian manifold by contraction of (17)

$$\begin{aligned} S^*(Y,Z) &= S(Y,Z)+3(1-a)(\varepsilon+a-1)g(Z,Y) \\ &+[2n(1-a^2)-3\varepsilon(1-a)[\varepsilon+a-1]]\eta(Y)\eta(Z). \end{aligned} \tag{18}$$

On contracting (18) with respect to Y and Z, we get

$$r^*=r+3(1-a)(2n+1+\varepsilon)(\varepsilon+a-1)+(2n(1-a)^2), \tag{19}$$

for scalar curvatures r and r^* of (M,g) and (M, g^*) respectively. By using equations (15) and (16) we can calculate,

$$
\begin{aligned}
(L_v^* g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) &= g^*(\nabla_X^* V, Y) + g^*(X, \nabla_Y^* V) \\
&\quad + 2S(X,Y) + [6(1-a)(\varepsilon + a - 1) + 2\lambda a] g(X,Y) \\
&\quad + 2(2n(1-a)^2 - 3\varepsilon(1-a)(\varepsilon + a - 1) + \lambda a(a-1))\eta(X)\eta(Y).
\end{aligned} \tag{20}
$$

Using (15) in the preceeding equation we obtain,

$$
\begin{aligned}
&(L_V^* g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) = \\
&a\{g(\nabla_X V, Y) + g(X, \nabla_Y V)\} + a(a-1)\{\eta(Y)\eta(\nabla_X V) + \eta(X)\eta(\nabla_Y V)\} \\
&+ a(a-1)\{\eta(V) g(\phi X, Y) + \eta(X) g(\phi V, Y) + \eta(Y) g(X, \phi V)\} \\
&+ 2S(X,Y) + [6(1-a)(\varepsilon + a - 1) + 2\lambda a] g(X,Y) \\
&+ 2(2n(1-a)^2 - 3\varepsilon(1-a)(\varepsilon + a - 1) + \lambda a(a-1))\eta(X)\eta(Y).
\end{aligned} \tag{21}
$$

If V $\perp$ ξ, it provides $\eta(\nabla_X V) = -g(\phi V, X)$, then the above equation reduces to

$$
\begin{aligned}
&(L_V^* g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) = \\
&a\{\{g(\nabla_X V, Y) + g(X, \nabla_Y V)\} + 2S(X,Y) + 2\lambda g(X,Y)\} \\
&+ 2(1-a) S(X,Y) + 6(1-a)(\varepsilon + a - 1) g(X,Y) \\
&+ 2(2n(1-a)^2 - 3\varepsilon(1-a)(\varepsilon + a - 1) + \lambda a(a-1))\eta(X)\eta(Y).
\end{aligned} \tag{22}
$$

Since g is Ricci soliton, (22) reduces to,

$$
\begin{aligned}
&(L_v^* g^*)(X,Y) + 2S^*(X,Y) + 2\lambda g^*(X,Y) = 2(1-a) S(X,Y) \\
&+ 6(1-a)(\varepsilon + a - 1) g(X,Y) + 2(2n(1-a)^2 \\
&- 3\varepsilon(1-a)(\varepsilon + a - 1) + \lambda a(a-1))\eta(X)\eta(Y).
\end{aligned} \tag{23}
$$

If g^* is Ricci Soliton then,

$$
S(X,Y) = -3(\varepsilon + a - 1) g(X,Y) + [2n(1-a) - 3\varepsilon(\varepsilon + a - 1) + \lambda a]\eta(X)\eta(Y). \tag{24}
$$

Replacing X and Y each by ξ we get,

$$
\lambda = \frac{3\varepsilon(\varepsilon + a - 1)[\varepsilon(n-1) + (1-a)] - 2n(1-a)}{a}. \tag{25}
$$

i.e., soliton is shrinking for a>3.

Theorem 3.1. *Let $(M,\phi^*,\xi^*,\eta^*,g^*)$ be obtained by D_a-homothetic deformation of ε-Sasakian manifold (M,ϕ,ξ,η,g) with the potential vector field V orthogonal to Reeb vector field. Then M is an η-Einstein manifold with the scalar curvature $r=-3(\varepsilon+a-1)(2n+1+\varepsilon(1-a))+2n(1-a)+\lambda a$.*

Now let us consider V as a pointwise collinear with the Reeb vector field ξ^*, i.e., $V=\beta\xi^*$, where β is a function on M.

$$g^*(\nabla^*_X\beta\xi^*,Y)+g^*(X,\nabla^*_Y\beta\xi^*)+2S^*(X,Y)+2\lambda g^*(X,Y)=0. \tag{26}$$

By substituting (15),(16) and (18) in (26), we get

$$\begin{aligned}&(X\beta)\eta(Y)+(a-1)X\beta\eta(Y)+(1-a)\beta g(\phi X,Y)+(Y\beta)\eta(X)\\&+(a-1)\eta(X)Y\beta+(1-a)\beta g(X,\phi Y)+2S(X,Y)\\&+2(3(1-a)(\varepsilon+a-1))g(X,Y)+2(2n(1-a)^2-\\&3\varepsilon(1-a)(\varepsilon+a-1))\eta(X)\eta(Y)+2\lambda ag(X,Y)+2\lambda a(a-1)\eta(X)\eta(Y)=0.\end{aligned} \tag{27}$$

Replacing $Y=\xi$ in (27) and by making use of (14), we obtain

$$(X\beta)a=-[a\xi\beta+2\varepsilon(n-1)+4n(1-a)^2+2\lambda a\varepsilon+2\lambda a(a-1)]\eta(X). \tag{28}$$

Setting $X=\xi$ in the above equation, we get

$$\xi\beta a=-[\varepsilon(n-1)+2n(1-a)^2+\lambda a\varepsilon+\lambda a(a-1)]. \tag{29}$$

Substituting the above equation in (28), we have

$$(X\beta)a=-[\varepsilon(n-1)+2n(1-a)^2+\lambda a(\varepsilon+a-1)]\eta(X). \tag{30}$$

Now, applying 'd' in the above equation results in

$$-[\varepsilon(n-1)+2n(1-a)^2+\lambda a(\varepsilon+a-1)]d\eta=0. \tag{31}$$

Since $d\eta\neq 0$, the above equation results in

$$\varepsilon(n-1)+2n(1-a)^2+\lambda a(\varepsilon+a-1)=0. \tag{32}$$

Substituting the above equation in (30), we conclude that β is constant. This leads to the following:

Theorem 3.2. *Let $(M,\phi^*,\xi^*,\eta^*,g^*)$ be obtained by D_a- homothetic deformation of ε-Sasakian manifold (M,ϕ,ξ,η,g) with the potential vector field $V=\beta\xi^*$, then the scalar function β becomes constant.*

Now taking the Lie derivative of g^* along V and by making use of (2) we obtain,

$$\begin{aligned}(33)\qquad &(L_V g^*)(X,Y)+2S^*(X,Y)+2\lambda g^*(X,Y)=\\ &a(L_V g)(X,Y)+a(a-1)\{(L_V\eta)(X)\eta(Y)+\eta(X)(L_V\eta)(Y)\}+\\ &2S(X,Y)+2[3(1-a)(\varepsilon+a-1)+\lambda a]g(X,Y)\\ &+2(2n(1-a)^2-3\varepsilon(1-a)(\varepsilon+a-1)+\lambda a(a-1))\eta(X)\eta(Y).\end{aligned}$$

Since g^* is Ricci soliton, we have,

$$\begin{aligned}(34)\qquad &a(L_V g)(X,Y)+a(a-1)\{(L_V\eta)(X)(Y)+\eta(X)(L_V\eta)(Y)\}+2S(X,Y)\\ &+2[3(1-a)(\varepsilon+a-1)+\lambda a]g(X,Y)\\ &+2(2n(1-a)^2-3\varepsilon(1-a)(\varepsilon+a-1)+\lambda a(a-1))\eta(X)\eta(Y)=0.\end{aligned}$$

On substitution of (2) in (34) we get,

$$\begin{aligned}(35)\qquad &2(1-a)S(X,Y)+a(a-1)\{(L_V\eta)(X)\eta(Y)+\eta(X)(L_V\eta)(Y)\}\\ &+6(1-a)(\varepsilon+a-1)g(X,Y)+2(2n(1-a)^2-3\varepsilon(1-a)(\varepsilon+a-1)+\\ &\lambda a(a-1))\eta(X)\eta(Y)=0.\end{aligned}$$

By setting Y=ξ and making use of (10), (5) in the above equation we obtain,

$$\begin{aligned}(36)\qquad &a(a-1)(L_V\eta)(X)+a(a-1)(L_V\eta)(\xi)\eta(X)\\ &+2((1-a)\varepsilon(n-1)+2n(1-a)^2+\lambda a(a-1))\eta(X)=0.\end{aligned}$$

Now we calculate $(L_V\eta)(\xi)$, by replacing X=ξ, and using (5) in preceeding equation we get,

$$(37)\qquad a(a-1)(L_v\eta)(\xi)=-(1-a)(\varepsilon(n-1)+2n(1-a)-\lambda a).$$

Substituting (37) in (36) we obtain,

$$(L_v\eta) = \left\{\frac{\varepsilon(n-1)+2n(1-a)}{a} - \lambda\right\}\eta. \tag{38}$$

Now considering (24) and (25), (1) takes the form,

$$\begin{aligned}(L_V g)(X,Y) = 2[3(\varepsilon+a-1)-\lambda]g(X,Y) - 2[2n(1-a)\\ -3\varepsilon(\varepsilon+a-1)+\lambda a]\eta(X)\eta(Y).\end{aligned} \tag{39}$$

Also with the use of (25), (38) becomes

$$L_V\eta = \left(\frac{\varepsilon(n-1)+4n(1-a)-3\varepsilon(\varepsilon+a-1)((n-1)+\varepsilon(1-a))}{a}\right)\eta \tag{40}$$

We know that $L_V\xi = \eta(L_V\xi)\xi$. Lie-differentiating $\eta(\xi) = 1$ along V we get,

$$(L_V\eta)(\xi) + \eta(L_V\xi) = 0. \tag{41}$$

Now Lie-differentiating $g(\xi,\xi) = 1$ along V we get,

$$(L_V g)(\xi,\xi) + 2g(L_V\xi,\xi) = 0. \tag{42}$$

Replacing X and Y by ξ in (2), making use of (5), (10) and using the resulting equation in (16) we obtain,

$$\eta(L_V\xi) = \varepsilon[(n-1)-\lambda]. \tag{43}$$

Substituting the value for λ in above equation and calculating for $L_V\xi$ we find that

$$L_V\xi = \varepsilon\left[(n-1) - \left(\frac{3\varepsilon(\varepsilon+a-1)[\varepsilon(n-1)+(1-a)]-2n(1-a)}{a}\right)\right]\xi. \tag{44}$$

Now Lie-differentiating $d\eta(X,Y) = g(X,\phi Y)$ along V and noting that L_V commutes with exterior operator 'd' we get,

$$(d(L_V\eta))(X,Y) = (L_v g)(X,\phi Y) + g(X,(L_V\phi)Y). \tag{45}$$

Using (39) and (41) in the above equation we obtain,

$$g(X,(L_V\phi)Y) = ((\frac{\varepsilon(n-1)+2n(1-a)}{a} - \lambda) - (2\varepsilon(\varepsilon+a-1)-2\lambda))g(X,\phi Y). \tag{46}$$

From above equation we get,

$$(L_V\phi)Y = (A-B)\phi Y, \tag{47}$$

where $A=\{\frac{\varepsilon(n-1)+2n(1-a)}{a}-\lambda\}$, $B=2\{\varepsilon(\varepsilon+a-1)-\lambda\}$ and
$\lambda=\frac{3\varepsilon(\varepsilon+a-1)[\varepsilon(n-1)+(1-a)]-2n(1-a)}{a}$.

Thus we have the following:

Theorem 3.3. *Let (M,g) be an ε-Sasakian manifold and (M,g^*) be obtained by D_a-homothetic deformation of (M,g). If (M,g,V,λ) and (M,g^*,V,λ) are both Ricci solitons, then V transforms ϕ,η and ξ according to the equations as following:*

1. $L_V\eta=\left(\frac{\varepsilon(n-1)+4n(1-a)-3\varepsilon(\varepsilon+a-1)((n-1)+\varepsilon(1-a))}{a}\right)\eta.$
2. $L_V\xi=\varepsilon\left[(n-1)-\left(\frac{3\varepsilon(\varepsilon+a-1)[\varepsilon(n-1)+(1-a)]-2n(1-a)}{a}\right)\right]\xi.$
3. $L_V\phi=(A-B)\phi$, *where* $A=\{\frac{\varepsilon(n-1)+2n(1-a)}{a}-\lambda\}$, $B=2\{\varepsilon(\varepsilon+a-1)-\lambda\}$ *and* $\lambda=\frac{3\varepsilon(\varepsilon+a-1)[\varepsilon(n-1)+(1-a)]-2n(1-a)}{a}$.

Now, Consider the concurrent vector field V on a Riemannian manifold (M,g),

Definition 3.1. [2] A vector field V on a Riemannian manifold is said to be concurrent if

$$(\nabla_X V)=\rho X,\quad \forall X, \tag{48}$$

where, ρ is a constant.

We know that,

$$(L_V g)(X,Y)=g(\nabla_X V,Y)+g(X,\nabla_Y V). \tag{49}$$

Making use of (48) in the above equation we get,

$$(L_V g)(X,Y)=2\rho g(X,Y). \tag{50}$$

Replacing Y=ξ in (50) yields,

$$(L_V\eta)(X)=g(X,L_V\xi)+2\rho\varepsilon\eta(X). \tag{51}$$

As $(L_V\xi)=\eta(L_V\xi)\xi$ and by substituting the value of $\eta(L_V\xi)$, the above equation reduces to,

$$(L_V\eta)(X)=\varepsilon(\varepsilon(n-1-\lambda)+2\rho)\eta(X). \tag{52}$$

Since g^* is Ricci soliton, (33) reduces to

$$S(X,Y)=Ag(X,Y)+B\eta(X)\eta(Y), \tag{53}$$

where $A=-[a(\rho+\lambda)+3(1-a)(\varepsilon+a-1))]$ and $B=-(a-1)[a\varepsilon(\varepsilon(n-1-\lambda)+2\rho)-2n(1-a)+3\varepsilon(\varepsilon+a-1)+\lambda a)]$.

On contraction of the above equation gives,

$$r=-[3(1-a)(\varepsilon+a-1)(2n+2)+(2n+1)a(\rho+\lambda) \\ +(a-1)(a\varepsilon(\varepsilon(n-1-\lambda)+2\rho)-2n(1-a)+\lambda a)]. \tag{54}$$

This leads to the following result:

Theorem 3.4. *Let (M,g,V,λ) be an ε-Sasakian manifold and (M,g^*,V,λ) be obtained by D_a-homothetic deformation of (M,g) with V as a concurrent vector field, then the manifold reduces to η-Einstein with a constant scalar curvature.*

Now replacing Y=ξ in (2) and by making use of (5) and (10) we get,

$$(L_V\eta)(X)=g(X,L_V\xi)-2\varepsilon(n-1+\lambda)\eta(X). \tag{55}$$

It is known that $L_V\xi=\eta(L_V\xi)\xi$ and considering g^* as a Ricci soliton, therefore (33) becomes

$$a(L_vg)(X,Y)+2S(X,Y)+2[(\varepsilon-(1-a))(1-a)+\lambda a]g(X,Y) \\ +2(1-a)[\varepsilon(1-a)(n-1)-\lambda a-a(1-2\varepsilon)(n-1+\lambda)]\eta(X)\eta(Y)=0. \tag{56}$$

As g is Ricci soliton the above equation yields,

$$S(X,Y)=-3[\varepsilon+a-1]g(X,Y) \\ -[2n(1-a)-3\varepsilon(\varepsilon+a-1)-\lambda a+a(\varepsilon(n-1)+3\varepsilon\lambda)]\eta(X)\eta(Y). \tag{57}$$

Plugging (57) in (18), making use of (15) we get,

$$S^*(X,Y)=-3(\varepsilon+a-1)g^*(X,Y)+[3(\varepsilon+a-1)(a-1)+\frac{1}{a^2}[2na(1-a) \\ -3\varepsilon(\varepsilon+a-1)(2-a)+\lambda a-a(\varepsilon(n-1)+3\varepsilon\lambda]]\eta^*(X)\eta^*(Y). \tag{58}$$

Therefore, we have the following:

Theorem 3.5. *Under D_a-homothetic deformation of ε-sasakian manifold, η-Einstein Ricci soliton remains η-Einstein Ricci soliton with constant scalar curvature.*

Now repacing Y=ξ in (4) and by making use of (5) and (10) we get,

$$(L_V\eta)(X) = g(X, L_V\xi) - 2\varepsilon[(n-1) - (\lambda + \rho R)]\eta(X). \tag{59}$$

It is known that $L_V\xi = \eta(L_V\xi)\xi$ and considering g^* as a Ricci soliton, (33) becomes,

$$\begin{aligned} &a(L_Vg)(X,Y) + 2S(X,Y) + 2[3(1-a)(\varepsilon + a - 1) + \lambda a]g(X,Y) \\ &+ 2[a(a-1)\varepsilon[(n-1)(\varepsilon - 2) - \lambda(\varepsilon + 2) - 2\rho R] \\ &+ 2n(1-a)^2 - 3\varepsilon(1-a)(\varepsilon + a - 1) + \lambda a(a-1)]\eta(X)\eta(Y) = 0. \end{aligned} \tag{60}$$

As g is Ricci soliton the above equation yields,

$$\begin{aligned} S(X,Y) = -3(\varepsilon + a - 1)g(X,Y) + [\varepsilon a((n-1)(\varepsilon - 2) - \lambda(\varepsilon + 2) - 2\rho R) \\ +2n(1-a) + 3\varepsilon(\varepsilon + a - 1) + \lambda a]\eta(X)\eta(Y). \end{aligned} \tag{61}$$

Substituting (61) in (18) and by making use of (15) we get,

$$\begin{aligned} S^*(X,Y) = -3(\varepsilon + a - 1)g^*(X,Y) + [3(a-1) + \frac{1}{a^2}[\varepsilon a(n-1)(\varepsilon - 2) \\ -\lambda(\varepsilon + 2) - 2\rho R) - 2na(1-a) + 3\varepsilon a(\varepsilon + a - 1) + \lambda a]]\eta^*(X)\eta^*(Y). \end{aligned} \tag{62}$$

Therefore, we state the following:

Theorem 3.6. *Under D_a-homothetic deformation of ε-Sasakian manifold, η-Einstein ρ Ricci soliton remains η-Einstein Ricci soliton with constant scalar curvature.*

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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