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ANALYSIS OF SOLUTION OF FRACTIONAL SUMMATION-DIFFERENCE EQUATION OF FINITE DELAY IN CONE METRIC SPACE

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Abstract. In this paper, we will investigate the solution of the Summation-Fractional Difference equation of finite delay with nonlocal condition in cone metric space. The result is obtained by applying some extensions of Banach's contraction principle in complete cone metric space, and an example is provided to demonstrate the main result.

Keywords: fractional difference equation; summation equation; existence of solution; cone metric space; Banach contraction principle.

2020 AMS Subject Classification: 54E50, 39A05, 45N05, 34K05, 47G20, 47H10.

1. Introduction

The purpose of this paper is to look at how to solve the Summation Fractional Difference equation in cone metric space with nonlocal conditions.

$$\Delta^{\alpha}x(t) = A(t)x(t) + f(t,x(t),x(t-1)) + \sum_{s=0}^{t} k(s,x(s)), \qquad t \in J = [0,b] \qquad (1.1)$$

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$$x(t-1) = \psi(t) \quad t \in [0,1] \tag{1.2}$$

$$x(0) + g(x) = x_0, \tag{1.3}$$

where $A(t)$ is a bounded linear operator on a Banach space X with domain $D(A(t))$, the unknown $x(\cdot)$ takes values in the Banach space X; $f : J \times X \times X \to X$, $k : J \times X \to X$, $g : C(J,X) \to X$ are appropriate continuous functions and x_0 is given element of X.
$\psi(t)$ is a continuous function for $[0,1]$ $\lim_{t\to(1-0)} \psi(t)$ exists, for which we denote by $\psi(1-0) = c_0$. We note that, if $t \in [0,1]$, the problem is reduced to Summation Difference equation

$$\Delta^{\alpha} x(t) = A(t)x(t) + f(t,x(t),\psi(t)) + \sum_{s=0}^{t} k(s,x(s)), \qquad t \in J = [0,b],$$

with initial condition $x(0) + g(x) = x_0$. Here, it is essential to obtain the solutions of (1.1)-(1.3) for [0,b].
Many authors have investigated the difficulties of existence, uniqueness, continuation, and other features of solutions of the difference equation, including R. Agrawal [2], W. G. Kelley, and A. C. Peterson[8], who created the theory of difference equations and its inequalities. Several existence, uniqueness, and comparison results on the difference and fractional difference equation and summation equation which can be found in [4, 5, 6, 7, 9, 10, 11, 12]. For undefined terms reader can reffer [3, 8].

2. Definitions and Preliminaries

Let us review the cone metric space concepts, and we refer the reader to [1], for more information.

Definition 2.1. Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,
1.P is closed, nonempty and $P \neq \{0\}$.
2.$a,b \in \mathbb{R}, a,b \geq 0, x,y \in P \Rightarrow ax+by \in P$.
3.$x \in P$ and $-x \in P \Rightarrow x = 0$.
Let $\leq$ be the partial ordering relation defined on P as, $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$. If there exist a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for every $x,y \in E$ then P is called as

normal cone. We always assume E is a real Banach space, P is a cone in E with $intP \neq \phi$, and $\leq$ is partial ordering with respect to P in the following way.

Definition 2.2. Let X a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d_1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$

(d_2) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(d_3) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y \in X$.

Here d is called as a cone metric on$X$ and (X,d) is referred to as a cone metric space.

Definition 2.3. Let X be an ordered set. A function $\Phi : X \to X$ is said to a comparison function if every $x,y \in X, x \leq y$, implies that $\Phi(x) \leq \Phi(y), \Phi(x) \leq x$ and $\lim_{n\to\infty} \|\Phi^n(x)\| = 0$, for every $x \in X$.

3. Main Result

Let X is a Banach space with norm $\|.\|$. $B = C(J,X)$ be the Banach space of all continuous function from J into X endowed with supremum norm

$$\|x\|_\infty = sup\{\|x(t)\| : t \in [0,b]\}$$

Let $P = \{(x,y) : x,y \geq 0\} \subset E = \mathbb{R}^2$, and define

$$d(f,g) = (\|f-g\|_\infty, \alpha\|f-g\|_\infty)$$

for every $f,g \in B$, then it is easily seen that (B,d) is a cone metric space.

Definition 3.1.[1] The function $x \in B$ satisfies the summation equation

case I: If $t \in [0,1]$ then, $x(t)$ is defined as,

$$x(t) = x_0 - g(x) + \sum_{s=0}^{t} K^\alpha(s-t)A(s)\Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\Big], \tag{3.1}$$

case II: If $t \in [1,b]$ then, $x(t)$ is defined as

$$\begin{aligned} x(t) = x_0 - g(x) &+ \sum_{s=0}^{1} K^\alpha(s-t)A(s)\Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\Big] \\ &+ \sum_{s=1}^{t} K^\alpha(s-t)A(s)\Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\Big], \end{aligned} \tag{3.2}$$

is called the mild solution of the equation $(1.1)-(1.3)$.

We need the following lemma for further discussion:

Lemma 3.1. Let (X,d) be a complete cone metric space,where P is a normal cone with normal constant K. Let $f: X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$ such that

$$d(f(x), f(y)) \leq \Phi(d(x,y))$$

for very $x, y \in X$.Then f has unique fixed point.

We list the following hypothesis for our convenience:

(H_1) A(t) is a bounded linear operator on X for each $t \in J$, the function $t \to A(t)$ is continuous in the uniform operator topology and hence there exists a constant P such that

$$P = \sup_{t \in J} \|A(t)\|.$$

(H_2) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a comparison function

(i)There exist continuous function $p_1, p_2 : J \to \mathbb{R}^+$ such that

case I: for $t \in [0,1]$

$$(\|f(t,x(t),\psi(t)) - f(t,y(t),\psi(t))\|, \alpha\|f(t,x(t),\psi(t)) - f(t,y(t),\psi(t))\|)$$
$$\leq p_1(t)\Phi(d(x,y));$$

case II: for $t \in [1,b]$

$$(\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|, \alpha\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|)$$
$$\leq p_2(t)\Phi(d(x,y)),$$

for every $t \in J$ and $x, y \in X$.

(ii)There exist continuous function $q : J \to \mathbb{R}^+$ such that

$$(\|k(t,x) - k(t,y)\|, \alpha\|k(t,x) - k(t,y)\|) \leq q(t)\Phi(d(x,y)),$$

for every $t \in J$ and $x, y \in X$.

(iii) There exists a positive constant G such that

$$(\|g(x) - g(y)\|, \alpha\|g(x) - g(y)\|) \leq G\Phi(d(x,y)),$$

for every $x, y \in X$.

$$(H_3) \qquad \sup_{t\in J}\left\{G + K^{\alpha}(2b)P\sum_{s=0}^{t}\Big[p_1(s)+p_2(s)+\sum_{\tau=0}^{s}q(\tau)\Big]\right\} = 1.$$

The following theorem encapsulates our main findings:

Theorem 3.1 If hypotheses $(H_1)-(H_3)$ hold then the evoluation equation (1.1)-(1.2) has a unique solution x on J.

Proof: The operartor $F : B \to B$ is defined by

case I: for $t \in [0,1]$

$$Fx(t) = x_0 - g(x) + \sum_{s=0}^{t} K^{\alpha}(s-t)A(s)\Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\Big], \tag{3.3}$$

case II: for $t \in [1,b]$

$$\begin{aligned} Fx(t) = x_0 - g(x) &+ \sum_{s=0}^{1} K^{\alpha}(s-t)A(s)\Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\Big] \\ &+ \sum_{s=1}^{t} K^{\alpha}(s-t)A(s)\Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\Big], \end{aligned} \tag{3.4}$$

By using the hypothesis $(H_1)-(H_3)$, we have

case I: for $t \in [0,1]$

$$\begin{aligned}
&(\|Fx(t)-Fy(t)\|, \alpha\|Fx(t)-Fy(t)\|) \\
&\le \Big(\|g(x)-g(y)\| \\
&\quad + \sum_{s=0}^{t}\|K^{\alpha}(s-t)\|\|A(s)\|\Big[\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\| + \sum_{\tau=0}^{s}\|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big], \\
&\quad \alpha\|g(x)-g(y)\| \\
&\quad + \alpha\sum_{s=0}^{t}\|K^{\alpha}(s-t)\|\|A(s)\|\Big[\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\| + \sum_{\tau=0}^{s}\|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]\Big) \\
&\le (\|g(x)-g(y)\|, \alpha\|g(x)-g(y)\|) \\
&\quad + \sum_{s=0}^{t} K^{\alpha}(2b)P(\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|, \alpha\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|) \\
&\quad + \sum_{s=0}^{t} K^{\alpha}(2b)P\sum_{\tau=0}^{s}(\|k(\tau,x(\tau))-k(\tau,y(\tau))\|, \alpha\|k(\tau,x(\tau))-k(\tau,y(\tau))\|) \\
&\le G\Phi(\|x-y\|, \alpha\|x-y\|) + \sum_{s=0}^{t} K^{\alpha}(2b)Pp_1(s)\Phi(\|x(s)-y(s)\|, \alpha\|x(s)-y(s)\|)
\end{aligned}$$

$$+\sum_{s=0}^{t} K^{\alpha}(2b)P\sum_{\tau=0}^{s} q(\tau)\Phi(\|x(\tau)-y(\tau))\|,\alpha\|x(\tau)-y(\tau))\|)$$

$$\leq G\Phi(\|x-y\|_{\infty},\alpha\|x-y\|_{\infty})+\Phi(\|x-y\|_{\infty},\alpha\|x-y\|_{\infty})\sum_{s=0}^{t} K^{\alpha}(2b)P\Big[p_1(s)+\sum_{\tau=0}^{s} q(\tau)\Big]$$

$$\leq G\Phi(d(x,y))+\Phi(d(x,y))\sum_{s=0}^{t} K^{\alpha}(2b)P\Big[p_1(s)+\sum_{\tau=0}^{s} q(\tau)\Big]$$

$$\leq \Phi(d(x,y))\Bigg\{G+\sum_{s=0}^{t} K^{\alpha}(2b)P\Big[p_1(s)+\sum_{\tau=0}^{s} q(\tau)\Big]\Bigg\}$$

$$\leq \Phi(d(x,y))\Bigg\{G+\sum_{s=0}^{t} K^{\alpha}(2b)P\Big[p_1(s)+p_2(s)+\sum_{\tau=0}^{s} q(\tau)\Big]\Bigg\}$$

$$\leq \Phi(d(x,y)) \tag{3.5}$$

case II: for $t\in[1,b]$

$$(\|Fx(t)-Fy(t)\|,\alpha\|Fx(t)-Fy(t)\|)$$

$$\leq \Big(\|g(x)-g(y)\|$$

$$+\sum_{s=0}^{1}\|K^{\alpha}(s-t)\|\|A(s)\|\Big[\|f(s,x(s),\psi(s))-f(s,y(s),\psi(s))\|+\sum_{\tau=0}^{s}\|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]$$

$$+\sum_{s=1}^{t}\|K^{\alpha}(s-t)\|\|A(s)\|\Big[\|f(s,x(s),x(s-1))-f(s,y(s),y(s-1))\|+\sum_{\tau=0}^{s}\|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]$$

$$\alpha\|g(x)-g(y)\|$$

$$+\alpha\sum_{s=0}^{1}\|K^{\alpha}(s-t)\|\|A(s)\|\Big[\|f(s,x(s),\psi(s))-f(s,y(s),\psi(s))\|+\sum_{\tau=0}^{s}\|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]$$

$$+\alpha\sum_{s=1}^{t}\|K^{\alpha}(s-t)\|\|A(s)\|\Big[\|f(s,x(s),x(s-1))-f(s,y(s),y(s-1))\|+\sum_{\tau=0}^{s}\|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]\Big)$$

$$\leq G\Phi(d(x,y))+\sum_{s=0}^{1} K^{\alpha}(2b)P\Big[p_1(s)\Phi(d(x,y))+\sum_{\tau=0}^{s} q(\tau)\Phi(d(x,y))\Big]$$

$$+\textstyle\sum_{s=1}^{t} K^{\alpha}(2b)P\Big[p_2(s)\Phi(d(x,y))+\sum_{\tau=0}^{s} q(\tau)\Phi(d(x,y))\Big]$$

$$\leq G\Phi(d(x,y))+\sum_{s=0}^{1} K^{\alpha}(2b)P\Big[(p_1(s)+p_2(s))\Phi(d(x,y))+\sum_{\tau=0}^{s} q(\tau)\Phi(d(x,y))\Big]$$

$$+\sum_{s=1}^{t} K^{\alpha}(2b)P\Big[(p_1(s)+p_2(s))\Phi(d(x,y))+\sum_{\tau=0}^{s} q(\tau)\Phi(d(x,y))\Big]$$

$$\leq \Phi(d(x,y))\Bigg\{G+\sum_{s=0}^{t} K^{\alpha}(2b)P\Big[p_1(s)+p_2(s)+\sum_{\tau=0}^{s} q(\tau)\Big]\Bigg\}$$

$$\leq \Phi(d(x,y));\ \forall\, x,\ y\in B. \tag{3.6}$$

This implies that $d(Fx,Fy) \leq \Phi(d(x,y)), \forall\, x,\ y \in B$. Now by using Lemma (3.1), the operator has a unique point in B and hence the equation (1.1)-(1.2) has unique solution.

4. Application

We use an example in this section to show how useful the result from the previous section is. Take a look at the evolution equation below.

$$\Delta x(t) = \frac{35}{44}e^{-t}x(t) + f(t,x(t),x(t-1)) + \sum_{s=0}^{t}\frac{sx(s)}{20},\ t \in J = [0,2],\ x \in X \tag{4.1}$$

$$x(0) + \frac{x}{8+x} = x_0, \tag{4.2}$$

Where,

$$f(t,x(t),x(t-1)) = \frac{te^{-t}x(t)}{(9+e^t)(1+x(t))}, \qquad t \in [0,1]$$

$$f(t,x(t),x(t-1)) = \frac{2te^{-(t-1)}x(t-1)}{(9+e^{t-1})(1+x(t-1))}, \quad t \in [1,2].$$

Therefore, we have

$$A(t) = \tfrac{35}{38}e^{-t},\ \ t \in J,$$

$$f(x,x(t),\psi(t)) = \tfrac{te^{-t}x(t)}{(9+e^t)(1+x(t))},\ \ (t,x) \in J \times X,$$

$$f(t,x(t),x(t-1)) = \tfrac{2te^{-(t-1)}x(t-1)}{(9+e^{t-1})(1+x(t-1))},\ \ (t,x) \in J \times X,$$

$$k(t,x(t)) = \tfrac{tx(t)}{20},\ \ (t,x) \in J \times X,$$

$$g(x) = \tfrac{x}{8+x},\ \ x \in X.$$

Now for $x,y \in C(J,X)$ and $t \in J$, we have

case I: for $t \in [0,1]$,

$$(\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|, \alpha\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|)$$

$$= \frac{te^{-t}}{9+e^t}(\|\frac{x(t)}{1+x(t)} - \frac{y(t)}{1+y(t)}\|, \alpha\|\frac{x(t)}{1+x(t)} - \frac{y(t)}{1+y(t)}\|)$$

$$= \frac{te^{-t}}{9+e^t}(\|\frac{x(t)-y(t)}{(1+x(t))(1+y(t))}\|, \alpha\|\frac{x(t)-y(t)}{(1+x(t))(1+y(t))}\|)$$

$$\leq \frac{te^{-t}}{9+e^t}(\|x(t)-y(t)\|, \alpha\|x(t)-y(t)\|)$$

$$\leq \frac{te^{-t}}{9+e^t}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$

$$\leq \frac{te^{-t}}{9+e^t}d(x,y).$$

$$\leq \frac{t}{10}\Phi(d(x,y)).$$

where $p_1(t) = \frac{t}{10}$, which is continuous function of J into $\mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(d(x,y)) = d(x,y)$.

case II: for $t \in [1,2]$,

$$(\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|, \alpha\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|)$$

$$= \frac{2te^{-(t-1)}}{9+e^{t-1}}(\|\frac{x(t-1)}{1+x(t-1)} - \frac{y(t-1)}{1+y(t-1)}\|, \alpha\|\frac{x(t-1)}{1+x(t-1)} - \frac{y(t-1)}{1+y(t-1)}\|)$$

$$= \frac{2te^{-(t-1)}}{9+e^{t-1}}(\|\frac{x(t-1)-y(t-1)}{(1+x(t-1))(1+y(t-1))}\|, \alpha\|\frac{x(t-1)-y(t-1)}{(1+x(t-1))(1+y(t-1))}\|)$$

$$\leq \frac{2te^{-(t-1)}}{9+e^{t-1}}(\|x(t-1)-y(t-1)\|, \alpha\|x(t-1)-y(t-1)\|)$$

$$\leq \frac{2te^{-(t-1)}}{9+e^{t-1}}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$

$$\leq \frac{2te^{-(t-1)}}{9+e^{t-1}}d(x,y)$$

$$\leq \frac{t}{5}\Phi(d(x,y)).$$

where $p_2(t) = \frac{t}{5}$, which is continuous function of J into $\mathbb{R}^+$ and a comparison function

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(d(x,y)) = d(x,y)$.

Similarly, we have

$(\|k(t,x)-k(t,y)\|, \alpha\|k(t,x)-k(t,y)\|)$

$$= (\|\tfrac{tx(t)}{20} - \tfrac{ty(t)}{20}\|, \alpha\|\tfrac{tx(t)}{20} - \tfrac{ty(t)}{20}\|)$$

$$\leq \tfrac{t}{20}(\|x(t)-y(t)\|, \alpha\|x(t)-y(t)\|)$$

$$\leq \tfrac{t}{20}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$

$$\leq \tfrac{t}{20} d(x,y)$$

$$\leq \tfrac{t}{20}\Phi(d(x,y)).$$

where $q(t) = \frac{t}{20}$, which is continuous function of J into$\mathbb{R}^+$ and the comparison function Φ defined as above. Also,

$(\|g(x)-g(y)\|, \alpha\|g(x)-g(y)\|)$

$$\leq 8\left(\tfrac{\|x-y\|}{(8+\|x\|)(8+\|y\|)}, \alpha\tfrac{\|x-y\|}{(8+\|x\|)(8+\|y\|)}\right)$$

$$\leq \tfrac{8}{64}(\|x-y\|, \alpha\|x-y\|)$$

$$\leq \tfrac{1}{8}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$

$$\leq \tfrac{1}{8}\Phi(d(x,y)).$$

where $G = \frac{1}{8}$, and the comparison function Φ defined as above. Hence the condition (H_1) holds with $K = \frac{35}{44}$. Moreover,

$$\sup_{t\in J}\left\{G + K\sum_{s=0}^{t}\left(p_1(s)+p_2(s)+\sum_{\tau=0}^{s} q(\tau)\right)\right\} = \sup_{t\in J}\left\{\frac{1}{8} + \frac{35}{44}\sum_{s=0}^{t}\left(\frac{s}{10}+\frac{s}{5}+\sum_{\tau=0}^{s}\frac{\tau}{20}\right)\right\}$$

$$
\begin{aligned}
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\sum_{s=0}^{t}\left(\frac{s}{10}+\frac{s}{5}+\left[\frac{\tau^{\underline{2}}}{40}\right]_0^{s+1}\right)\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\sum_{s=0}^{t}\left(\frac{3s}{10}+\frac{(s+1)^{\underline{2}}}{40}\right)\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\left[\frac{3s^{\underline{2}}}{20}+\frac{(s+1)^{\underline{3}}}{120}\right]_0^{t+1}\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\left[\frac{3(t+1)^{\underline{2}}}{20}+\frac{(t+2)^{\underline{3}}}{120}\right]\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\left[\frac{3(t+1)t}{20}+\frac{(t+2)(t+1)t}{120}\right]\right\}\\
&= \left[\frac{1}{8}+\frac{35}{44}\times\left(\frac{18}{20}+\frac{4}{20}\right)\right]=\left[\frac{1}{8}+\frac{35}{44}\times\frac{11}{10}\right]=\left[\frac{1}{8}+\frac{7}{8}\right]=1.
\end{aligned}
$$

Since all the conditions of Theorem 3.1 are satisfied, the problem (4.1)-(4.2) has a unique solution x on J.

5. Conclusion

The existence of fractional Summation-Difference type finite delay equations in cone metric spaces was investigated in this paper, and the solution was proven to be unique. By extending the Banach contraction principle, we were able to demonstrate this result in complete cone metric space. We put the above result into action as well.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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