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ON ROLF NEVANLINNA PRIZE WINNERS COLLABORATION GRAPH - III

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Abstract. The problem of determining the collaboration graph of co-authors of Paul Erdos is a challenging task. Here we take up this problem for the case of Rolf Nevanlinna Prize Winners. Even though the number of the prize winners as on date is 8, the collaboration graph has 23 vertices and 48 edges and posses several interesting properties. In this paper we have obtained this graph and determined standard graph parameters for both this graph and its complement besides probing its structural properties. Several new results were obtained.

Keywords: Three-step iterative scheme with errors, Banach spaces, ϕ-strongly quasi-accretive operators, common fixed point, strongly accretive.

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1. Introduction

In the past decade, graph theory has gone through a remarkable shift and a profound transformation. The change is in large part due to humongous amount of information

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Dedicated to Prof.Dr.R.Balakrishnan as a mark of Deep Respect

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that are confronted with. A main way to sort through massive data sets is to build and examine the network formed by interrelations. For example, Google's successful web search algorithms are based on the www graph, which contain all WebPages as vertices and hyper links as edges. These are sorts of information networks, such as biological networks built from biological databases and social networks formed by email, phone calls, instant messaging and various other types of physical networks. Of particular interest to mathematicians is the collaboration graph, which is based on the data from Mathematical Reviews. In the collaboration graph, every mathematician is a vertex, and two mathematicians who wrote a joint paper are connected by an edge.

The graph considered in his paper is finite, simple and undirected. For any undefined terms see [1] and [11]. For any graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G respectively. The collaboration graph G has as vertices all researchers (dead or alive) from all academic disciplines with an edge joining vertices u and v if u and v have jointly published a paper or book. The distance between two vertices u and v denoted $d(u, v)$, is the number of edges in the shortest path between u and v in case if such a path exists and ∞ otherwise. Clearly $d(u, u) = 0$. We now consider the collaboration subgraph centered at Paul Erdos (1913-1996). For a researcher v, the number $d(Erdos, v)$ is called the Erdos number of v. That is, Paul Erdos himself has Erdos number 0, and his coauthors have Erdos number 1. People not having Erdos number 0 or 1 but who have published with some one with Erdos number 1 have Erdos number 2, and so on. Those who are not linked in this way to Paul Erdos have Erdos number ∞. The collection of all individuals with a finite Erdos number constitutes the Erdos component of G. 511 people have Erdos number 1, and over 5000 have Erdos number 2. In the history of scholarly publishing in Mathematics, no one has ever matched Paul Erdos's number of collaborators or papers (about 1500, almost 70% of which were joint works). Many important people in academic areas other than mathematics proper-as diverse as physics, chemistry, crystallography, economics, finance, biology, medicine, biophysics, genetics, metrology, astronomy, geology, aeronautical engineering, electrical engineering, computer science, linguistics, psychology and philosophy do indeed have finite Erdos numbers. Also see [13] for more details.

Problem: For the sake of brevity we denote the Rolf Nevanlinna Prize Winners Collaboration Graph by G^*. In this paper we consider the problem of 1) obtainting G^*; 2) determining for G^* and its complement, certain standard graph parameters; and 3) investigating the structural properties of G^*.

We have already described in [23, 24, 25]about the Rolf Nevanlinna Prize, History of Rolf Nevanlinna Prize

2. Construction of G^*

G^* is constructed as follows: G^* has twenty three vertices and forty eight edges. $V(G^*) = \{u_1, u_2, \ldots, u_{23}\}$ here u_1 = Paul Erdos, u_2 = Maria Margarat Klawe, u_3 = Siemion Fajtlowicz, u_4 = Robert Robinson, u_5 = George Gunthar Lorentz, u_6 = Endre Szemeredi, u_7 = Laszlo Lovasz, u_8 = Nathan Linial, u_9 = Alon Noga, u_{10} = Boris Aronov, u_{11} = Andrej Ehrenfeucht, u_{12} = Mark Jerrum, u_{13} = Alok Aggarwal, u_{14} = Robert Endre Tarjan, u_{15} = Leslie Valiant, u_{16} = A.A. Razborov, u_{17} = Avi Wigderson, u_{18} = Peter W. Shor, u_{19} = Madhu Sudan, u_{20} = Jon Kleinberg, u_{21} = Mario Szegedy, u_{22} = Lance J. Fortnow, u_{23} = Daniel Spielman. Note that the chronological order of prize winners are defined in order by u_j, $j = 14$ to 20, 23, $E(G^*) = \{e_1, e_2, \ldots, e_{48}\}$ where $e_1 = (u_1, u_2)$, $e_2 = (u_1, u_3)$, $e_3 = (u_1, u_4)$, $e_4 = (u_1, u_5)$, $e_5 = (u_1, u_6)$, $e_6 = (u_1, u_7)$, $e_7 = (u_1, u_8)$, $e_8 = (u_1, u_9)$, $e_9 = (u_1, u_{10})$, $e_{10} = (u_2, u_8)$, $e_{11} = (u_2, u_{13})$, $e_{12} = (u_2, u_{14})$, $e_{13} = (u_2, u_{17})$, $e_{14} = (u_2, u_{18})$, $e_{15} = (u_3, u_{11})$, $e_{16} = (u_4, u_{12})$, $e_{17} = (u_5, u_{16})$, $e_{18} = (u_6, u_9)$, $e_{19} = (u_6, u_{16})$, $e_{20} = (u_6, u_{17})$, $e_{21} = (u_7, u_8)$, $e_{22} = (u_7, u_9)$, $e_{23} = (u_7, u_{17})$, $e_{24} = (u_7, u_{18})$, $e_{25} = (u_8, u_9)$, $e_{26} = (u_8, u_{13})$, $e_{27} = (u_8, u_{17})$, $e_{28} = (u_8, u_{18})$, $e_{29} = (u_9, u_{10})$, $e_{30} = (u_9, u_{17})$, $e_{31} = (u_9, u_{19})$, $e_{32} = (u_{10}, u_{13})$, $e_{33} = (u_{11}, u_{15})$, $e_{34} = (u_{12}, u_{15})$, $e_{35} = (u_{13}, u_{17})$, $e_{36} = (u_{13}, u_{18})$, $e_{37} = (u_{13}, u_{19})$, $e_{38} = (u_{13}, u_{20})$, $e_{39} = (u_{16}, u_{17})$, $e_{40} = (u_{17}, u_{19})$, $e_{41} = (u_{19}, u_{20})$, $e_{42} = (u_1, u_{21})$, $e_{43} = (u_7, u_{21})$, $e_{44} = (u_9, u_{21})$, $e_{45} = (u_{17}, u_{22})$, $e_{46} = (u_{19}, u_{21})$, $e_{47} = (u_{21}, u_{22})$, $e_{48} = (u_{22}, u_{23})$. None of the eight RNPW'S have Erdos number 1. Out of the 511 direct co-authors of Paul Erdos, with Erdos Number 1, only ten members are connected by a path of length 1 with the

RNPW'S. Out of the eight RNPW'S only five members namely $u_{14}, u_{16}, u_{17}, u_{18}, u_{19}$ have Erdos number 2, the remaining members namely $u_{15}, u_{20}, u_{22}, u_{23}$ have Erdos number 3. G^* is shown in Figure 1.

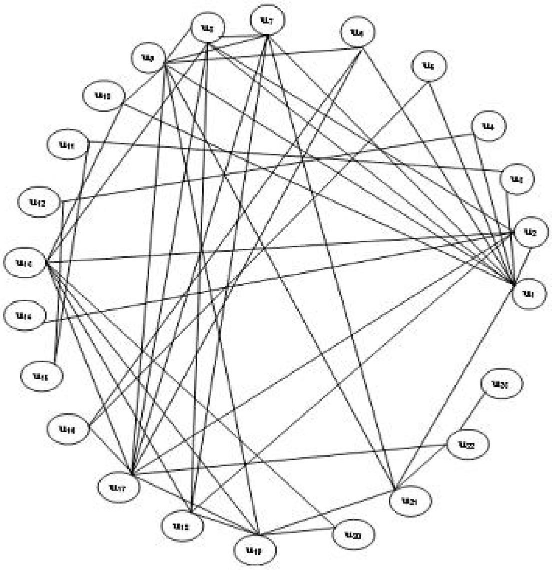


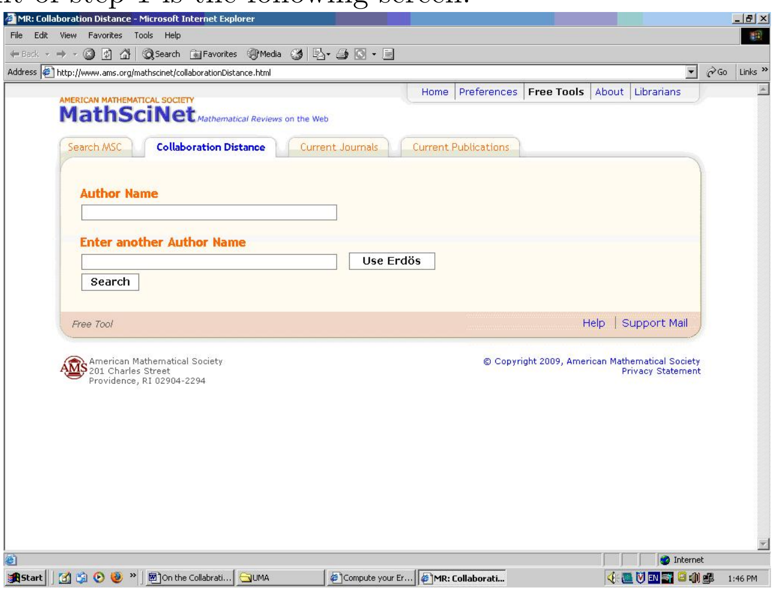
FIGURE 1. G^*

The method of obtaining the G^* is described as follows:

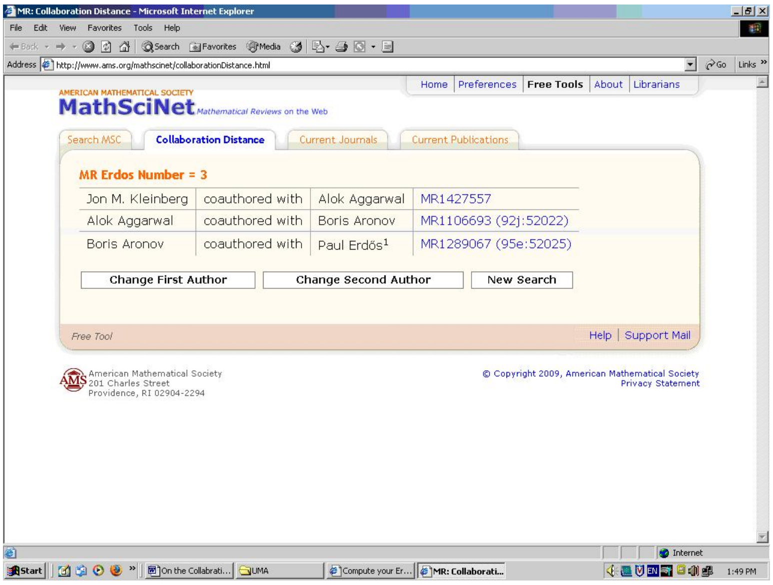
: Step 1: Click on the link:

http://www.ams.org/mathscinet/collaborationDistance.html

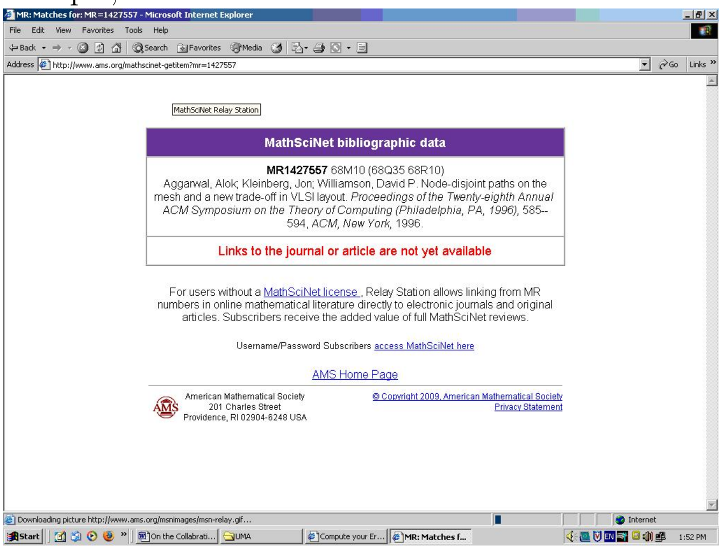
The result of step 1 is the following screen:



: Step 2: Enter the Author name and Enter another author name or click on the use Erdos icon. For example, if the author name is: Jon.M. Kleinberg and the another author name is: Paul Erdos then we obtain the following screen:



To know more details about the joint work of these authors, just click on the respective MR number. For example, if we click on MR1427557 then we derive the following screen:



Proceeding like this, one can obtain all the eight RNPW'S collaboration details one by one. Since the number of RNPW'S is a small number, the above procedure is recommended. It is vital to record a fact that, if there is no co author relationship at all between two persons say X and Y, then the result of our action of doing the Step 2 will be: "No path found". We have thoroughly checked all possible combinations.

That is, first, we have checked the co author relationship between any of the RNPW'S with any of the 10 applicable co-authors at level 1 with Erdos number 1. This action leads to $5 \times 10 + 3 \times 10$ (where 5×10 stands for the possible collaboration between 5 RNPW's at level 2 with any of the 10 possible collaborators at level 1 and 3×10 stands for the possible collaboration between 3 RNPW's at level 3 with any of the 10 possible collaborators at level 1) combinations. Then we have considered the possible collaboration of the 4 non RNPW's at level 2 with any of the 10 possible collaborators at level 1. This action leads to 4×10 combinations. Then we have to consider possible collaborators between themselves of both 5 RNPW's and 4 non RNPW's at level 2. This action leads to $\binom{5}{2} + \binom{4}{2} + 5 \times 4$ combinations. Then we have to consider the possible collaborators between 3 of the RNPW's at level 3 with 5 RNPW's at level 2 and with 4 non RNPW's at level 2. This action leads to $3 \times 5 + 3 \times 4$ combinations. Finally we have to consider all possible combinations between themselves of the 3 RNPW's at level 3. This action leads to $\binom{3}{2}$ combinations. A scrupulous implementation of the above said procedure has led to the graph G^* in Figure 1.

3. G^* - its Certain Coloring Parameters and their Properties

Graph coloring is an important area of theoretical and practical research in combinatorics. By a coloring we mean an assignment of colors to the vertices or edges. More formally, a coloring of a graph $G(V, E)$ is a function f from $V(G)$ or $E(G)$ to the set of all natural numbers. Here we restrict our attention to only vertex colorings. Hence, the range of the coloring is only a finite subset; and if the graph is colored with k-colors, without loss of generality, we can assume the range of the coloring to the $\{1, \ldots, k\}$. A coloring of a graph G is called proper if no two adjacent vertices are assigned the same color. The minimum number of colors used in such a coloring is what is called the chromatic number of G, denoted by $\chi(G)$. A coloring (not necessarily proper) of a graph G is called a pseudocomplete coloring if for every pair of distinct colors, say, i, j there exists an edge $e = (u, v) \in E(G)$ such that u is colored i and v is colored j. The maximum number of colors used in a pseudocomplete coloring of a graph G is called the pseudoachromatic number, $\psi^*(G)$. The maximum number of colors used in a proper complete coloring of a graph G is the achromatic number, $\psi(G)$. (Note that the chromatic number of G is the minimum of colors used in a proper pseudocomplete coloring of G). Further it is easy to see that $\chi(G) \leq \psi(G) \leq \psi^*(G)$.

Proposition 3.1. $\chi(G^*) \leq \binom{\omega(G^*)+1}{2}$

Proof. Look at G^*. As $\{u_2, u_8, u_{13}, u_{17}\}$ constitutes the complete graph on four vertices as an induced subgraph of G^*, we have $\chi(G^*) \geq 4$. Now color the vertices $u_2, u_3, u_4, u_5, u_6, u_7, u_{10}, u_{19}, u_{23}$ with color a; u_8, u_{11}, u_{12}, u_{14}, u_{16}, u_{20}, u_{22} color the vertices with color b; color the vertices u_1, u_{13}, u_{15}, u_{21} with color c; color the vertices u_9, u_{17}, u_{18} with d. This gives raise to a chromatic 4-coloring of G^*. This implies that $\chi(G^*) \leq 4$. Hence $\chi(G^*) = 4$. Further, it is easy to check that G^* contains no K_5, the complete graph on 5 vertices as an induced subgraph. Therefore $\omega(G^*) = 4$; As $4 = \chi(G^*) \leq 10 = \binom{4+1}{2}$, the proposition follows. □

We know that if a graph G does not contain $2K_2$ as an induced subgraph then $\chi(G) \leq \binom{\omega(G)+1}{2}$.

Proposition 3.2. *It is not necessary that a graph G satisfying the inequality $\chi(G) \leq \binom{\omega(G)+1}{2}$ should not contain $2K_2$ as an induced subgraph.*

Proof. Clearly (u_{13}, u_{20}) and (u_{22}, u_{23}) constitutes $2K_2$ as an induced subgraph of G^*. The result now follows from Proposition 3.1. □

Proposition 3.3. $10 \leq \chi(\overline{G^*}) \leq 20$, *where $\overline{G^*}$ denotes the complement of G^*.*

Proof. As no two of the vertices $\{u_2, u_3, u_4, u_5, u_6, u_7, u_{10}, u_{15}, u_{19}, u_{22}\}$ is an independent set in G^*, it induce a K_{10} in $\overline{G^*}$. Therefore $\chi(\overline{G^*}) \geq 10$. To obtain the upper bound we appeal to the famous Nordhaus and Gaddum inequality. $\chi(G)+\chi(G^*) \leq |V(G^*)|+1$. □

G^* and its chromatic polynomial

We know that any given graph G on n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G and is defined as follows: The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors. Let r_i be the different ways of properly coloring G using exactly i colors. Since i colors can be chosen out of λ colors in $\binom{\lambda}{i}$ different ways, there are $\binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors. Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms; that is, $P_n(\lambda) = \sum_{i=1}^{n} r_i \begin{pmatrix} \lambda \\ i \end{pmatrix}$. Clearly $r_1 = 0$, as any graph with non empty edge set requires at least two colors for properly coloring its vertices. Now Consider G^*. $r_{23} = 23!$ as G^* can be properly colored in $23!$ ways using 23 different colors. As $\chi(G^*) = 4$, it is easy to deduce that $r_2 = r_3 = 0$. We leave it to the readers to determine r_i for $4 \le i \le 23$. Hence,

Theorem 3.4. *The chromatic polynomial of* G^* *is* $P_{23}(\lambda) = \sum_{i=4}^{23} r_i \begin{pmatrix} \lambda \\ i \end{pmatrix} + \prod_{i=0}^{23} (\lambda - i)$.

G^* and its Partitions

The Vertex-arboricity $a(G)$ of a graph G is the fewest number of subsets in a partition of the vertex set of G such that each subset induces an acyclic subgraph. Clearly $a(G) \le \chi(G)$ for any graph G.

Proposition 3.5. $a(G^*) = 2$.

Proof. As $a(G^*) \le \chi(G^*) = 4$, we have $a(G^*) \le 4$. Partition the vertex set of G^* as $V(G) = \bigcup_{i=1}^{3} V_i$ with $V_1 = \{u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6,\ u_7,\ u_{10},\ u_{11},\ u_{12},\ u_{14},\ u_{17},\ u_{18},\ u_{19},\ u_{22}\}$, $V_2 = \{u_8, u_9, u_{13}, u_{15}, u_{16}, u_{20}, u_{21}, u_{23}\}$. Note that each V_i, $1 \le i \le 2$ induces an acyclic subgraph. Now it is easy to see the result. □

Proposition 3.6. $2 \le \overline{a(G^*)} \le 11$.

Proof. Mitchem [17] proved that for any G of order p, 1) $\sqrt(p) \leq a(G^*) + a(\overline{G^*}) \leq \frac{p+3}{2}$; 2) $\frac{p}{4} \leq a(G^*)a(\overline{G^*}) \leq \left(\frac{(p+3)}{4}\right)^2$. In view of this we get for our G^*, $4.8 \leq a(G^*) + a(\overline{G^*}) \leq 13$ and this inturn yields that $2 \leq a(\overline{G^*}) \leq 11$. □

G^* and connectivity properties

A set A of vertices of a graph G is a separator if $G - A$ has at least two connected components. If A induces a clique in G then we call A a clique separator. G^* has a number of clique separators. For example, $\{u_1, u_2\}$, $\{u_1, u_7, u_8\}$, $\{u_2, u_8, u_{13}, u_{17}\}$ are all clique separators of different cardinality. Further the vertices u_{14}, u_{20} and u_{23} are simplicial vertices, as the set of vertices adjacent to them respectively induces a clique in G^*. That is $adj(u_{14}) = \{u_2\}$, a K_1, the complete graph on one vertex and $adj(u_{20}) = \{u_{13}, u_{19}\}$, a K_2, $adj(u_{23}) = \{u_{22}\}$, a K_1, where $adj(u) = \{v : (u, v) \in E(G)\}$. It is interesting to note that the simplicial vertices need not be clique separators, as $\omega(G^*) = \omega(G^* - u_{14})$ and $\omega(G^*) = \omega(G^* - u_{23})$.

Proposition 3.7. *G^* is not a chordal graph.*

Proof. We call a graph G, chordal, if every cycle in G of length at least 4 has a chord. G^* is not a chordal graph, because, the set of vertices of G^*, namely, $\{u_{17}, u_{19}, u_{21}, u_{22}\}$, even though induces a C_4, has no chord edge between the non adjacent pair of vertices (u_{17}, u_{21}), (u_{19}, u_{22}). □

Proposition 3.8. *G^* is not self complementary.*

Proof. We know that a graph G is called a self complementary graph if $G \cong \overline{G}$. Also Clapham [7] has shown that every self-complementary graph has a Hamiltonian cycle. As G^* has a pendent vertices u_{14}, u_{23}, it has no Hamiltonian cycle. □.

Proposition 3.9. $\kappa(G^*) = \kappa'(G^*) = 1$, *$\kappa(G^*)$, $\kappa'(G^*)$ are the vertex and the edge connectivity of G^*.*

Proof. As G^* has pendent vertices u_{14}, u_{23}, we have $\kappa(G^*) \leq \kappa(G^*) \leq 1$. Now $\kappa(G^*) = 1$ as $\omega(G^* - u_2) \neq \omega(G^*)$ and $\kappa'(G^*) = 1$ as $\omega(G^* - (u_{22}, u_{23})) \neq \omega(G^*)$. □

Observation 3.10. *It is quite interesting to observe that a vertex disjoint clique decomposition of G^* account for only fifteen edges out of a total of forty eight edges which is one third of $q(G)$. That is $V(G) = \bigcup_{j=1}^{12} H_j$, where $H_1 = \{u_2, u_8, u_{13}, u_{17}\} \cong K_4$; $H_2 = \{u_1, u_6, u_9\} \cong K_3$; $H_3 = \{u_3, u_{11}\} \cong K_2$; $H_4 = \{u_4, u_{12}\} \cong K_2$; $H_5 = \{u_5, u_{16}\} \cong K_2$; $H_6 = \{u_7, u_{18}\} \cong K_2$; $H_7 = \{u_{19}, u_{20}\} \cong K_2$; $H_8 = \{u_{22}, u_{23}\}$; $H_9 = \{u_{10}\},$; $H_{10} = \{u_{14}\}$; $H_{11} = \{u_{15}\}$; $H_{12} = \{u_{21}\}$ all $H_i \cong K_1$, $9 \leq i \leq 12$. By a clique graph $cl(G)$ of a given graph G, we mean the graph, whose vertices are the vertex-disjoint cliques of G and the edge set is constructed as follows: Introduce an edge between two clique vertices, if any vertex of one clique is adjacent to any vertex of the other clique. The clique graph $cl(G)$ of G^* is given in Figure 2.*

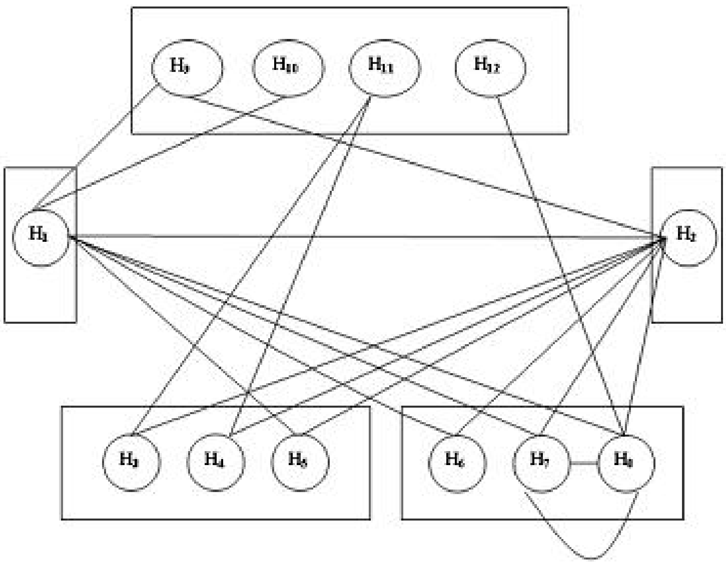


FIGURE 2. $cl(G^*)$

Observation 3.11. *We call an open walk that includes all the edges of a graph without retracing any edge a unicursal line or an open Euler line. A connected graph that has a unicursal line will be called a unicursal graph. We know that if a connected graph G has exactly $2k$ odd vertices then there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph. Consider G^*. It has 10 ($= 2k$) odd degree vertices (with $k = 5$) $u_8, u_{10}, u_{13}, u_{14}, u_{16}, u_{17}, u_{19}, u_{21}, u_{22}, u_{23}$. Now add 5 edges* *to G^* between the vertex pairs (u_8, u_{19}), (u_{10}, u_{16}), (u_{13}, u_{21}), (u_{17}, u_{23}), (u_{14}, u_{22}) to form a new graph (G^*). Since every vertex of (G^*) is of even degree, (G^*) consists of an Euler line ρ: Remove from ρ the 5 edges we just added. Then ρ will be split into 5 walks, each of which is a unicursal line. The first removal will leave a single unicursal line; the second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are 5 of them.*

Proposition 3.12. $\beta_0(\overline{G^*}) \leq 14$.

Proof. We know from [16] that if G and $\overline{G}$ are two complementary graphs of finite order p then 1) $\beta_0(G) + \beta_0(\overline{G}) \leq p + 1$ and 2) $\beta_0(G)\beta_0(\overline{G}) \leq \left\lfloor \frac{p+1}{2} \right\rfloor \left\lceil \frac{p+1}{2} \right\rceil$. In view of this we deduce that $\beta_0(\overline{G^*}) \leq 14$ as $\beta_0(\overline{G^*}) = 10$ and $p = 23$. □

Proposition 3.13. $8 \leq \beta_1(G^*) \leq 11$. *If* $\overline{\beta_1} = \max\{\beta_1, \overline{\beta_1}\}$ *else* $1 \leq \overline{\beta_1} \leq 12$.

Proof. We know from Chartrand and Schuster [6] that for a pair of complementary graphs G and $\overline{G^*}$ of finite order p, 1) $\left\lfloor \frac{p}{2} \right\rfloor \leq \beta_1(G) + \beta_1(\overline{G^*}) \leq 2 \left\lfloor \frac{p}{2} \right\rfloor$ and 2) $0 \leq \beta_1(G)\beta_1(\overline{G^*}) \leq \left\lfloor \frac{p}{2} \right\rfloor^2$. In view of this, we have $11 \leq \beta_1(G^*) + \beta_1(\overline{G^*}) \leq 22$ and hence $8 \leq \beta_1(\overline{G^*}) \leq 11$. But the results of Cockayne and Lorimer [8] and Erdos and Schuster [9] imply, moreover, that $\left\lfloor \frac{(p+1)}{3} \right\rfloor \leq \max\{\beta_1(G^*)\beta_1(\overline{G^*})\} \leq \left\lfloor \frac{p}{2} \right\rfloor$. So, $8 \leq \max\{\beta_1(G^*)\beta_1(\overline{G^*})\} \leq 11$. □

Proposition 3.14. $12 \leq \alpha_1(\overline{G^*}) \leq 15$.

Proof. We know from Lasker and Aucrbach [15] that if G and G^* are complementary graphs of order p then

1) $2 \left\lfloor \frac{(p+1)}{2} \right\rfloor \leq \alpha_1(G) + \alpha_1(\overline{G^*}) \leq \left\lceil \frac{3p}{2} \right\rceil - 2$;

2) $\left\lfloor \frac{(p+1)}{2} \right\rfloor^2 \leq \alpha_1(G)\alpha_1(\overline{G^*}) \leq \left\lfloor \frac{(\lceil \frac{3p}{2} \rceil - 2)}{2} \right\rfloor \left\lceil \frac{(\lceil \frac{3p}{2} \rceil - 2)}{2} \right\rceil$;

3) $\left\lfloor \frac{(p+1)}{2} \right\rfloor \leq \min\{\alpha_1(G)\alpha_1(\overline{G^*})\} \leq \left\lfloor \frac{(2p+1)}{3} \right\rfloor$; In view of this we have $24 \leq \alpha_1(G^*) + \alpha_1(\overline{G^*}) \leq 33$; $144 \leq \alpha_1(G^*)\alpha_1(\overline{G^*}) \leq 272$;

$12 \leq \min\{\alpha_1(G^*), \alpha_1(\overline{G^*})\} \leq 15$. Hence $12 \leq \alpha_1(G^*) \leq 15$. □

4. G^*, a Bounded Fragmentation Graph

We now proceed to check whether G^* is a bounded fragmentation graph or not? It is quite a recent interesting property introduced by Mohammad Taghi Hajiaghayi and Mahdi Hajiaghayi in [18]. We know that connectivity can be considered as a measure of the reliability of a network. Suppose that a network N is represented by an undirected graph G, in which two computers, namely nodes of the network, can communicate if and only if there is a path in G from one to other. If G is k-connected, then after removing at most $k-1$ vertices of G, the rest of G (which has $n-k+1$ vertices) is still connected. This means that if at most $k-1$ nodes of the network fail, the rest of the nodes of the network can communicate with each other. Now we define a bounded fragmentation graph. A graph G is a $(k, g(k))$-bounded fragmentation graph if $|\zeta(G[V-S])| \leq |g(k)|$ for every $S \subseteq V(G)$ of size at most k, where g is a function of k. A graph G is a totally $g(k)$-bounded fragmentation graph if it is a $(k, g(k))$-bounded fragmentation graph for all $0 \leq k \leq n$. Here $\zeta(G)$ denote the number of components of G, where each element of $\zeta(G)$ is a connected graph. We remark that a bounded fragmentation can play a similar role in the reliability of a network like connectivity. That is, if G is a $(k, g(k))$-bounded fragmentation graph, then thereafter removing at most k vertices, we still have at least one component which has $\Omega(n)$ vertices. The reason is that after removing at most k vertices the rest of the nodes fall into at most a constant number of connected components $(g(k))$ and thus one component has at least $\Omega(n)$ vertices. Thus, after the failure of at most $k-1$ nodes of N, $\Omega(n)$ nodes in the rest of N (and not necessarily $n-k$) still can communicate with each other. So by grouping these facts, we conclude that bounded fragmentation can be considered as a generalization of connectivity. It also has another application in the reliability of a network. Suppose that we need to repair the network N temporarily by adding several links between the current nodes of the network (not by adding any new node because of its high cost) when the number of failing nodes in the networks is at most k. If G is a $(k, g(k))$-bounded fragmentation graph, then we can simply repair the network by adding at most $g(k)-1$ number of links, which is constant. Here after removing the failure nodes, we find the connected components of G in $O(|V(G)|)$ time. Then we can connect these at most $g(k)-1$ edges among them. These two simultaneous properties of bounded fragmentation graphs cause their corresponding networks to be more reliable and robust.

Proposition 4.1. *G^* is a $10k$ bounded fragmentation graph.*

Proof. Clearly the maximum degree of G, viz., $\Delta(G^*) = 10$, is a constant. So after removing any k vertices, $0 \leq k \leq 23$, the number of connected components is at most $g(k) = 10k$. $\square$

Proposition 4.2. *G^* is totally 10-bounded fragmentation graph.*

Proof. For any set $S \subseteq V(G)$ of size k, $0 \leq k \leq 23$, at least one vertex from each connected component of $G[V-S]$ is contained in any maximum independent set. Since the size of the maximum independent set is 10, we see that the number of connected components is bounded above by 6, as well. So, G is totally 10-bounded fragmentation graph. $\square$

Proposition 4.3. *G^* is a totally $(k+5)$-bounded fragmentation graphs.*

Proof. G^* has 5 disjoint paths viz., $u_1u_2u_{14}$, $u_3u_{11}u_{15}u_{12}u_4$, $u_5u_{16}u_6u_9u_{10}$, $u_{21}u_7u_8u_{17}u_{22}u_{23}$, $u_{20}u_{19}u_{13}u_{18}$. Now the removal of a vertex from a path splits the path into at most two sub paths and thus at most two connected components. Thus, removing any k vertices, $0 \leq k \leq 23$, can add at most k connected components. Thus we have at most $(k+5)$-connected components. $\square$

We say that a vertex u of G covers an edge e if u is incident with e (and conversely, e covers u). The minimum number of vertices (edges) covering all the edges (vertices) of G is called vertex-(edge) covering number of G and denoted by $\alpha_0(G)[\alpha_1(G)]$. Similarly a set A of vertices [edges] of G is said to be independent if no edge [vertex] of G is incident with more than one vertex [edge] in A. The maximum cardinality of an independent set of vertices [edges] of G is called vertex-[edge-]independence number of G and denoted by $\beta_0(G)[\beta_1(G)]$. For G^*, $\beta_0 = 10$, and the vertices are: $\{u_2, u_3, u_4, u_5, u_6, u_7, u_{10}, u_{15}, u_{19}\}$. We know that $\alpha_0 + \beta_0 = p$ where $p = |V(G)|$ and hence $\beta_0(G^*) = 10$, $p = 23$ implies $\alpha_0(G^*) = 13$, and the set of vertices which cover all the edges of G^* are $\{u_1, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{16}, u_{17}, u_{18}, u_{20}\}$. Further we also have a result that $\alpha_1 + \beta_1 = p$ and hence we now calculate either of these parameters for G^* to find the other. Here again $\beta_1(G^*) = 10$ and the set of independent edges are $\{(u_1, u_{10}), (u_2, u_{14}), (u_3, u_{11}), (u_4, u_{12}), (u_5, u_{16}), (u_6, u_{17}), (u_7, u_9), (u_8, u_{13}), (u_{19}, u_{20})\}$. So $\alpha_1(G^*) = 13$, and the set of edges which cover all the vertices of G^* are $\{(u_1, u_{10}), (u_2, u_{14}), (u_3, u_{11}), (u_4, u_{12}), (u_5, u_{16}), (u_6, u_9), (u_7, u_8), (u_{13}, u_{17}), (u_{12}, u_{15}), (u_{13}, u_{18}), (u_{19}, u_{20}), (u_1, u_{21}), (u_{22}, u_{23}\}$.

5. G^*, ITS DIAMETER, RADIUS, ECCENTRICITY ETC.

We know that in a graph G, the distance between two vertices u and v, denoted by $d_G(v)$ is the length of the shortest path between u and v in G. The distance of a vertex v in G is defined $d_G(v) = \sum d_G(u, v)$. A vertex of minimum distance is called a median vertex of G. The median is the subgraph of G induced by its median vertices and is denoted by $M(G)$. The eccentricity of a vertex v in G denoted $e(v)$ is the number $\max_{u \in v(G)} d_G(u, v)$. The subgraph of G induced by the vertices of minimum eccentricity is the center $C(G)$ of G. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the $diam(G)$, the diameter of G is the maximum eccentricity. A vertex v is called a peripheral vertex if $e(v) = diam(G)$, and the periphery is the set of all such vertices.

Proposition 5.1. *$M(G^*) = K_1$, where $M(G^*)$ is the median graph of G^*.*

Proof. Compute the distance of each vertex of G^*.
$d(u_1) = \sum_{v \in V(G)} d(u_1, v) = d(u_1, u_2) + d(u_1, u_3) + d(u_1, u_4) + d(u_1, u_5) + d(u_1, u_6) + d(u_1, u_7) + d(u_1, u_8) + d(u_1, u_9) + d(u_1, u_{10}) + d(u_1, u_{11}) + d(u_1, u_{12}) + d(u_1, u_{13}) + d(u_1, u_{14}) + d(u_1, u_{15}) + d(u_1, u_{16}) + d(u_1, u_{17}) + d(u_1, u_{18}) + d(u_1, u_{19}) + d(u_1, u_{20}) + d(u_1, u_{21}) + d(u_1, u_{22}) + d(u_1, u_{23}) = 1+1+1+1+1+1+1+1+1+2+2+2+2+3+2+2+2+2+3+1+2+3 = 37$.
Similarly one can compute that $d(u_2) = 46$, $d(u_3) = 54$, $d(u_4) = 54$, $d(u_5) = 55$, $d(u_6) = 51$, $d(u_7) = 46$, $d(u_8) = 44$, $d(u_9) = 43$, $d(u_{10}) = 50$, $d(u_{11}) = 73$, $d(u_{12}) = 71$, $d(u_{13}) = 58$, $d(u_{14}) = 68$, $d(u_{15}) = 85$, $d(u_{16}) = 65$, $d(u_{17}) = 53$, $d(u_{18}) = 60$, $d(u_{19}) = 56$, $d(u_{20}) = 64$, $d(u_{21}) = 50$, $d(u_{22}) = 72$, $d(u_{23}) = 94$. From this, it clears that u_1 is the

median vertex, as $d(u_1) = \min\{d(u_i) : 1 \leq i \leq 23\} = 37$. As the median vertex for our G^* is unique; the median graph $M(G^*) = K_1$. $\square$

Proposition 5.2. *$C(G^*) = K_1$, where $C(G^*)$ is the center of G^*.*

Proof. We compute the eccentricity of every vertex of G^*. $e(u_1) = \max\{d(u_1, u_j) : 2 \leq j \leq 23\} = d(u_1, u_{15}) = 3$; $e(u_2) = d(u_2, u_{15}) = d(u_2, u_{23}) = 4$; $e(u_3) = d(u_3, u_{20}) = d(u_3, u_{23}) = 4$; $e(u_4) = d(u_4, u_{20}) = d(u_4, u_{23}) = 4$; $e(u_5) = d(u_5, u_{15}) = d(u_5, u_{23}) = 4$; $e(u_6) = d(u_6, u_{15}) = d(u_6, u_{23}) = 4$; $e(u_7) = d(u_7, u_{15}) = 4$; $e(u_8) = d(u_8, u_{16}) = d(u_8, u_{23}) = 4$; $e(u_9) = d(u_9, u_{15}) = 4$; $e(u_{10}) = d(u_{10}, u_{15}) = d(u_{10}, u_{23}) = 4$; $e(u_{11}) = d(u_{11}, u_{20}) = d(u_{11}, u_{23}) = 5$; $e(u_{12}) = d(u_{12}, u_{19}) = d(u_{12}, u_{23}) = 5$; $e(u_{13}) = d(u_{13}, u_{15}) = d(u_{13}, u_{23}) = 5$; $e(u_{14}) = d(u_{14}, u_{15}) = d(u_{14}, u_{23}) = 5$; $e(u_{15}) = d(u_{15}, u_{20}) = 6$; $e(u_{16}) = d(u_{16}, u_{15}) = d(u_{16}, u_{23}) = 5$; $e(u_{17}) = d(u_{17}, u_{15}) = d(u_{17}, u_{23}) = 5$; $e(u_{18}) = d(u_{18}, u_{15}) = d(u_{18}, u_{23}) = 5$; $e(u_{19}) = d(u_{19}, u_{15}) = 6$; $e(u_{20}) = d(u_{20}, u_{15}) = 6$, $e(u_{21}) = d(u_{21}, u_{20}) = 4$, $e(u_{22}) = d(u_{22}, u_{15}) = d(u_{22}, u_{18}) = d(u_{22}, u_{19}) = d(u_{22}, u_{20}) = 5$, $d(u_{23}, u_{15}) == d(u_{23}, u_{20}) = 6u_1$ is the only vertex having minimum eccentricity 3, we deduce that $C(G^*) = K_1$. $\square$

Theorem 5.3. *$M(G^*) = C(G^*)$.*

Proof. It follows from Proposition 5.1 and Proposition 5.2.

Corollary 5.4. *$r(G^*) = 3$, where $r(G^*)$ is the radius of G^*.*

Corollary 5.5. *$diam(G^*) = 6$, where $diam(G^*)$ is the diameter of G^*.*

Corollary 5.6. *The periphery of G^* is an empty set.*

Proof. As G^* has no peripheral vertex the proof follows. $\square$

Proposition 5.7. *G^* is the extremal graph for the inequality*
$r(G) \leq diam(G) \leq 2r(G)$.

Proof. It follows from Corollary 5.4 and Corollary 5.5. $\square$

Proposition 5.8. *$diam(\overline{G^*}) = r(\overline{G^*}) \leq 6$.*

Proof. Bosak [2] have proved that, if G^* and $\overline{G^*}$ are two connected complementary graphs of order $p \geq 6$, then 1) $4 \leq diam(G^*) + diam(\overline{G^*}) \leq p+1$ 2) $4 \leq diam(G^*)diam(\overline{G^*}) \leq 2p-2$. In view of this, we have that $diam(\overline{G^*}) \leq \frac{42}{7} = 6.3 = 6$. The proof now follows from Corollary 23.1 and Corollary 23.2. □

The average eccentricity of a graph G is the mean eccentricity of a vertex in G. That is, $avec(G) = \dfrac{1}{n} \displaystyle\sum_{v \in V(G)} e(v)$. It is easy to see that $r(G) \leq avec(G) \leq diam(G)$. Now we compute the $avec(G^*)$.

$avec(G^*) = \frac{1}{23}[3+4+4+4+4+4+4+4+4+4+5+5+5+5+6+5+5+5+6+6+4+5+6] = \frac{107}{23} = 4.6$ and it lies between the radius and diameter of G^*, i.e., $r(G) = 3 \leq 4.6 \leq 6 = diam(G)$. Similarly the average distance is defined as $\mu(G) = \frac{1}{n(n-1)} \sum_{v \in V(G)} d_G(v)$ where $d_G(v)$ is the distance of a vertex v in G. Now

$$\begin{aligned}\mu(G^*) = \frac{1}{506}\left[\sum_{j=1}^{23} d(u_j)\right] &= \frac{1}{506}[37+46+54+54+55+51+46+44+43 \\ &\quad +50+73+71+58+68+85+65+53+60+56+64+50+72+94] \\ &= \frac{1}{506} \times 1349 = 2.66.\end{aligned}$$

Note that for any graph G, $avec(G) \leq \frac{1}{n}d(C(G)) + r(G)$. One can check that this graph the inequality.

Note 5.9. *Nestled between the minimum eccentricity and maximum eccentricity is the average eccentricity. It was introduced by Buckley and Harary* [4]*. This new parameter has a practical relevance. For example, consider a communications network modeled by a graph with vertices representing the nodes of the network and edges representing the links between them. One might want to minimize the average, taken over all the nodes in the system, of the maximum time delay of a message emanating from it. This is the average eccentricity of the corresponding graph.*

Deleting an edge from a graph may cause its diameter to increase or stay the same, but it cannot decrease. A graph G is diameter-minimal if for all the edges $e \in E(G)$,

$diam(G - e) > diam(G)$. Any edge that can be removed from G without affecting the diameter is called superfluous. Note that diameter-minimal graphs have no superfluous edges. For, Let G be a diameter-minimal graph with diameter 2. Then every superfluous edge $e = (u, v)$ is contained in a triangle. Suppose not, then the removal of e would make $diam(G) \geq d(u, v) \geq 3$.

6. G^* and Distance Degree Sequence

For a vertex v in a connected graph G, let $n_i(v)$ be the number of vertices at distance i from v. The distance degree sequence of vertex v is $dd_s(v) = (n_0(v), n_1(v), \ldots, n_{e(v)}(v))$. Clearly $n_0(v) = 1$ for all v; $n_1(v) = deg(v)$. The length of the sequence $dd_s(v)$ is one more than the eccentricity of v; $\sum n_i(v) = p$. The distance degree sequence $dd_s(G)$ of a graph G consists of sequences $dd_s(v)$ of its vertices, listed in numerical order. If a particular dd_s appears k times, we list it once with k as an exponent to indicate the multiplicity. For G^*, $dd_s(u_1) = (1, 10, 9, 3)$; $dd_s(u_2) = (1, 6, 11, 4, 1)$; $dd_s(u_3) = (1, 2, 10, 9, 1)$; etc. Similarly one can have corresponding to the distance of each vertex, a special distance sequence $sds(G)$ of a connected graph G as the list of its distance values arranged in non decreasing order. The distance values need not be consecutive integers; There need not be two vertices with maximum distance value: $sds(G)$ is derivable from $dd_s(G)$: For the sequence $dd_s(v) = (n_0(v), n_1(v), \ldots, n_{e(v)}(v))$, we have $d(v) = \sum_{i=1}^{e(v)} in_i(v)$. For instance we have for G^*, $d(u_1) = 1 \times n_1(u_1) + 2 \times n_2(u_2) + 3 \times n_3(u_3) = 37$.

A graph G is geodesic if every pair of vertices u and v are joined by a unique path of length $d(u, v)$. One can see [4] for more.

Proposition 6.1. *G^* is not a geodesic.*

Proof. We know that if every cycle of G is odd, then G is a geodesic. As G^* contains an even cycle: $u_1u_2u_8u_9u_1$, it is not geodesic. □

We know that, if G is geodesic, then every cycle of G of smallest length is odd. But the converse is not true. For example, every cycle of G^* of smallest length is 3, an odd number, but G^* is not a geodesic.

Proposition 6.2. *G^* must contain a cycle with a diagonal.*

Proof. We know that a graph G with $p \geq 4$ vertices and $2p-3$ edges must contain a cycle with a diagonal. As G^* has $p = 23$ vertices and greater than or equal to $2p-3$ edges, it must contain a cycle with a diagonal. Obviously, $u_{13}u_{18}u_2u_8u_{13}$ is a cycle in G^* with a diagonal $u_{13}u_2$. □

Proposition 6.3. *G^* contains two cycles with no edges in common.*

Proof. We know that a graph G with $p \geq 5$ vertices and $p+4$ edges contains two cycles with no edges in common. As G^* has $p = 23$ vertices and greater than or equal to $p+4$ edges, it must contain two cycles with no edges in common. Clearly $u_8u_9u_{10}u_{13}u_8$ and $u_6u_{16}u_{17}u_6$ are two cycles with no edges in common. □

Proposition 6.4. *G^* contains two cycles with no vertices in common even though it has only less than $3p-5$ edges.*

Proof. We know that a graph G with $p \geq 6$ vertices and $3p-5$ edges contains two cycles with no vertices in common. It is not necessary that the converse of the above stated result be true. Clearly G^* has $p = 23$ vertices and less than or equal to $3p-5$ edges and it has two vertex disjoint cycles $u_8u_9u_{10}u_{13}u_8$ and $u_6u_{16}u_{17}u_6$. □

We know that if G is connected with diameter d, then $2d-3-\left\lfloor\left(\frac{(d^2-d-4)}{p}\right)\right\rfloor \leq \left\lfloor\frac{(p^2-2q)}{p}\right\rfloor$. It is easy to check that G^* satisfies the inequality as L.H.S = 8 with $d = 6$ and R.H.S = 18 with $p = 23$, $q = 48$.

Proposition 6.5. *$g(G^*) \leq 2\ diam(G^*) + 1$, where $g(G^*)$ is the girth of G^*.*

Proof. We know that if G is connected and not a tree, then $g(G) \leq 2diam(G) + 1$. As G^* is connected and has a number of cycles as induced subgraph, it is not a tree. Further as the length of the shortest cycle in G^*, namely, the girth is 3, and $diam(G^*) = 6$, the result follows. □

7. Domination and Total Domination

A set S of vertices in a graph $G = (V, E)$ is a dominating set of G if every vertex in $V - S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A total dominating set S of G is a dominating set such that the induced subgraph $\langle S \rangle$ has no isolates, or if any vertex v of G is adjacent to at least one vertex of the set S. The total domination number $\gamma_T(G)$ of G is the minimum cardinality of a total dominating set. The maximum degree of a graph G, denoted by $\Delta(G) = \max\limits_{v \in V(G)} deg(v)$.

Theorem 7.1. $\gamma(G^*) = 6$.

Proof. First we claim that no dominating set of G^* can have cardinality 5. Suppose not, then it means there exists a dominating set S with cardinality exactly 5. The degree distribution of vertices of G^* reveals the fact that S should contain u_1 as $deg(u_1) = \Delta(G^*) = 10$, either u_2 or u_{14}, u_{22} or u_{23} as $deg(u_{14}) = 1 = deg(u_{23})$. Since $deg(u_2) > deg(u_{14})$ and $deg(u_{22}) > deg(u_{23})$, it is clear that $u_2, u_{22} \in S$. Now look at all the 2-degree vertices of G^*. They are $u_3, u_4, u_5, u_{11}, u_{12}, u_{15}, u_{20}$. We find no single vertex of G^* adjacent with all the above given 2-degree vertices, to be included as, the fourth choice of element of S. However, as u_{11}, u_{12} are both adjacent only to u_{15}, we allow u_{15} as the fourth preferred element of S. This is because, if we did not allow u_{15} as an element of S, then, in order to accommodate u_{11}, u_{12}, we need to include u_3, u_4 in S, in which case, $S = \{u_1, u_2, u_3, u_4\}$ and there exists an element u_{16} which is not adjacent to any element of S. So $S = \{u_1, u_2, u_{22}, u_{15}\}$. Now the fifth element of S can be any one of the remaining nineteen vertices of G^*. But it is clear that, none of the remaining 2-degree vertices can be an element of S. Therefore, we have the number of choices reduced from nineteen to thirteen. But obviously u_{14} cannot be the fifth element of S. So it is enough to consider only the vertices $u_6, u_7, u_8, u_9, u_{10}, u_{13}, u_{16}, u_{17}, u_{18}, u_{19}, u_{21}, u_{23}$. Now as $(u_i, u_{16}) \notin E(G^*)$ for $i = 7, 8, 9, 10, 13, 18, 19, 21, 23$; $(u_{17}, u_{20}) \notin E(G^*)$; $(u_{16}, u_{19}) \notin E(G^*)$; $(u_6, u_{19}) \notin E(G^*)$ we see that none of the vertices among $u_6, u_7, u_8, u_9, u_{10}, u_{13}, u_{16}, u_{18}, u_{19}, u_{21}$ and u_{23} can be an element of S. Hence it follows that a five element subset cannot be a dominating set of G^*. This means that a probable dominating set of G^* must contain at least 6 elements.

Now let S be a subset of $V(G)$ and $v \in S$. Then the Private Neighbourhood of v with respect to the set S is denoted by $Pr[v, S]$ and is defined as:

$Pr[v, S] = \{w \in V(G) : N(w) \cap S = \{v\}\}$. Note that 1) if $w \in V(G) - S$ and w is adjacent to only $v \in S$, then $w \in Pr[v, S]$, 2) if $w \in S$ and $w \neq v$, then $w \notin Pr[v, S]$, 3) If $w = v$ is not adjacent to any vertex of S, then $w \in Pr[v, S]$. Haynes, Hedetniemi and Slater [12] have showed that a dominating set in a graph G is a minimal dominal set if and only if if for every vertex $v \in S$, $Pr[v, S] \neq \phi$. Now consider the set $S_1 = \{u_1, u_2, u_{17}, u_{15}, u_{22}, u_{13}\}$, a subset of $V(G^*)$. As (u_i, u_1) for $i = 2, 3, \ldots, 10$; (u_{21}, u_1), (u_{22}, u_{23}), (u_i, u_{15}) for $i = 11, 12$; (u_i, u_{13}), for $i = 17, 18, 19, 20$ are all edges of G^*, we see that S_1 is a dominating set of G^*. As $N(u_4) \cap S_1 = \{u_1\}$; $N(u_{20}) \cap S_1 = \{u_{13}\}$; we see that $u_4 \in Pr[u_1, S_1]$ and hence $Pr[u_1, S_1] \neq \phi$. Similarly $N(u_{11}) \cap S_1 = \{u_{15}\}$; $N(u_{16}) \cap S_1 = \{u_{17}\}$; $N(u_{23}) \cap S_1 = \{u_{22}\}$ implies that $Pr[u_j, S_1] \neq \phi$ for $j = 2, 13, 15, 17, 22$. Therefore S_1 is a minimal dominating set. Note that G^* has no minimum dominating set. That is, we show that $S_2 = \{u_1, u_{14}, u_{13}, u_{15}, u_{17}, u_{22}\}$ is another minimal dominating set. S_2 is a dominating set, since, (u_i, u_1) for $i = 2, 3, \ldots, 10, 21$; (u_i, u_{15}) for $i = 11, 12$; (u_{14}, u_2); (u_{16}, u_{17}); (u_i, u_{13}) for $i = 18, 19, 20$ are all edges of G^*. Moreover, $u_4 \in Pr[u_1, S_2]$; $u_2 \in Pr[u_{14}, S_2]$; $u_{20} \in Pr[u_{13}, S_2]$; $u_{11} \in Pr[u_{15}, S_2]$; $u_{16} \in Pr[u_{17}, S_2]$; $u_{23} \in Pr[u_{22}, S_2]$ and hence $Pr[u_i, S_2] \neq \phi$ for all $i \in S_2$. Therefore, S_2 is a minimal dominating set. Hence $\gamma(G^*) = |S_1|$ (or $= |S_2|$) $= 6$. □

Note 7.2. *It is interesting to note that even though $deg(u_{14}) < deg(u_2)$ and $deg(u_{23}) < deg(u_2)$, u_{14} and u_{23} can also become an element of a dominating set in general and a minimal dominating set in particular.*

Theorem 7.3. $\gamma_T(G^*) = 7$

Proof. By Theorem 7.1, we have seen that a dominating set of G^* should have at least six elements. Now we claim that any total dominating set of G^* must have at least seven elements. To see this, let us first start with an arbitrary set $S \subseteq V(G^*)$ with indispensable elements as dictated by the structure of G^*. By Theorem 7.1, the compulsory elements of G^* are $\{u_1, u_2, u_{15}, u_{22}, u_6\}$ (or) $\{u_1, u_2, u_{15}, u_{22}, u_{16}\}$ (or) $\{u_1, u_2, u_{15}, u_{22}, u_{17}\}$ (or) $\{u_1, u_{14}, u_{15}, u_{22}, u_6\}$ (or) $\{u_1, u_{14}, u_{15}, u_{22}, u_{16}\}$ (or) $\{u_1, u_{14}, u_{15}, u_{22}, u_{17}\}$. Suppose $\{u_1, u_2, u_{15}, u_{22}, u_6\}$ then the all possible dominating sets are $S_1 = \{u_1, u_2, u_6, u_{13}, u_{15}, u_{22}\}$, $S_2 = \{u_1, u_2, u_6, u_{15}, u_{19}, u_{22}\}$, $S_3 = \{u_1, u_2, u_6, u_{15}, u_{20}, u_{22}\}$. This is because $\{u_1, u_2, u_6, u_{15}, u_{22}, u_j\}$ is not a dominating set as $(u_j, u_{20}) \notin E(G^*)$ for the following different possible combinations: $(u_1, u_2, u_3, u_6, u_{15}, u_{22})$ (or) $(u_1, u_2, u_4, u_6, u_{15}, u_{22})$ (or) $(u_1, u_2, u_5, u_6, u_{15}, u_{22})$ (or) $(u_1, u_2, u_6, u_7, u_{15}, u_{22})$ (or) $(u_1, u_2, u_6, u_8, u_{15}, u_{22})$ (or) $(u_1, u_2, u_6, u_9, u_{15}, u_{22})$ (or) $(u_1, u_2, u_6, u_{10}, u_{15}, u_{22})$ (or) $(u_1, u_2, u_6, u_{11}, u_{15}, u_{22})$ (or) $(u_1, u_2, u_6, u_{12}, u_{15}, u_{22})$ (or) $(u_1, u_2, u_6, u_{14}, u_{15}, u_{22})$ (or) $(u_1, u_2, u_6, u_{15}, u_{16}, u_{22})$ (or) $(u_1, u_2, u_6, u_{15}, u_{17}, u_{22})$ (or) $(u_1, u_2, u_6, u_{15}, u_{18}, u_{22})$. Again if $S = \{u_1, u_2, u_{15}, u_{22}, u_{16}\}$ (or) $\{u_1, u_2, u_{15}, u_{22}, u_{17}\}$ (or) $(u_1, u_6, u_{14}, u_{15}, u_{22})$ (or) $(u_1, u_{14}, u_{15}, u_{17}, u_{22})$ then the all possible dominating sets are $S_1 = \{u_1, u_2, u_6, u_{13}, u_{15}, u_{22}\}$, $S_2 = \{u_1, u_2, u_6, u_{15}, u_{19}, u_{22}\}$, $S_3 = \{u_1, u_2, u_6, u_{15}, u_{20}, u_{22}\}$, $S_4 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{22}\}$, $S_5 = \{u_1, u_2, u_{15}, u_{17}, u_{19}, u_{22}\}$, $S_6 = \{u_1, u_2, u_{15}, u_{17}, u_{20}, u_{22}\}$, $S_7 = \{u_1, u_6, u_{13}, u_{14}, u_{15}, u_{22}\}$, $S_8 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}, u_{22}\}$, $S_9 = \{u_1, u_{13}, u_{14}, u_{15}, u_{17}, u_{22}\}$ for the same reason given above. In view of this, the total number of distinct dominating sets possible for G^* is twelve. It is easy to see that none of these eight dominating sets can be a total dominating set. This is because, the u_{22} is not adjacent with any of the elements of any of these twelve dominating sets. Hence we infer that a total dominating set of G^* must have at least seven elements.

Now by the definition of a total dominating set, we infer that every element in the total dominating set S must be adjacent with at least one element of the S. The presence of u_{15} as an indispensable element in the construction of a total dominating set reveals the fact that the sixth element must be either u_{11} or u_{12}. Hence the all possible total dominating sets of G^* are $T_1 = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}, u_{22}\}$, $T_2 = \{u_1, u_2, u_{11}, u_{15}, u_{17}, u_{19}, u_{22}\}$, $T_3 = \{u_1, u_2, u_{12}, u_{13}, u_{15}, u_{17}, u_{22}\}$, $T_4 = \{u_1, u_2, u_{12}, u_{15}, u_{17}, u_{22}\}$. Let G be a graph and S a subset of $V(G)$. For $v \in S$, the total private neighbourhood of v with respect to S in G, denoted $TPr[v, S]$ is defined as: $TPr[v, S] = \{w \in V(G) : N[w] \cap S = \{v\}\}$. Note that, 1) if $w \in V(G) - \{v\}$ and w is adjacent to only $v \in S$, then $w \in TPr[v, S]$. 2) if $w = v \in S$, then $w \notin TPr[v, S]$. We claim that a total dominating set S in G is a minimal total dominating set if and only if for every vertex $v \in S$, $TPr[v, S] \neq \phi$. Let S be a minimal total dominating set in G and $v \in S$ be any arbitrary vertex. So there exists a $w \in V(G)$ such that w is not adjacent to any vertex $S - v$. If $w = v$, then w is not adjacent to any vertex of S and in which case, S will turn out to be a non total dominating set, a contradiction. Now let $w \neq v \in V(G)$. As S is a total dominating set and w is adjacent to only v in S, we see that $w \in TPr[v, S]$. That is, $TPr[v, S] \neq \phi$ for any $v \in S$. Conversely, suppose that S is a total dominating set in G for any vertex $v \in S$, $TPr[v, S] \neq \phi$. Let $S_1 = S - \{v\}$ and $w \in TPr[v, S]$. Then $w \neq v$ is adjacent to only v in S. Also w is not adjacent to any vertex of S. That is, S_1 is not a total dominating set in G. This is true, of course, for any vertex v of S. Hence S is a minimal total dominating set in G. Clearly $S = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}, u_{22}\}$ is a minimum total dominating set. This is because (u_i, u_1) for $i = 3, \ldots, 10, 21$; (u_{12}, u_{15}); (u_{14}, u_2); (u_{16}, u_{17}); (u_i, u_{13}) for $i = 18, 19, 20$; (u_i, u_{22}) for $i = 21, 23$; (u_1, u_2); (u_2, u_{17}); (u_{11}, u_{15}); (u_{13}, u_{17}); (u_{17}, u_{22}) are all edges of G^* or the subgraph induced by S, namely $\langle S \rangle$, has no isolates. Further note that u_4 is adjacent to only $u_1 \in S$; u_{13} is adjacent to $u_2 \in S$; u_{15} is adjacent to only $u_{11} \in S$; u_{19} is adjacent to only $u_{13} \in S$; u_{12} is adjacent to only $u_{15} \in S$; u_{16} is adjacent to only $u_{17} \in S$, u_{23} is adjacent to only $u_{22} \in S$. This shows that $TPr[u_i, S] \neq \phi$ for all $u_i \in S$. For the same reason we infer that $S_1 = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}, u_{22}\}$ is a minimal total dominating set in G^*. Therefore we infer that G^* has no minimum total dominating set. Hence we conclude that $\gamma_T(G^*) = |S| = 7$. □

Note 7.4. *In Theorem 7.1, we have seen two dominating sets. In the course of the proof of Theorem 7.3, we have decisively found all the dominating sets of G^*.*

Note 7.5. *We have found in the course of the proof of Theorem 7.3 all the minimal total dominating sets of G^*. They are $T_1 = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}, u_{22}\}$, $T_2 = \{u_1, u_2, u_{11}, u_{15}, u_{17}, u_{19}, u_{22}\}$, $T_3 = \{u_1, u_2, u_{12}, u_{13}, u_{15}, u_{17}, u_{22}\}$, $T_4 = \{u_1,\ u_2,\ u_{12},\ u_{15}, u_{17}, u_{19}, u_{22}\}$.*

Let us present here an algorithm to find a dominating set of any graph G.

Algorithm

(1) Pick a vertex $u \in V(G)$ and color it A.

(2) Color all uncolored neighbours of all vertices with color A with the color B.

(3) Color all uncolored neighbours of all vertices with color B with the color A.

(4) If there are uncolored vertices, go to step 2.

(5) Let $S_A = \{u \in V(G) : color(u) = A\}$ and $S_B = \{u \in V(G) : color(u) = B\}$.

If $|S_A| > |S_B|$ then $S = S_B$; else $S = S_A$. S_A is the set of vertices with color A, S_B is the set of vertices with color B. The resulting dominating set is not necessarily minimal but $|S| \leq n/2$, where $n = V(G)$. We use the above algorithm and construct a dominating set of G^*. Pick u_1 and let $col(u_1) = A$ where $c : V(G^*) \to \{A, B\}$ is a color function. So initially $S_A = \{u_1\}$. Assign all the neighbours of u_1 with color B. As $N(u_1) = \{u_i : 2 \leq i \leq 10, 21\}$, we get $S_B = \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{21}\}$. Now color all the uncolored neighbours of all the elements of S_B with the color A. Then the initial S_A gets revised to $S_A = \{u_1, u_{11}, u_{12}, u_{13}, u_{14}, u_{16}, u_{17}, u_{18}, u_{19}, u_{23}\}$. Now color all the uncolored neighbours of all the elements of S_A with the color B. This gives revised S_B with $S_B = \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{15}, u_{20}, u_{21}, u_{22}\}$. As $|S_B| > |S_A|$, we get $S = S_A$ and it is a dominating set. Also note that $|S_A| = 10 < 11.5$ $(= 23/2)$.

We observe that the above algorithm is not an efficient one to produce a minimal dominating set. However, for networks of enormous size, this algorithm proves useful in producing an initial dominating set, which can be pruned later into a minimal one by employing other heuristic or greedy approaches.

8. Global Domination and Total Global Domination

A dominating set S of G is a global dominating set if S is also a dominating set of G^C. The global domination number $\gamma_g(G)$ of G is the minimum cardinality of a global dominating set. A global dominating set S of G is a total global dominating set, if S is also a total dominating set of G^C. The total global domination number $\gamma_{tg}(G)$ is the minimum cardinality of a total global dominating set. Note that $\gamma(G)$ and $\gamma_g(G)$ are defined for any graph G, while $\gamma_t(G)$ is defined only for those G with $\delta(G) \geq 1$ and $\gamma_{tg}(G)$ is defined only for those G with $\delta(G) \geq 1$ and $\delta(G^C) \geq 1$.

Theorem 8.1. $\gamma_g(G^*) = 6$

Proof. In the course of the proof of Theorem 7.3, we have found all the dominating sets of G^*. They are $S_1 = \{u_1, u_2, u_6, u_{13}, u_{15}, u_{22}\}$, $S_2 = \{u_1, u_2, u_6, u_{15}, u_{19}, u_{22}\}$, $S_3 = \{u_1, u_2, u_6, u_{15}, u_{20}, u_{22}\}$, $S_4 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{22}\}$, $S_5 = \{u_1, u_2, u_{15}, u_{17}, u_{19}, u_{22}\}$, $S_6 = \{u_1, u_2, u_{15}, u_{17}, u_{20}, u_{22}\}$, $S_7 = \{u_1, u_6, u_{13}, u_{14}, u_{15}, u_{22}\}$, $S_8 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}, u_{22}\}$, $S_9 = \{u_1, u_{13}, u_{14}, u_{15}, u_{17}, u_{22}\}$. As $(u_{13}, u_j) \in E(G^*)^C$ for $j = 3, 4, 6, 7, 9, 11, 21$; $(u_{14}, u_j) \in E(G^*)^C$ for $j = 8, 10, 13, 16, 17, 18, 19, 20, 23$; (u_2, u_{15}); (u_{12}, u_{16}) are edges in $(G^*)^C$; we find that S_1 is a global dominating set. Moreover, we find S_j is a global dominating set for $j = 1$ to 12. In view of this, we infer that there is no minimum global dominating set. Hence $\gamma_g(G^*) = 5$. $\square$

Theorem 8.2. $\gamma_{tg}(G^*) = 7$

Proof. In the course of the proof of Theorem 7.3, we have found all the total dominating sets of G^*. They are $T_1 = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}, u_{22}\}$, $T_2 = \{u_1, u_2, u_{11}, u_{15}, u_{17}, u_{19}, u_{22}\}$, $T_3 = \{u_1, u_2, u_{12}, u_{13}, u_{15}, u_{17}, u_{22}\}$, $T_4 = \{u_1, u_2, u_{12}, u_{15}, u_{17}, u_{19}, u_{22}\}$. Consider the total dominating set $T_1 = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}, u_{22}\}$ of G^*. We claim that T_1 is also a total global dominating set of G^*. As (u_1, u_i) for $i = 11, 13, 14, 17, 18, 22$; (u_2, u_i) for $i = 3, 4, 5, 6, 7, 9, 10, 12, 15, 16, 19, 20, 21, 23$; (u_8, u_{11}); are all edges of $(G^*)^C$, the claim follows. By Theorem 7.3, $\gamma_t(G^*) = 7$ with T_1 as the minimum total dominating set. Moreover, we find T_j is a global total dominating set for $j = 1$ to 4. In view of this, we infer that there is no minimum total global dominating set and $\gamma_{tg}(G^*) = 7$. $\square$

Theorem 8.3. *The global dominating set and total global dominating set of G^* are distinct with different cardinality.*

Proof. In [1], we proved that $diam(\overline{G^*}) = 3$. Kulli and Janakiram [14] have showed that if G and G^C both have no isolated vertices and $diam(G) \geq 5$ then a set $S \subseteq V(G)$ with $\delta(\langle S \rangle) \geq 1$ is a global dominating set if and only if S is a total global dominating set. Here both G^* and $(G^*)^C$ have no isolated vertices and $diam(G^*) = 3$. This implies that G^* has global dominating set distinct from its total global dominating set. This result was further confirmed by Theorem 8.1 and Theorem 8.2. □

Theorem 8.4. *It is not necessary that diameter of a graph should be at least 5 to have the same total dominating set and total global dominating set.*

Proof. Kulli and Janakiram [14] have showed that for a graph G with $diam(G) \geq 5$, a set $S \subseteq V(G)$ is a total dominating set iff S is a total global dominating set. But we have proved in Theorem 8.2, that $\gamma_{tg}(G^*) = 7$ and $\gamma_t(G^*) = 7$ with the same set $S = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}, u_{22}\}$ serving as a total global dominating set and a total dominating set. Also in [1], we have proved that $diam(G^*) = 3$. Hence the result follows. □

Observation 8.5. *Kulli and Janakiram* [14] *have proved: A total dominating set S of G is a total global dominating set iff for each $v \in V$ there exists a vertex $u \in S_1 = \{u_1, u_2, u_{13}, u_{15}, u_6, u_{22}\}$ such that v is not adjacent to u. The graph G^* satisfies this result. This can be seen by noticing that for u_i, $i = 3, 4, 5, 6, 7, 11, 12, 15, 16$ there exists a vertex $u_2 \in S =$ and for u_i, $i = 1, 2, 8, 9, 10, 13, 14, 17, 18, 19, 20$ there exists a vertex $u_{11} \in S$ such that u_i are not adjacent to u_2, u_{11} for respective i's. Also we have established by Theorem 7.3 and 8.2 above that S is the same for both total domination and total global domination.*

Observation 8.6. *Kulli and Janakiram* [14] *have proved: If G is a graph such that neither itself nor its complement has an isolated vertex, then $\gamma_{tg}(G) = |V(G)|$ iff G is isomorphic to one of P_4, mK_2, $(mK_2)^C$, $m \geq 2$. As G^* is different from these graphs we conclude that $\gamma_{tg}(G^*) \neq |V(G^*)| = 23$. Moreover we have proved in Theorem 8.2 that $\gamma_{tg}(G^*) = 7$.*

9. Connected Domination

Sampath Kumar and Walikar [20] defined a connected dominating set S to be a dominating set S whose induced subgraph is connected. Since a dominating set of a graph G must contain at least one vertex from each component of G, it follows that only connected graphs have connected dominating set. A connected dominating set S is said to be a minimal connected dominating set of G if and only if for any $v \in S$, $S - v$ is not a connected dominating set.

Theorem 9.1. $\gamma_C(G^*) = 9$

Proof. Let S be a connected dominating set in G and $v \in S$. Then note that 1) if $w \in V(G) - S$ and w is adjacent to only $v \in S$, then $w \in Pr[v, S]$ 2) if $w \in S$ and S contains at least two vertices, then $w \notin Pr[v, S]$, 3) if $S = \{v\}$, then $v \in Pr[v, S]$.

First we claim that a connected dominating set S in G is a minimal connected dominating set if and only if at least one of the following conditions are satisfied by every vertex $v \in S$. (i) $Pr[v, S] \neq \phi$, (ii) v is a cut vertex of S. Let S be a minimal connected dominating set in G and $v \in S$. Then, $S - v$ is not a connected dominating set. So, either the subgraph induced by $S - v$ is not connected or $S - v$ is not a dominating set.

Case-1 The subgraph induced by $S - v$ is connected. In this case, v is a cut-vertex of S.

Case-2 $S - v$ is not a dominating set in G. In this case, there exists a vertex $w \in V(G) - \{S - v\}$ such that w is not adjacent to any vertex of $S - v$. That is, w is adjacent to only one vertex of S. If $w = v$, then w is not adjacent to every vertex of S and therefore the subgraph induced by S is not connected, a contradiction. So, $w \neq v$ and w is adjacent to only $v \in S$. This means $w \in Pr[v, S]$. That is, $Pr[v, S] \neq \phi$. Conversely, suppose that S is a connected dominating set in G and at least one of the following conditions are satisfied by every vertex $v \in S$, viz., (i) $Pr[v, S] \neq \phi$, (ii) v is a cut vertex of S. Now two cases arise.

Case-1 v is not a cut vertex of S. Then the subgraph induced by $S - v$ is connected. That is, S contains at least two vertices. Let $w \in Pr[v, S]$. Then $w \notin S$ and w is adjacent to only v in S. Let $S_1 = S - v$. Then S_1 is not a dominating set in G. Thus, if a vertex $v \in S$ is not a cut-vertex of S, then $S - v$ is not a connected dominating set.

Case-2 v is not a cut vertex of S and the subgraph induced by $S - v$ is not connected. In this case, $S - v$ is not a connected dominating set. To sum up, in all the above cases, we have that for any $v \in V(G)$, $S - v$ is not a connected dominating set. Hence S is a minimal connected dominating set in G.

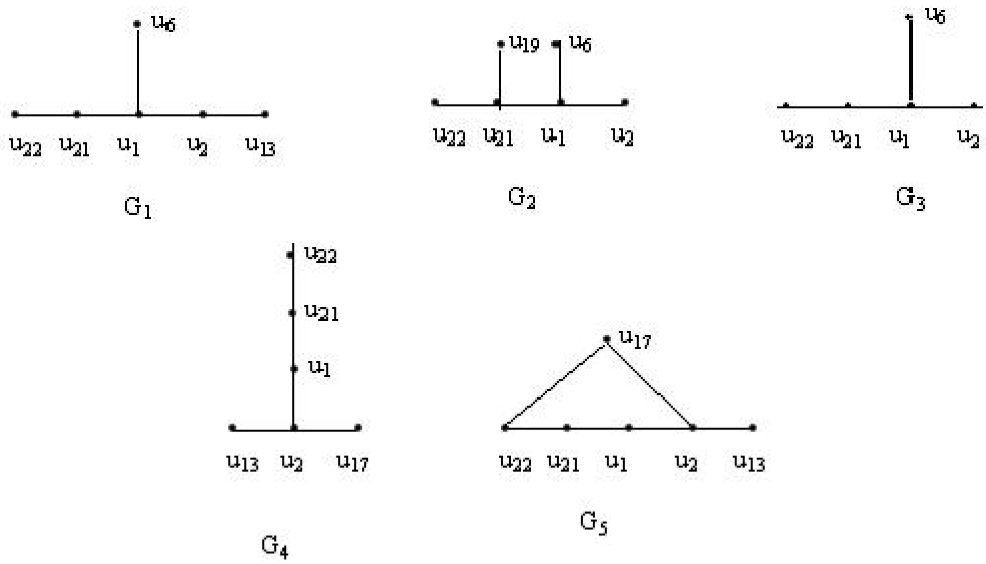


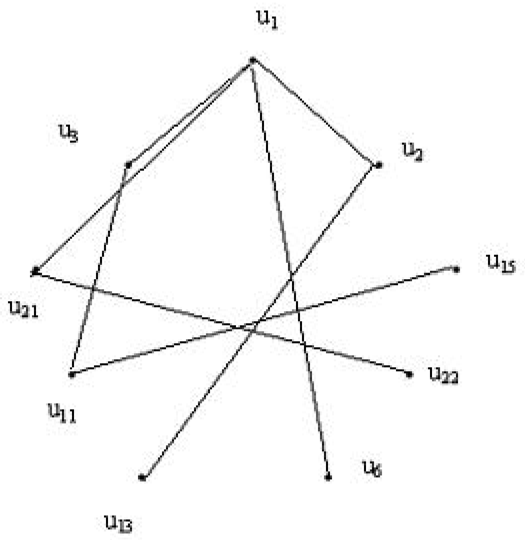
FIGURE 3

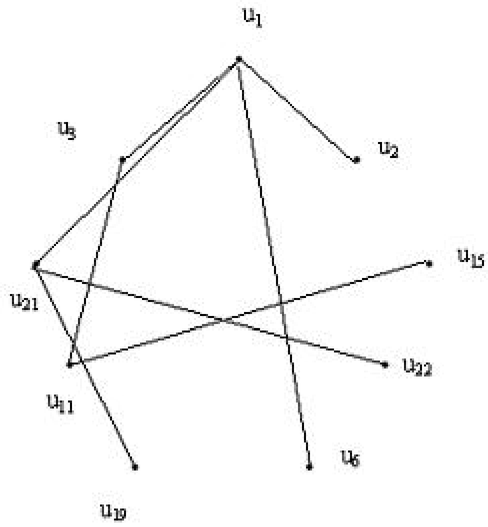
By Theorem 7.1 and Theorem 7.3 it follows that the dominating sets of G^* are $S_1 = \{u_1, u_2, u_6, u_{13}, u_{15}, u_{22}\}$, $S_2 = \{u_1, u_2, u_6, u_{15}, u_{19}, u_{22}\}$, $S_3 = \{u_1, u_2, u_6, u_{15}, u_{20}, u_{22}\}$, $S_4 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{22}\}$, $S_5 = \{u_1, u_2, u_{15}, u_{17}, u_{19}, u_{22}\}$, $S_6 = \{u_1, u_2, u_{15}, u_{17}, u_{20}, u_{22}\}$, $S_7 = \{u_1, u_6, u_{13}, u_{14}, u_{15}, u_{22}\}$, $S_8 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}, u_{22}\}$, $S_9 = \{u_1, u_{13}, u_{14}, u_{15}, u_{17}, u_{22}\}$. First we establish a connected dominating set must contain at least 9 elements. The presence of u_{15} and u_{22} in each S_j, for $j = 1$ to 9 indicates that u_{11} or u_{12} and u_{21} must be there in every S_j, for $j = 1$ to 9. This means $T_j = S_j \cup \{u_{11}, u_{21}\}$ or $W_j = S_j \cup \{u_{12}, u_{21}\}$ and in both cases $|T_j| = |W_j| = 8$. We observe that S_j at this stage does not induce a connected graph for any j. This is because, the subgraphs induced by T_j for $j = 1$ to 9 are: $G_1 \cup K_2$, $G_2 \cup K_2$, $G_3 \cup K_1 \cup K_2$, $G_4 \cup K_2$, $P_6 \cup K_2$ (where $P_6 = u_{22}u_{21}u_1u_2u_{17}u_{19}$), $G_5 \cup K_2$, $P_4 \cup 2K1 \cup K_2$ (where $P_4 = u_{22}u_{21}u_1u_6$ and $K_1 = u_{14}$, $K_1 = u_{13}$), $P_3 \cup 3K_1 \cup K_2$ (where

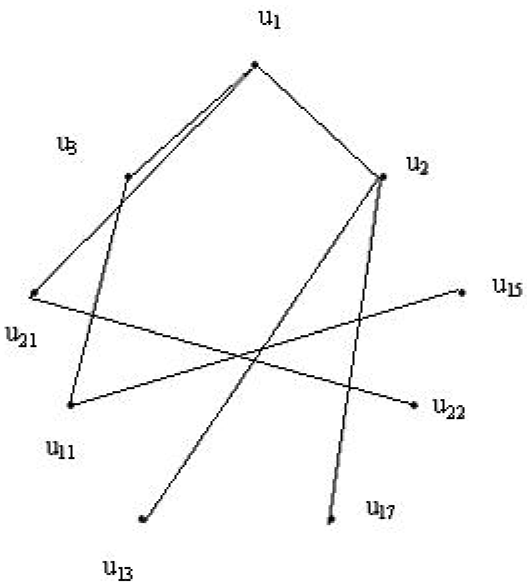
$P_3 = u_{22}u_{21}u_1$ and $K_1 = u_{13}$, $K_1 = u_{16}$, $K_1 = u_{14}$), $P_3 \cup 2K_2 \cup K_1$ (where $P_3 = u_{22}u_{21}u_1$ and $K_1 = u_{14}$, $K_2 = u_1u_{15}$, $K_2 = u_{13}u_{17}$) where $G_1, G_2, \ldots$ are shown in Figure 3.

Where the K_2 in $\langle T_j \rangle$ for $j = 1$ to 8 is (u_{11}, u_{15}). Further, we also observe on similar lines that the subgraphs induced by W_j for $j = 1$ to 9 are almost the same as one induced by T_j except for the only change where instead of u_{11}, we have now u_{12}. In view of the above discussion we deduce that a connected dominating set of G^* must contain at least 9 elements. Now consider the subgraphs H_i induced by $U_j = T_j \cup \{u_3\}$ or $V_j = W_j \cup \{u_4\}$ for $j = 1, 2, 4, 5$ as shown in Figure 4, for $i = 1$ to 4 and obviously H_i for every I are connected.

Next consider T_j and W_j for $j = 3, 6, 7, 8, 9$. Clearly for the nineth element of both T_j and W_j, there are $\binom{15}{2}$ choices each. Suppose that $U_3 = T_3 \cup \{u_m\}$ for $m = 3$ to $10, 12$ to $14, 17$ to $19, 23$. Consider $\langle u_3 \rangle$. The presence of an isolated vertex u_{20} for all values of m except 13, 19 indicates that $\langle u_3 \rangle$ is not connected. Again for $m = 13, 19$ the presence of an isolated edge (u_{11}, u_{15}) indicates that $\kappa(\langle u_3 \rangle) \geq 2$. Next suppose that $U_6 = T_6 \cup \{u_m\}$ for $m = 3$ to 10, 12 to 14, 16, 18, 19, 23. For the same reason as above, we observe that $\kappa(\langle u_6 \rangle) \geq 2$. Next suppose that $U_7 = T_7 \cup \{u_m\}$ for $m = 2$ to 5, 7 to 10, 12, 16 to 20, 23. Here we infer that the presence of an isolated vertex u_{14} indicates that $\kappa(\langle u_7 \rangle) \geq 2$ for all values of m except 2. Again the presence of an isolated edge (u_{11}, u_{15}) yields that $\kappa(\langle u_7 \rangle) \geq 2$. Next suppose that $U_8 = T_8 \cup \{u_m\}$ for $m = 2$ to 10, 12, 17 to 20, 23. For the same reason as given for the case U_7 we derive that $\kappa(\langle u_8 \rangle) \geq 2$. Finally consider $U_9 = T_9 \cup \{u_m\}$ for $m = 2$ to 10, 12, 16, 18 to 20, 23. Here also by proceeding on similar lines as in the case U_7 we get $\kappa(\langle u_9 \rangle) \geq 2$. Further by repeating the above procedure for W for $j = 3$, 6 to 9 as in T_j, we observe that the $\kappa(\langle V_j \rangle) \geq 2$ where $V_j = W_j \cup \{u_m\}$ with u_m assuming any one of the $\binom{15}{1}$ appropriate choices for the respective j's. Hence the above analysis allows us to omit from our consideration the subgraphs induced by U_j and V_j for $j = 3$, 6 to 9. This is because we are interested in the computation of connected domination number which stands for the minimum cardinality of a connected dominating set.


u_1
u_3
u_2
u_{15}
u_{21}
u_{22}
u_{11}
u_6
u_{13}
H_1


u_1
u_3
u_2
u_{15}
u_{21}
u_{22}
u_{11}
u_6
u_{19}
H_2


u_1
u_3
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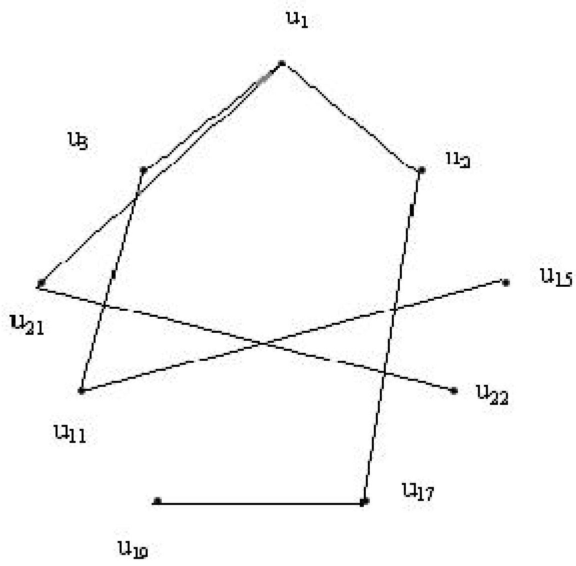
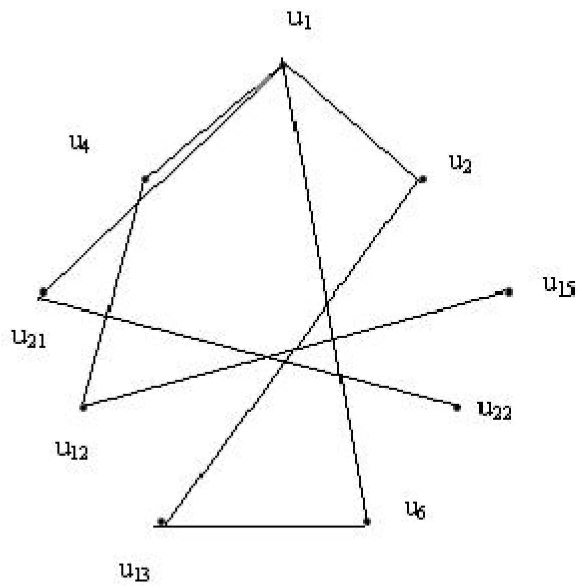
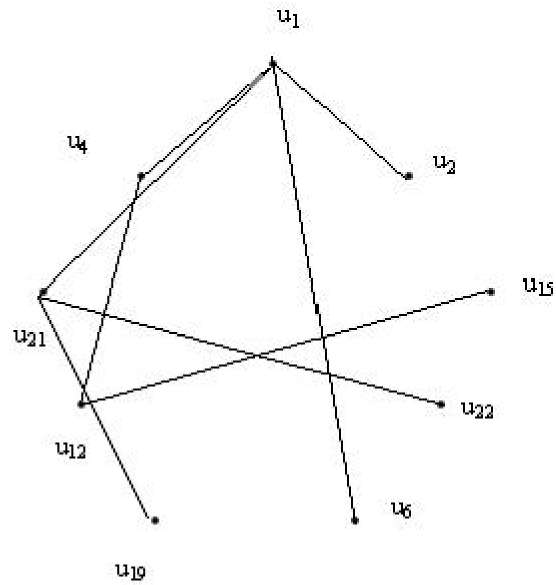
u_{17}
u_{13}
H_3

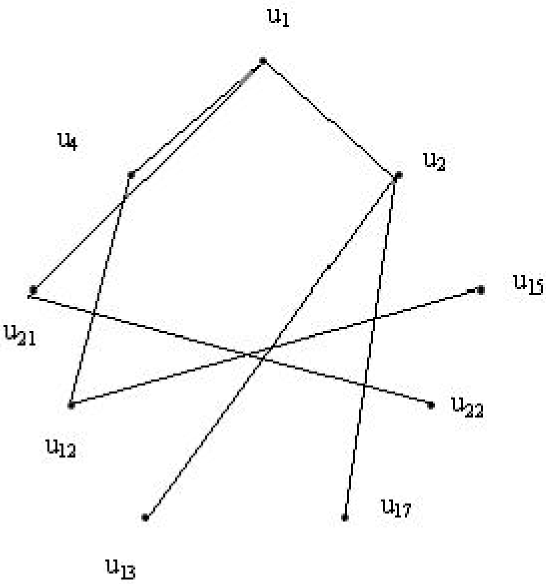

u_1
u_3
u_2
u_{15}
u_{21}
u_{22}
u_{11}
u_{17}
u_{19}
H_4


u_1
u_4
u_2
u_{15}
u_{21}
u_{22}
u_{12}
u_6
u_{13}

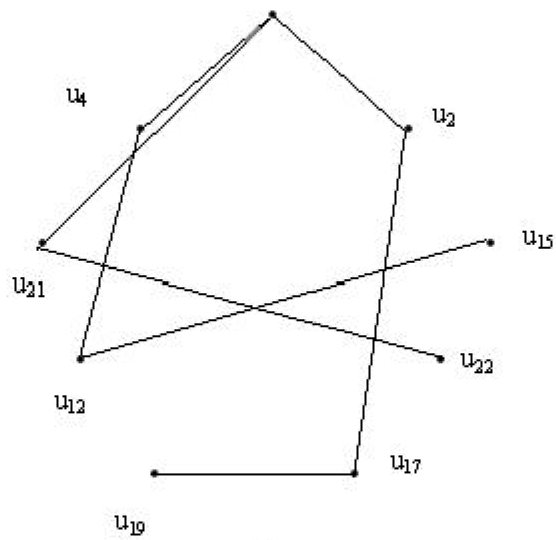
H_5


u_1
u_4
u_2
u_{15}
u_{21}
u_{22}
u_{12}
u_6
u_{19}

H_6


u_1
u_4
u_2
u_{15}
u_{21}
u_{22}
u_{12}
u_{17}
u_{13}

H_7


u_4
u_2
u_{15}
u_{21}
u_{22}
u_{12}
u_{17}
u_{19}

H_8

Now let us check how many of U_j and V_j for $j = 1, 2, 4, 5$ are minimal connected dominating sets. First consider U_1. Here u_i for $i = 1, 2, 3, 11, 21, 23$ is a vertex. Further $u_{12} \in Pr[u_{15}, U_1]$, $u_{16} \in Pr[u_6, U_1]$ and $u_{20} \in Pr[u_{13}, U_1]$. This shows that U_1 is a minimal connected dominating set. Next consider U_2. Here u_i. For $i = 1, 3, 11, 21$ is a cut vertex. Further $u_{18} \in Pr[u_2, U_2]$, $u_{12} \in Pr[u_{15}, U_2]$, $u_{23} \in Pr[u_{22}, U_2]$, $u_{16} \in Pr[u_6, U_2]$ and $u_{20} \in Pr[u_{19}, U_2]$. That is U_2 is a minimal connected dominating set. Next consider U_4. Here u_i for $i = 1, 2, 3, 11, 21$ is cut vertex. Further $u_{12} \in Pr[u_{15}, U_4]$, $u_{23} \in Pr[u_{22}, U_4]$, $u_{16} \in Pr[u_{17}, U_4]$ and $u_{20} \in Pr[u_{13}, U_4]$. So U_4 is a minimal connected dominating set. Next consider U_5. Here u_i for $i = 1, 2, 3, 11, 17, 21$ is cut vertex. Further $u_{12} \in Pr[u_{15}, U_5]$, $u_{23} \in Pr[u_{22}, U_5]$ and $u_{20} \in Pr[u_{19}, U_5]$. So U_5 is a minimal connected dominating set. Next consider V_1. Here u_i for $i = 1, 2, 4, 12, 21, 22$ is cut vertex. Further $u_{16} \in Pr[u_6, V_1]$, $u_{20} \in Pr[u_{13}, V_1]$ and $u_{11} \in Pr[u_{15}, V_1]$. So V_1 is a minimal connected dominating set. Next consider V_2. Here u_i for $i = 1, 4, 12, 21$ is cut vertex. Further $u_{18} \in Pr[u_2, V_2]$, $u_{11} \in Pr[u_{15}, V_2]$, $u_{23} \in Pr[u_{22}, V_2]$, $u_{16} \in Pr[u_6, V_2]$ and $u_{20} \in Pr[u_{19}, V_2]$. So V_2 is a minimal connected dominating set. Next consider V_4. Here u_i for $i = 1, 4, 12, 21$ is cut vertex. Further $u_{11} \in Pr[u_{15}, V_4]$, $u_{23} \in Pr[u_{22}, V_4]$, $u_{16} \in Pr[u_{17}, V_4]$ and $u_{20} \in Pr[u_{13}, V_4]$. So V_4 is a minimal connected dominating set. Finally consider V_5. Here u_i for $i = 1, 4, 12, 17, 21$ is cut vertex. Also $u_{11} \in Pr[u_{15}, V_5]$, $u_{23} \in Pr[u_{22}, V_5]$ and $u_{20} \in Pr[u_{19}, V_5]$. So V_5 is a minimal connected dominating set. We find here that all connected dominating sets are minimal. This implies that there exists no minimum connected dominating set. Hence $\gamma_C(G^*) = 9$. □

10. k-Domination

The concept of k-domination is stronger than the concept of domination. There are dominating sets which are not k-dominating for $k \geq 2$. Let G be a graph and k be a positive integer. A subset S of $V(G)$ is said to be a k-dominating set in the graph G if every vertex $v \in V(G) - S$ is adjacent to at least k vertices of S. A k-dominating set S in G is said to be minimal k-dominating set if for any $v \in S$, $S - v$ is not a k-dominating set. A k-dominating set in G with minimum cardinality is called a minimum k-dominating set in G. The minimum cardinality of a k-dominating set, denoted $\gamma_k(G)$ is called the k-dominating number. If S is a k-dominating set in G, $\gamma_k(G) \leq |S|$. If $k = 1$, then $\gamma_1(G) = \gamma(G)$. If S is a k-dominating set in G then it is also j-dominating set for $1 \leq j \leq k$, and $\gamma_j(G) = \gamma_k(G)$.

Theorem 10.1. $\gamma_2(G^*) = 12$

Proof. Let G be any graph and S a subset of $V(G)$. Let $v \in S$ and $k \geq 1$. The Private k-neighbourhood of v with respect to S, denoted $PR_k[v, S]$ is defined as: $PR_k[v, S] = \{w \in V(G) - S : w \text{ is adjacent to exactly } k \text{ vertices of } S \text{ including } v\} \cup \{v : v \text{ is adjacent to at most } k - 1 \text{ vertices of } S\}$. First we claim that a k-dominating set S in a graph G is a minimal k-dominating set if and only if $PR_k[v, S] \neq \phi$, $\forall\ v \in S$. Let S be a minimal k-dominating set in G. Let $v \in S$ be any arbitrary vertex. Then $S - v$ is not a k-dominating set in G. So there exists a vertex $w \in V(G) - (S - v)$ which is adjacent to at most $(k - 1)$ vertices of $S - v$. If $w = v$ and is adjacent to at most $(k - 1)$ vertices of S. Then $v \in PR_k[v, S]$ and $PR_k[v, S] \neq \phi$. If $w \neq v \in V(G) - (S - v)$ then as S is k-dominating and w is adjacent to at most $(k - 1)$ vertices of $S - v$, w must be adjacent to v. This means, w is adjacent to exactly k vertices of S and so $w \in PR_k[v, S]$ and and $PR_k[v, S] \neq \phi$. Conversely, suppose that $PR_k[v, S] \neq \phi$ for every $v \in S$. Let $v \in S$ be any arbitrary vertex and $w \in PR_k[v, S]$. If $w = v$ and w is adjacent to at most $(k - 1)$ vertices of S. Then w is adjacent to at most $(k - 1)$ vertices of $S - v$. That is, $S - v$ is not a k-dominating set in G. If $w \neq v$ then w is adjacent to exactly k vertices of S including v. That is, w is adjacent to exactly $(k - 1)$ vertices of $S - v$ and $S - v$ is not k-dominating set in G. This means S is a minimal k-dominating set in G.

Now let us contruct a minimal 2-connected dominating set S. First we allow all vertices of degree one as the element of S for obvious reasons. Therefore $S_1 = \{u_{14}, u_{23}\}$. Next let us include all vertices of degree two in S. Then S_1 gets refined to $S_2 = \{u_{14}, u_{23}, u_3, u_4, u_5, u_{11}, u_{12}, u_{15}, u_{20}\}$. Clearly S_2 is not a 2-connected dominating set as u_6 is not adjacent to any element of S_2. Now as u_{11} is adjacent to u_3, u_{15} and u_{12} is adjacent to u_4, u_{15}, we can conveniently drop u_{11} and u_{12} from S_2 to include other vital elements to produce a 2-connected dominating set. So S_2 gets refined to $S_3 = \{u_{14}, u_{23}, u_3, u_4, u_5, u_{15}, u_{20}, \}$. We find that the inclusion of u_2 and u_{22} are mandatory, else,

$Pr_2(u_{14}, S)$ and $Pr_2(u_{23}, S)$ will become empty by the defintion of a minimal 2-connected dominating set. So S_3 gets modified into $S_4 = \{u_{14}, u_{23}, u_3, u_4, u_5, u_{15}, u_{20}, u_2, u_{22}\}$. Now as u_6 to u_{10} and u_{21} are adjacent to u_1, the inclusion of u_1 will ensure at least one vertex of adjacency to these vertices in the proposed minimal 2-connected dominating set. So S_4 gets refined to $S_5 = \{u_1, u_2, u_3, u_4, u_5, u_{14}, u_{15}, u_{20}, u_{22}, u_{23}\}$. Similarly as u_{17} is adjacent to u_6 to u_9, u_{13}, u_{16} and u_{19}. For the same reason as above, we allow u_{17} into our proposed set. Hence S_5 gets modified into $S_6 = \{u_1, u_2, u_3, u_4, u_5, u_{14}, u_{15}, u_{17}, u_{20}, u_{22}, u_{23}\}$. But S_6 is still not a 2-connected dominating set. This is because, u_{10} is adjacent to only one vertex in S_6. Again as u_{13} is adjacent to u_{10}, u_{18} and u_{20}, we allow u_{13} into the proposed set. So S_6 gets modified into $S = \{u_1, u_2, u_3, u_4, u_5, u_{13}, u_{14}, u_{15}, u_{17}, u_{20}, u_{22}, u_{23}\}$. Now as u_6 to u_9 are adjacent to u_1 and u_{17}, u_{10} is adjacent to u_1 and u_{13}, u_{11} is adjacent to u_3 and u_{15}, u_{12} is adjacent to u_4 and u_{15}, u_{16} is adjacent to u_5 and u_{17}, u_{18} is adjacent to u_2 and u_{13}, u_{19} is adjacent to u_{13} and u_{17}, u_{21} is adjacent to u_1 and u_{22} we see that S is a 2-connected dominating set.

Moreover we find that $u_{10} \in Pr_2[u_1, S]$, $u_{18} \in Pr_2[u_2, S]$, $u_{11} \in Pr_2[u_3, S]$, $u_{12} \in Pr_2[u_4, S]$, $u_{16} \in Pr_2[u_5, S]$, $u_{10} \in Pr_2[u_{13}, S]$, $u_{11} \in Pr_2[u_{15}, S]$, $u_7 \in Pr_2[u_{17}, S]$, $u_{21} \in Pr_2[u_{22}, S]$, u_{14} is adjacent to only u_2 in S, u_{20} is adjacent to only u_{13} in S, u_{23} is adjacent to only u_{22} in S we see that $Pr_2[u_j, S] \neq \phi$ for all $j = 1$ to $5, 13$ to $15, 17, 20, 22$ and 23. Hence we deduce that S is a minimal 2-connected dominating set.

Interestingly we find another minimal 2-connected dominating set S' by just adding u_{16} to S and dropping u_5 from S. That is $S' = \{u_1, u_2, u_3, u_4, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{20}, u_{22}, u_{23}\}$ is a minimal 2-connected dominating set as $u_5 \in Pr_2[u_{16}, S']$. In view of this we conclude that there exists no minimum or a unique 2-connected dominating set with twelve elements. Hence $\gamma_2(G^*) = 12$. □

Note 10.2. *It is quite cute to note the existence of one another 2-connected dominating set with 12 elements but fails to be a minimal 2-connected dominating set. That is* $S'' = \{u_1, u_3, u_4, u_{10}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18}, u_{20}, u_{21}, u_{23}\}$ *a 2-connected dominating set. But* $Pr_2[u_{14}, S''] = \phi$ *and* $Pr_2[u_{23}, S''] = \phi$.

Note 10.3. *It is easy to see from the structure of G^* that there exists no k-dominating set for $k \geq 3$ as there are a number of vertices (7 to be exact) with maximum degree equal to 2.*

Note 10.4. *In [23] we have found the vertex independence number β_0 of G^*. $\beta_0 = 10$ and $I = \{u_2, u_3, u_4, u_5, u_6, u_7, u_{10}, u_{15}, u_{19}, u\}$ is an independent set. It is interesting to observe that an independent set need not be a 2-dominating set. That is, I is not 2-dominating as u_{14} is adjacent to only u_2 and out of the two adjacent elements u_{13}, u_{19} of u_{20} in G^* only $u_{19} \in I$.*

11. Strong Domination

For a graph $G = (V, E)$, a set $S \subseteq V$ is a strong dominating set if every vertex $v \in V - S$ has a neighbour u in S such that the degree of u is not smaller than the degree of v. The minimum cardinality of a strong dominating set of G is the strong domination number, $\gamma_{strong}(G)$.

Theorem 11.1. $\gamma_{strong}(G^*) = 6$

Proof. We know from Theorem 7.1, 7.3 that a dominating set of G^* must have atleast six elements and the possible dominating sets are $S_1 = \{u_1, u_2, u_6, u_{13}, u_{15}, u_{22}\}$, $S_2 = \{u_1, u_2, u_6, u_{15}, u_{19}, u_{22}\}$, $S_3 = \{u_1, u_2, u_6, u_{15}, u_{20}, u_{22}\}$, $S_4 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{22}\}$, $S_5 = \{u_1, u_2, u_{15}, u_{17}, u_{19}, u_{22}\}$, $S_6 = \{u_1, u_2, u_{15}, u_{17}, u_{20}, u_{22}\}$, $S_7 = \{u_1, u_6, u_{13}, u_{14}, u_{15}, u_{22}\}$, $S_8 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}, u_{22}\}$, $S_9 = \{u_1, u_{13}, u_{14}, u_{15}, u_{17}, u_{22}\}$. Let us now determine how many of these are strong dominating sets. It turns out that S_j is a strong dominating set for $j = 1, 4, 5, 6, 9$ are strong dominating sets. This is because, for S_1: u_j is adjacent to u_1 with $deg(u_1) > deg(u_j)$; for $j = 3$ to 5, 7 to 10; u_j is adjacent to u_{15} for $j = 11, 12$; with $deg(u_{15}) > deg(u_j)$; u_{14} is adjacent to u_2 with $deg(u_2) > deg(u_{14})$; u_{16} is adjacent to u_6 with $deg(u_6) > deg(u_{16})$; u_j is adjacent to u_{13} for $j = 17$ to 20 with $deg(u_{13}) > deg(u_j)$; u_{21} is adjacent to u_1 with $deg(u_1) > deg(u_{21})$; u_{23} is adjacent to u_{22} with $deg(u_{22}) > deg(u_{23})$; For S_4, u_j is adjacent to u_1 with $deg(u_1) > deg(u_j)$; for $j = 3$ to 10; u_j is adjacent to u_{15} for

$j = 11, 12$; with $deg(u_{15}) \geq deg(u_j)$; u_{16} is adjacent to u_6 with $deg(u_6) > deg(u_{16})$; u_j is adjacent to u_{13} for $j = 17$ to 20 with $deg(u_{13}) > deg(u_j)$; u_{21} is adjacent to u_1 with $deg(u_1) > deg(u_{21})$; u_{23} is adjacent to u_{22} with $deg(u_{22}) > deg(u_{23})$; For S_5, u_j is adjacent to u_1 with $deg(u_1) > deg(u_j)$; for $j = 3$ to 10; u_j is adjacent to u_{15} for $j = 11, 12$; with $deg(u_{15}) \geq deg(u_j)$; u_{13} is adjacent to u_{17} with $deg(u_{17}) > deg(u_{13})$; u_{14} is adjacent to u_2 with $deg(u_2) > deg(u_{14})$; u_{16} is adjacent to u_{17} with $deg(u_{17}) > deg(u_{16})$; u_{18} is adjacent to u_2 with $deg(u_2) > deg(u_{18})$; u_{20} is adjacent to u_{19} with $deg(u_{19}) > deg(u_{20})$; u_{21} is adjacent to u_1 with $deg(u_1) > deg(u_{21})$; u_{23} is adjacent to u_{22} with $deg(u_{22}) > deg(u_{23})$; For S_6, u_j is adjacent to u_1 with $deg(u_1) > deg(u_j)$; for $j = 3$ to 10; u_j is adjacent to u_{15} for $j = 11, 12$; with $deg(u_{15}) \geq deg(u_j)$; u_{13} is adjacent to u_{17} with $deg(u_{17}) > deg(u_{13})$; u_{14} is adjacent to u_2 with $deg(u_2) > deg(u_{14})$; u_{16} is adjacent to u_{17} with $deg(u_{17}) > deg(u_{16})$; u_{18} is adjacent to u_2 with $deg(u_2) > deg(u_{18})$; u_{19} is adjacent to u_{17} with $deg(u_{17}) > deg(u_{19})$; u_{21} is adjacent to u_1 with $deg(u_1) > deg(u_{21})$; u_{23} is adjacent to u_{22} with $deg(u_{22}) > deg(u_{23})$; For S_9, u_j is adjacent to u_1 with $deg(u_1) > deg(u_j)$; for $j = 2$ to 10; u_j is adjacent to u_{15} for $j = 11, 12$; with $deg(u_{15}) \geq deg(u_j)$; u_{16} is adjacent to u_{17} with $deg(u_{17}) > deg(u_{16})$; u_{18} is adjacent to u_{13} with $deg(u_{13}) > deg(u_{18})$; u_{19} is adjacent to u_{17} with $deg(u_{17}) > deg(u_{19})$; u_{20} is adjacent to u_{13} with $deg(u_{13}) > deg(u_{20})$; u_{21} is adjacent to u_1 with $deg(u_1) > deg(u_{22})$; u_{23} is adjacent to u_{22} with $deg(u_{22}) > deg(u_{23})$; Now S_j is not a strong dominating set for $j = 2, 3, 7, 8$ because, the degree of u_{17} an element of S_j has degree more than all the elements of S_j. Further as there exists more than one strong dominating set, we conclude that G^* has no minimum strong dominating set. Hence $\gamma_{strong}(G^*) = 6$. □

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