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CONSTRAINED PROBABILISTIC MULTI-SOURCE INVENTORY MODEL BY DECREASING HOLDING COST WITH WEIBULL AND LINDLEY-WEIBULL DISTRIBUTIONS

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Abstract: The main goal of this work is to minimize the expected total cost for a multi-item, multi-source (MIMS) probabilistic continuous review inventory model with constraint on the expected decreasing holding cost utilizing the Lagrange multiplier technique. The demand is a continuous random variable. The optimal order quantity and the optimal reorder point for the i^{th} item and s^{th} source which achieve the objective are obtained when lead time demand follows Weibull and Lindley-Weibull distributions. Also, an application is analyzed and reach the goal of minimizing the expected total cost.

Keywords: continuous review; decreasing holding cost; Lindley-Weibull distribution; mixture shortage; Weibull distribution.

2020 AMS Subject Classification: 62P20, 90B05.

1. INTRODUCTION

The system is continuous review, so the stock level is always known and the demands are recorded as they arise. When the stock level hits a certain reorder point r, an order quantity of size Q is placed once each cycle. (Q and r are two independent decision variables). Hadley and Whitin [15] discussed probabilistic (Q,r) models with constant units of cost and lead-time demand is a

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random variable. Azoury and Brill [3] studied an application of the system-point method to inventory models under continuous review. MIMS inventory management system is the most common sort of procurement system, which can be explained as follows: To fulfil average demand rates, a specific amount of items are kept in inventory. $\overline{D}_1, \overline{D}_2, \overline{D}_3, \ldots, \overline{D}_i$. Unrestricted probabilistic (MIMS) inventory system was investigated by Fabrycky and Banks [8]. Gupta and Hira [14] explained Lagrange multipliers method. The majority of probabilistic inventory model assume that either one of these units of cost is variable or that all of these units are constant. Models for achieving this under numerous conditions and presumptions have been presented in hundreds of papers and books. With two constraints, Abuo-EL-Ata et al. [1] explored the probabilistic MISS inventory model with variable order cost. Cruz et al. [7] explained Analysis of second order properties of production–inventory systems with lost sales. With two linear constraints, Fergany and Elwakeel [10] determined a probabilistic single-item inventory problem with variable order cost. MISS mixture inventory model with random lead time and demand, as well as a budget limitation and surprise function, was described by Bera et al. [4]. Geometric programming was used to study the multi-item EOQ model with changing holding costs in [17]. Singer and Khmelnitsky [21] introduced a production-inventory model with price-sensitive demand. Fergany and Gomaa [12] provided a model of shortfall multi-source inventory with probabilistic mixtures and changing holding costs under constraint. Probabilistic multi-item inventory model with variable mixture shortage cost under constraints is solved by Fergany [9]. Fergany and El-Saadani [11] considered the Constrained Probabilistic inventory model with continuous distributions and varying holding cost. Inventory–forecasting: Mind the gap derived by Goltsos et al. [13]. Kourentzes et al. [18] examined Optimising forecasting models for inventory planning. Cordeiro et al. [6] explained The Lindley Weibull distribution: properties and applications. Selen and Gamze [20] examined The Lindley family of distributions: properties and applications. Statistical inference of the lifetime performance index for Lindley distribution under progressive first-failure censoring scheme applied to HPLC data derived by Hassanein [16]. Metiri et al. [19] considered On the Characterization of X-Lindley Distribution by Truncated Moments: Properties and Application. Reliability Models Using the Composite Generalizers of Weibull Distribution described by Aryal et al. [2]. The Weibull-G family of probability distributions first described by Bourguignon [5].

Here, we study the multi-item, multi-source (MIMS) probabilistic (Q, r) with mixture deficit and decreasing holding cost, subject to the expected constraint of decreasing holding cost. Three elements make up the expected overall cost of the inventory system (the expected setup cost, the expected decreasing holding cost, and the expected mixture penalty cost). The order quantity (Q^*_{is}) and the reorder point (r^*_{is}) are the optimal solutions that minimize the expected total cost $E(TC(Q_{is}, r_{is}))$, utilizing Lagrange method, are deduced mathematically. Additionally, these ideal values are introduced when the lead time demand is governed by the Weibull and Lindley-Weibull distributions. The model is illustrated with an application.

2. Assumptions and Notations

The following notations are used to construct our model:

$\overline{D}_i$	The i^{th} item's average annual demand.
Q_{is}	The decision variable for the i^{th} item and s^{th} source's order quantity each cycle.
Q_{is}^*	The optimal cycle order quantity for the i^{th} item and s^{th} source.
r_{is}	The decision variable for the i^{th} item and s^{th} source's reorder point each cycle.
r_{is}^*	The i^{th} item and s^{th} source's optimal ordering point per cycle.
C_{ois}	The i^{th} item and s^{th} source's order cost each cycle per unit.
C_{his}	The holding cost for the i^{th} item and s^{th} source per unit every cycle.
$C_{his}(Q_{is})$	The decreasing holding cost each cycle per unit for the i^{th} item and s^{th} source = $C_{his}Q_{is}^{-\beta}$, β is a constant real number selected to provide the best fit of estimated expected total cost function.
C_{bi}	The i^{th} item's backorder cost per unit per cycle.
C_{li}	The i^{th} item's lost sales cost per unit per cycle.
k_i	The restriction on the i^{th} item's expected decreasing annual holding cost.
λ_{is}	Multiplier of Lagrange for the i^{th} item and s^{th} source.
λ_{is}^*	The i^{th} item and s^{th} source's ideal Lagrange multiplier values.
γ_i	The i^{th} item's backorder fraction, $0 < \gamma_i < 1$.
$E_{is}(OC)$	The expected order cost for the i^{th} item and s^{th} source.
$E_{is}(HC)$	The expected decreasing holding cost for the i^{th} item and s^{th} source.

$E_{is}(BC)$	The expected backorder cost for the i^{th} item and s^{th} source.
$E_{is}(LC)$	The expected lost sales cost for the i^{th} item and s^{th} source.
$E_{is}(SC)$	The expected shortage cost = E_{is}(BC) + E_{is}(LC) for the i^{th} item and s^{th} source.
$\bar{S}$(r)	The quantity of a cycle's expected shortage.
$min_s\ E(TC_{is})$	Minimum expected total cost for s^{th} sources of i^{th} item.
min E(TC)	The minimum projected annual total cost for all item and s^{th} sources where min E(TC) = $\sum_{i=1}^{m} min_s\ E(TC_{is})$.

3. The Model Analysis

For the purpose of creating the mathematical model, the following presumptions are made:

(1) The expected order cost is given by:

$$E_{is}(OC) = C_{ois}\frac{\overline{D}_i}{Q_{is}}$$

(2) The expected decreasing holding cost is given by:

$$E_{is}(HC) = C_{his}\,(Q_{is})n_i\overline{H}_i$$

$$= C_{his}Q_{is}{}^{-\beta}\left[\frac{Q_{is}}{2} + r_{is} - \mu + (1-\gamma_i)\bar{S}(r_{is})\right]$$

where, H_i reflects the average stock level throughout the cycle. $\overline{H}_i = \frac{H_i}{n_i}$, n_i is annual cycle average and $n_i = \frac{\overline{D}_i}{Q_{is}}$

(3) The following are the mixture of the expected backorder cost, expected lost sales cost, and expected shortage cost:

$$E_{is}(SC) = E_{is}(BC) + E_{is}(LC)$$

Where,

$E_{is}(BC)= C_{bi}\,\gamma_i\frac{\overline{D}_i}{Q_{is}}\,\overline{S}(r_{is})$ and $E_{is}(LC) = C_{li}(1-\gamma_i)\,\frac{\overline{D}_i}{Q_{is}}\overline{S}(r_{is})$

Minimizing the applicable expected total cost function is the goal (i.e., the sum of the expected order cost, the expected decreasing holding cost, and the expected mixture shortage cost) which, based on the model's earlier assumptions is:

E(TC($Q_{is},\ r_{is}$)) =$\sum_{i=1}^{m}[\,E_{is}\,(OC) + E_{is}(HC) + E_{is}(SC)]$

min E (TC) = $\sum_{i=1}^{m} min_s\ E(TC_{is})$

$$= \sum_{i=1}^{m} \left[\begin{array}{c} \frac{C_{ois}\overline{D}_i}{Q_{is}} + C_{his}Q_{is}^{-\beta} \left[\frac{Q_{is}}{2} + r_{is} - \mu + \right. \\ \left. (1-\gamma_i)\overline{S}(r_{is})\right] + C_{bi}\frac{\gamma_i\overline{D}_i}{Q_{is}}\overline{S}(r_{is}) + C_{li}(1-\gamma_i)\frac{\overline{D}_i}{Q_{is}}\overline{S}(r_{is}) \end{array} \right] \quad (1)$$

Depending on how the expected decreasing holding cost limitation plays out:

$$\sum_{i=1}^{m} [E_{is}(HC) \le k_i] \quad (2)$$

Then to solve equation (1), under the constraint of equation (2), the Lagrange multipliers method ought to be utilized as follows:

$$L = \sum_{i=1}^{m} [E(TC_{is}) + \lambda_{is}\{E_{is}(HC) - k_i\}], \quad \lambda_{is} > 0$$

$$= \sum_{i=1}^{m} \left[\frac{C_{ois}\overline{D}_i}{Q_{is}} + C_{his}Q_{is}^{-\beta}\left\{\frac{Q_{is}}{2} + r_{is} - \mu + (1-\gamma_i)\overline{S}(r_{is})\right\} + \frac{C_{bi}\gamma_i\overline{D}_i}{Q_{is}}\overline{S}(r_{is}) + \right.$$

$$\left. \frac{C_{li}(1-\gamma_i)\overline{D}_i}{Q_{is}}\overline{S}(r_{is}) + \lambda_{is}\left[C_{his}Q_{is}^{-\beta}\left\{\frac{Q_{is}}{2} + r_{is} - \mu + (1-\gamma_i)\overline{S}(r_{is})\right\} - k_i\right] \right] \quad (3)$$

Set the appropriate initial partial derivatives of (3) with respect to the two variables of decisions equate to zero, it is possible to compute (Q_{is}^*) and (r_{is}^*) that will reduce E(TC) for the i^{th} item and the s^{th} source:

$$\frac{\partial L}{\partial Q_{is}} = -\frac{C_{ois}\overline{D}_i}{Q_{is}^2} - \beta\, C_{his}Q_{is}^{-(\beta+1)}\left[r_{is} - \mu + (1-\gamma_i)\overline{S}(r_{is})\right] + \frac{(1-\beta)C_{his}Q_{is}^{-\beta}}{2}$$

$$- \frac{C_{bi}\gamma_i\overline{D}_i}{Q_{is}^2}\overline{S}(r_{is}) - \frac{C_{li}(1-\gamma_i)\overline{D}_i}{Q_{is}^2}\overline{S}(r_{is}) - \beta\lambda_{is}C_{his}Q_{is}^{-(\beta+1)}[r_{is} - \mu$$

$$+ (1-\gamma_i)\overline{S}(r_{is})] + \frac{(1-\beta)\lambda_{is}C_{his}Q_{is}^{-\beta}}{2},$$

Hence,

$$(1-\beta)A(Q_{is}^*)^{(2-\beta)} = 2A\beta\{r_{is}^* - \mu + (1-\gamma_i)\overline{S}(r_{is}^*)\}(Q_{is}^*)^{1-\beta} + 2(B + M\bar{S}(r_{is}^*))$$

where $A = (1+\lambda_{is})C_{his}$, $B = C_{ois}\overline{D}_i$, $M = [C_{bi}\gamma_i\overline{D}_i + C_{li}(1-\gamma_i)\overline{D}_i]$

i.e.

$$(1-\beta)A(Q_{is}^*)^{2-\beta} - 2A\beta\{r_{is}^* - \mu + (1-\gamma_i)\overline{S}(r_{is}^*)\}(Q_{is}^*)^{1-\beta} - 2(B + M\bar{S}(r_{is}^*)) = 0 \quad (4)$$

also,

$$\frac{\partial L}{\partial r_{is}} = c_{his}Q_{is}^{-\beta}[1 - (1-\gamma_i)R(r_{is})] - \frac{c_{bi}\gamma_i\overline{D}_i}{Q_{is}}R(r_{is})$$

$$- \frac{C_{li}(1-\gamma_i)\overline{D}_i}{Q_{is}}R(r_{is}) + \lambda_{is}[c_{his}Q_{is}^{-\beta}\{1 - (1-\gamma_i)R(r_{is})\}]$$

$$R(r_{is}^*) = \frac{A(Q_{is}^*)^{1-\beta}}{[M + A(1-\gamma_i)(Q_{is}^*)^{1-\beta}]} \quad (5)$$

and we can prove that:

$$\begin{cases} \left[\frac{\partial^2 L}{\partial {Q_{is}}^2}\right]\left[\frac{\partial^2 L}{\partial {r_{is}}^2}\right] - \left[\frac{\partial^2 L}{\partial Q_{is}\partial r_{is}}\right]^2 & > 0 \\ \frac{\partial^2 L}{\partial {Q_{is}}^2} \quad or, \quad \frac{\partial^2 L}{\partial {r_{is}}^2} & > 0 \end{cases}$$

It is obvious that there is no closed form solution for equations (4) and (5). So, we can use the iteration procedure below to calculate min E(TC).

Algorithm I:

1: Input all of the inventory model data, such as order unit cost, holding unit cost, value of expected demand, mean, etc. at a single value of β and assumption value of λ, respectively, setting r_0= μ as an initial value to make S_0= 0. Then, determine the first Q_1 by computing S_0= 0.

2: Use the values determined in step 1 to determine r_1 and S_1.

3: Utilise the calculated r_1 and S_1 in step 2 to calculate a new Q_2.

4: Repetition of steps 1 and 2 When two computed order quantity values are equivalent, the Q* and r* are at their optimum.

5: Determining the minimum expected total cost using both calculated $Q^*$and r^*.

6: Repeat each step whenever the value of λ changes to indicate that the condition is met. If the condition holds true, min E(TC) at this value of β is true.

7: Repeat each step for different values of β.

4. The Model With Continuous Distributions

Assume for the purposes of our model that the demand for lead time satisfies the following Weibull and Lindley-Weibull distributions:

4.1 The model with Weibull distribution:

If x follows the Weibull distribution with parameters a, b, the density function is:

$$f(x;a,b) = \frac{a}{b}\,(\frac{x}{b})^{a-1}\, e^{-(\frac{x}{b})^a}, \quad x > 0\,, \quad a,\ b > 0 \tag{6}$$

Moreover, the reliability function is provided by:

$$R(r) = \int_r^\infty f(x)dx = \ e^{-(\frac{r}{b})^{\,a}} \tag{7}$$

and the expected shortage quantity is given by:

$$s(r) = \int_r^\infty (x-r)f(x)dx = b\,\left(\frac{1}{a}\right)!\ e^{-\left(\frac{r}{b}\right)^{a}} \sum_{k=0}^{\frac{1}{a}} \frac{\left[\left({r}/{b}\right)^{a}\right]^k}{k!} - re^{-(\frac{r}{b})^{\,a}} \tag{8}$$

It is achievable to determine (${Q_{is}}^*$) and (${r_{is}}^*$) that will minimize E(TC) for the i^{th} item and the s^{th} source by inserting from (8) and (7) in to (4) and (5), respectively. It is realized that the optimal values of Q_{is}^* and r_{is}^* are given by:

$$(1\text{-}\beta)A(Q_{is}^*)^{2-\beta} - 2A\beta\{ r_{is}^* - \mu + (1-\gamma_i)\{b\left(\frac{1}{a}\right)!\, e^{-\left(\frac{r}{b}\right)^a} \sum_{k=0}^{\frac{1}{a}} \frac{\left[(r/_b)^a\right]^k}{k!} - re^{-\left(\frac{r}{b}\right)^a}\}\}(Q_{is}^*)^{1-\beta} - 2(B + M\{b\left(\frac{1}{a}\right)!\, e^{-\left(\frac{r}{b}\right)^a} \sum_{k=0}^{\frac{1}{a}} \frac{[(r/_b)^a]^k}{k!} - re^{-\left(\frac{r}{b}\right)^a}\}) = 0 \quad (9)$$

$$R(r_{is}^*) = \frac{A(Q_{is}^*)^{1-\beta}}{[M + A(1-\gamma_i)(Q_{is}^*)^{1-\beta}]} = e^{-\left(\frac{r}{b}\right)^a} \quad (10)$$

4.2 The model with Lindley-Weibull distribution:

When x follows the Lindley-Weibull distribution with parameters θ, a, b, the density function is:

$$f(x;\theta,a,b) = \frac{a\theta^2}{b(\theta+1)}\left(\frac{x}{b}\right)^{a-1}\left(1+\left(\frac{x}{b}\right)^a\right) e^{-\left(\frac{x}{b}\right)^a\theta},\ x>0, \quad a,b,\theta>0 \quad (11)$$

as well as the reliability role being provided by:

$$R(r) = \int_r^\infty f(x)dx = \left(1 + \frac{\theta}{\theta+1}\left(\frac{r}{b}\right)^a\right) e^{-\left(\frac{r}{b}\right)^a\theta} \quad (12)$$

and the expected shortage quantity is given by:

$$\bar{S}(r) = \int_r^\infty (x-r)f(x)dx$$

$$= \frac{\theta}{\theta+1}\left[\frac{b}{\theta^{\frac{1}{a}}}\left(\frac{1}{a}\right)!\, e^{-\left(\frac{r}{b}\right)^a\theta} \sum_{k=0}^{\frac{1}{a}} \frac{\left(\left(\frac{r}{b}\right)^a\theta\right)^k}{k!} + \frac{b}{\theta^{\frac{1}{a}+1}}\left(\frac{1}{a}+1\right)!\, e^{-\left(\frac{r}{b}\right)^a\theta} \sum_{k=0}^{\frac{1}{a}+1} \frac{\left(\left(\frac{r}{b}\right)^a\theta\right)^k}{k!} - \frac{r}{\theta}\, e^{-\left(\frac{r}{b}\right)^a\theta}\left(1+\left(\frac{r}{b}\right)^a\theta\right) - r\, e^{-\left(\frac{r}{b}\right)^a\theta}\right] \quad (13)$$

So, for every i^{th} item and s^{th} source, it is possible to mathematically reduce the expected total cost by inserting from (13) and (12) in to (4) and (5), respectively. It is discovered that the ideal values for Q_{is}^* and r_{is}^* are given by:

$$(1\text{-}\beta)A(Q_{is}^*)^{2-\beta} - 2A\beta\{ r_{is}^* - \mu + (1-\gamma_i)\{ \frac{\theta}{\theta+1}\left[\frac{b}{\theta^{\frac{1}{a}}}\left(\frac{1}{a}\right)!\, e^{-\left(\frac{r}{b}\right)^a\theta} \sum_{k=0}^{\frac{1}{a}} \frac{\left(\left(\frac{r}{b}\right)^a\theta\right)^k}{k!} + \frac{b}{\theta^{\frac{1}{a}+1}}\left(\frac{1}{a}+1\right)!\, e^{-\left(\frac{r}{b}\right)^a\theta} \sum_{k=0}^{\frac{1}{a}+1} \frac{\left(\left(\frac{r}{b}\right)^a\theta\right)^k}{k!} - \frac{r}{\theta}\, e^{-\left(\frac{r}{b}\right)^a\theta}\left(1+\left(\frac{r}{b}\right)^a\theta\right) - r\, e^{-\left(\frac{r}{b}\right)^a\theta}\right]\}\}(Q_{is}^*)^{1-\beta} - 2(B + M\{ \frac{\theta}{\theta+1}\left[\frac{b}{\theta^{\frac{1}{a}}}\left(\frac{1}{a}\right)!\, e^{-\left(\frac{r}{b}\right)^a\theta} \sum_{k=0}^{\frac{1}{a}} \frac{\left(\left(\frac{r}{b}\right)^a\theta\right)^k}{k!} + \frac{b}{\theta^{\frac{1}{a}+1}}\left(\frac{1}{a}+1\right)!\, e^{-\left(\frac{r}{b}\right)^a\theta} \sum_{k=0}^{\frac{1}{a}+1} \frac{\left(\left(\frac{r}{b}\right)^a\theta\right)^k}{k!} - \frac{r}{\theta}\, e^{-\left(\frac{r}{b}\right)^a\theta}\left(1+\left(\frac{r}{b}\right)^a\theta\right) - r\, e^{-\left(\frac{r}{b}\right)^a\theta}\right]\}) = 0 \quad (14)$$

$$R(r_{is}^*) = \frac{A(Q_{is}^*)^{1-\beta}}{[M + A(1-\gamma_i)(Q_{is}^*)^{1-\beta}]} = \left(1 + \frac{\theta}{\theta+1}\left(\frac{r}{b}\right)^a\right) e^{-\left(\frac{r}{b}\right)^a\theta} \quad (15)$$

5. APPLICATION

According to the model assumptions, a manager of an electronics a firm in Egypt chose to order three electronic gadgets (three goods) from three distinct vendors. To reduce the anticipated total cost, he wants to obtain the best possible policy. Its problem is identical to the model being studied. The parameters for multi-item, multi-source are given in Table 1 and Table 2. We consider the parameters (a = 1, b = 3) for Weibull distribution and (θ = 2.84, a = 1, b = 3) for Lindley-Weibull distribution (see [6]).

TABLE 1. The parameters of the model which independent on the sources

	Item 1	Item 2	Item 3
$\overline{D_i}$	20	20	20
C_{bi}	5	9	11
C_{li}	8	12	14
γ_i	0.7	0.7	0.56
K_i	7	11	16

TABLE 2. The order cost and the holding cost per unit

Cost	Item	Source 1	Source 2	Source 3
C_{ois}	1	10	13	16
	2	14	17	20
	3	18	21	24
C_{his}	1	0.5	0.6	0.7
	2	0.9	1	1.1
	3	1.3	1.4	1.5

Consider $0.01 \leq \beta \leq 0.1$ which is appropriate in the case of the distributions under study for equations (9) and (10) applied for Weibull distribution as well as (14) and (15) for Lindley-Weibull distribution. Tables 1, 2, 3 and 5 are used to obtain about λ^*, Q^*, r^* and $min_s\, E(TC_{is})$ using the Mathematica program V 12.3. Tables 4 and 6 show the results for Weibull and Lindley-Weibull distributions respectively assuming for various values of the parameter β. The best minimum projected overall cost for the i^{th} item and s^{th} source when β = 0.1 for Weibull and Lindley-Weibull distributions which displayed in Table 7. Also, the minimum expected total cost can be illustrated against different values of β for each item and three sources for Weibull and Lindley-Weibull distributions as shown in Figure 1 and 2 respectively.

TABLE 3. The value of λ^* for Item 1 and the first Source at β = 0.01 for the Weibull distribution

λ	E(HC)	E(TC_i)
0	9.749	0.543
1	7.202	0.787
1.131	7.001	0.818
1.1317	7.000	0.818
1.1318	**6.999**	**0.818**

TABLE 4. The results for the Weibull distribution

β	Source	Item 1				Item 2				Item 3			
		λ_1	Q_1	r_1	ETC_{1i}	λ_2	Q_2	r_2	ETC_{2i}	λ_3	Q_3	r_3	ETC_{3i}
0.01	1	1.1318	22.832	5.005	0.818	1.4439	19.349	4.883	1.628	1.1572	19.330	4.901	2.177
	2	2.0533	20.163	3.836	1.229	2.0566	18.229	4.127	2.078	1.5668	18.520	4.342	2.595
	3	2.9959	18.310	2.948	1.732	2.3195	18.816	3.813	2.185	1.9859	17.798	3.860	3.059
0.02	1	1.0699	23.623	5.082	0.771	1.3761	19.987	4.956	1.536	1.0959	19.977	4.972	2.056
	2	1.9802	20.789	3.903	1.161	1.9829	18.787	4.194	1.965	1.4999	19.113	4.409	2.452
	3	2.9177	18.826	3.006	1.641	2.2426	19.385	3.878	2.066	1.9141	18.343	3.923	2.893
0.03	1	1.0081	24.460	5.162	0.726	1.3087	20.659	5.032	1.449	1.0344	20.663	5.047	1.939
	2	1.9071	21.449	3.971	1.096	1.9079	19.378	4.262	1.855	1.4325	19.739	4.478	2.315
	3	2.8375	19.371	3.066	1.554	2.1647	19.987	3.945	1.951	1.8418	18.919	3.987	2.733
0.04	1	0.9459	25.351	5.245	0.683	1.2411	21.375	5.109	1.364	0.9728	21.390	5.123	1.828
	2	1.8332	22.148	4.042	1.034	1.8329	20.002	4.333	1.750	1.3649	20.402	4.549	2.183
	3	2.7561	19.946	3.127	1.469	2.0866	20.622	4.013	1.840	1.7691	19.528	4.053	2.580
0.05	1	0.8841	26.296	5.329	0.642	1.1727	22.130	5.190	1.284	0.9116	22.160	5.202	1.721
	2	1.7587	22.889	4.115	0.973	1.7577	20.661	4.405	1.649	1.2974	21.103	4.622	2.057
	3	2.6745	20.550	3.190	1.388	2.0071	21.296	4.085	1.734	1.6959	20.172	4.121	2.433
0.06	1	0.8221	27.304	5.417	0.603	1.1051	22.932	5.273	1.207	0.8502	22.979	5.284	1.619
	2	1.6837	23.673	4.191	0.916	1.6814	21.361	4.481	1.552	1.2295	21.848	4.698	1.936
	3	2.5907	21.193	3.256	1.309	1.9275	22.009	4.158	1.632	1.6225	20.853	4.192	2.291
0.07	1	0.7605	28.375	5.508	0.562	1.0371	23.787	5.359	1.133	0.7892	23.848	5.368	1.521
	2	1.6081	24.507	4.268	0.861	1.6054	22.102	4.558	1.460	1.1619	22.636	4.776	1.820
	3	2.5071	21.869	3.323	1.234	1.8474	22.764	4.234	1.535	1.5488	21.575	4.264	2.156

TABLE 4. The results for the Weibull distribution (continued)

β	Source	Item 1				Item 2				Item 3			
		λ_1	Q_1	r_1	ETC_{1i}	λ_2	Q_2	r_2	ETC_{2i}	λ_3	Q_3	r_3	ETC_{3i}
0.08	1	0.6993	29.516	5.601	0.530	0.9692	24.696	5.447	1.063	0.7283	24.774	5.455	1.428
	2	1.5324	25.392	4.349	0.808	1.5283	22.892	4.639	1.371	1.0939	23.476	4.857	1.709
	3	2.4209	22.589	3.393	1.161	1.7669	23.566	4.312	1.441	1.4746	22.342	4.340	2.026
0.09	1	0.6381	30.737	5.697	0.496	0.9019	25.663	5.538	0.996	0.6676	25.761	5.543	1.339
	2	1.4564	26.334	4.432	0.757	1.4514	23.730	4.722	1.286	1.0261	24.368	4.941	1.604
	3	2.3355	23.348	3.465	1.091	1.6858	24.419	4.393	1.352	1.4006	23.155	4.418	1.902
1	1	**0.5778**	**32.036**	**5.796**	**0.463**	**0.8342**	**26.699**	**5.633**	**0.932**	**0.6074**	**26.812**	**5.638**	**1.254**
	2	1.3795	27.340	4.518	0.709	1.3744	24.621	4.807	1.205	0.9584	25.319	5.027	1.503
	3	2.2485	24.157	3.539	0.025	1.6046	25.326	4.476	1.266	1.3262	24.021	4.498	1.784

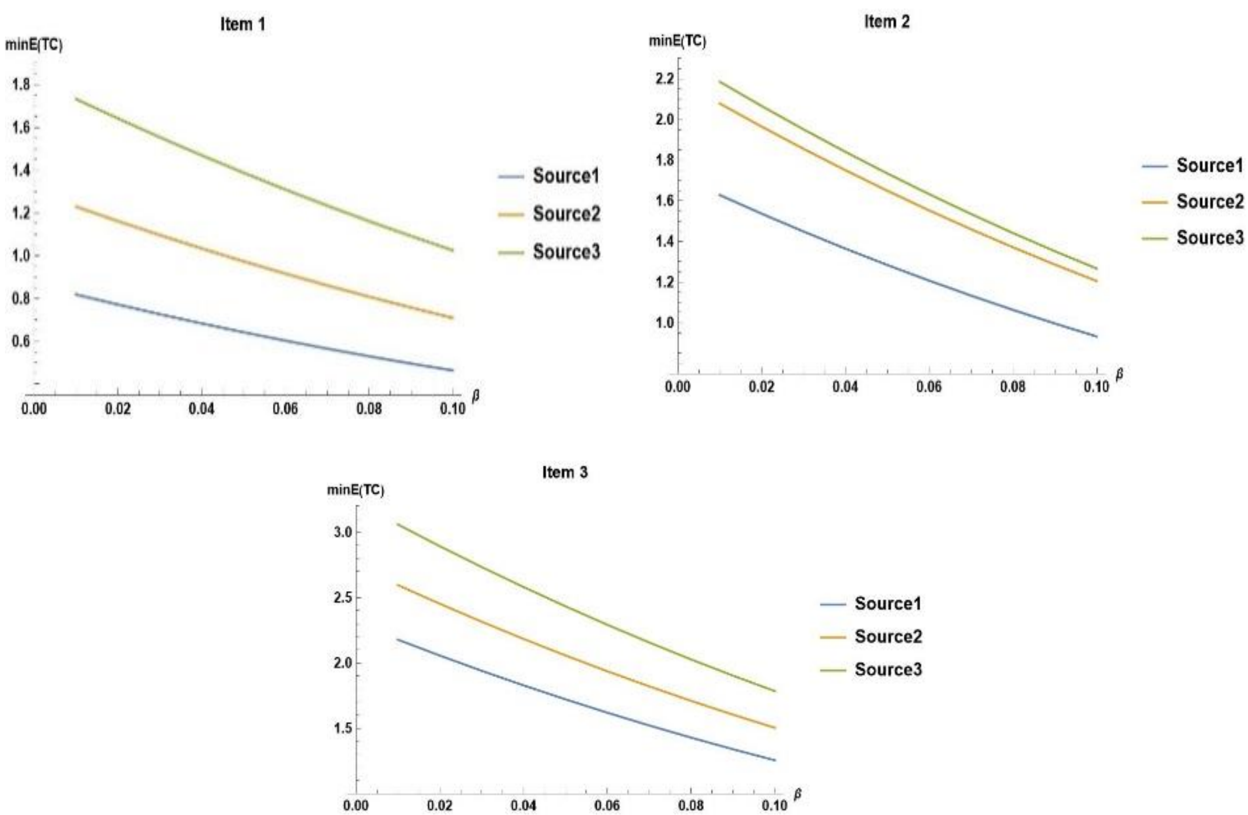

FIGURE 1. The min expected total cost of item 1,2,3 and three sources for the model with Weibull distribution.

TABLE 5. The value of λ^* for Item 1 and the first source at $\beta = 0.01$ for the Lindley-Weibull distribution

λ	E(HC)	$E(TC_i)$
0	8.203	0.511
0.3	7.272	0.585
0.412	7.001	0.612
0.4125	7.000	0.612
0.4126	**6.999**	**0.612**

TABLE 6. The results for the Lindley-Weibull distribution

β	Source	Item 1				Item 2				Item 3			
		λ_1	Q_1	r_1	ETC_{1i}	λ_2	Q_2	r_2	ETC_{2i}	λ_3	Q_3	r_3	ETC_{3i}
0.01	1	0.4126	25.537	2.585	0.612	0.5309	21.809	2.592	1.165	0.3415	21.946	2.589	1.565
	2	1.1550	21.681	2.054	0.943	1.0201	19.942	2.239	1.521	0.6579	20.606	2.327	1.890
	3	2.0011	18.996	1.633	1.378	1.5526	18.426	1.939	1.946	0.9966	19.444	2.097	2.261
0.02	1	0.3735	26.450	2.616	0.579	0.4900	22.560	2.621	1.103	0.3056	22.705	2.618	1.484
	2	1.1030	22.394	2.082	0.892	0.9706	20.593	2.267	1.441	0.6159	21.294	2.354	1.791
	3	1.9370	19.577	1.658	1.305	1.4950	18.999	1.965	1.844	0.9488	20.072	2.123	2.143
0.03	1	0.3349	27.414	2.648	0.547	0.4494	23.353	2.652	1.044	0.2696	23.508	2.648	1.406
	2	1.0498	23.152	2.111	0.843	0.9213	21.279	2.295	1.364	0.5739	22.021	2.383	1.697
	3	1.8720	20.190	1.685	1.236	1.4371	19.605	1.992	1.746	0.9007	20.735	2.150	2.030
0.04	1	0.2959	28.440	2.681	0.517	0.4086	24.193	2.683	0.987	0.2337	24.358	2.679	1.330
	2	0.9971	23.951	2.141	0.797	0.8717	22.006	2.324	1.290	0.5317	22.790	2.412	1.605
	3	1.8070	20.837	1.711	1.169	1.3791	20.243	2.019	1.652	0.8527	21.435	2.178	1.921
0.05	1	0.2573	29.527	2.715	0.488	0.3681	25.082	2.716	0.933	0.1979	25.258	2.711	1.258
	2	0.9448	24.794	2.172	0.752	0.8219	22.775	2.355	1.218	0.4898	23.603	2.443	1.518
	3	1.7410	21.521	1.739	1.104	1.3208	20.917	2.048	1.562	0.8045	22.176	2.207	1.816
0.06	1	0.2188	30.681	2.749	0.459	0.3278	26.024	2.749	0.880	0.1624	26.212	2.743	1.189
	2	0.8918	25.690	2.203	0.709	0.7723	23.589	2.386	1.150	0.4478	24.465	2.474	1.434
	3	1.6750	22.243	1.768	1.042	1.2625	21.628	2.077	1.475	0.7563	22.959	2.237	1.715
0.07	1	0.1805	31.909	2.786	0.433	0.2878	27.022	2.783	0.831	0.127	27.224	2.777	1.123
	2	0.8392	26.638	2.235	0.668	0.7227	24.451	2.418	1.084	0.4061	25.378	2.505	1.353
	3	1.6090	23.006	1.797	0.982	1.2040	22.382	2.107	1.391	0.7083	23.788	2.267	1.619

TABLE 6. The results for the Lindley-Weibull distribution (continued)

β	Source	Item 1				Item 1				Item 3			
		λ_1	Q_1	r_1	ETC_{1i}	λ_2	Q_2	r_2	ETC_{2i}	λ_3	Q_3	r_3	ETC_{3i}
0.08	1	0.1425	33.214	2.822	0.407	0.2477	28.086	2.818	0.783	0.0917	28.301	2.811	1.059
	2	0.7866	27.645	2.269	0.628	0.6732	25.366	2.451	1.022	0.3644	26.347	2.538	1.276
	3	1.5430	23.813	1.827	0.925	1.1450	23.182	2.138	1.311	0.6601	24.668	2.299	1.526
0.09	1	0.1050	34.600	2.859	0.383	0.2080	29.216	2.853	0.737	0.0567	29.446	2.847	0.998
	2	0.7339	28.716	2.303	0.590	0.6239	26.336	2.484	0.962	0.3229	27.377	2.572	1.202
	3	1.4760	24.672	1.858	0.870	1.0863	24.029	2.169	1.234	0.6121	25.602	2.331	1.437
0.1	1	**0.0675**	**36.084**	**2.899**	**0.359**	**0.1685**	**30.419**	**2.890**	**0.693**	**0.0219**	**30.666**	**2.883**	**0.940**
	2	0.6815	29.854	2.338	0.554	0.5745	27.370	2.519	0.904	0.2815	28.473	2.607	1.132
	3	1.4090	25.582	1.890	0.817	1.0277	24.927	2.202	1.160	0.5642	26.594	2.364	1.353

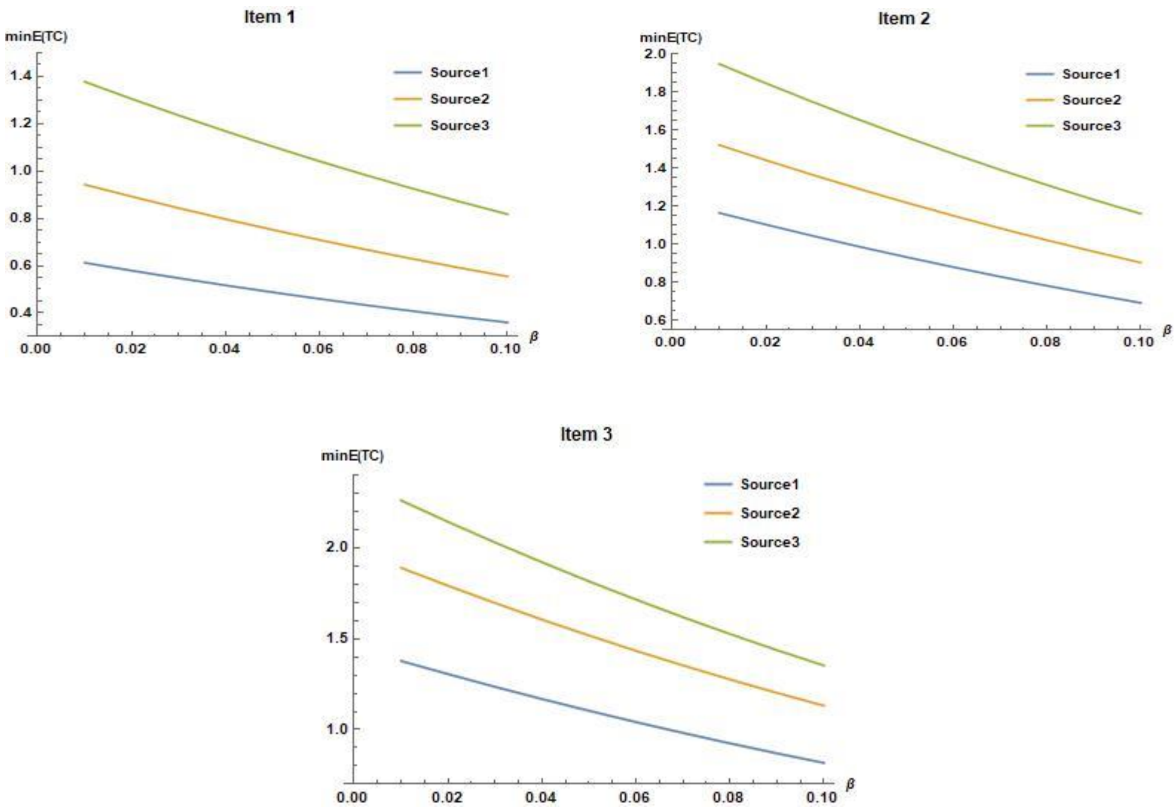

FIGURE 2. The min expected total cost of item 1,2,3 and three sources for the model with Lindley-Weibull distribution.

TABLE 7. MIMS's optimal policy model at β = 0.1for Weibull distribution and Lindley-Weibull distribution

	Weibull distribution					Lindley-Weibull distribution				
Item	${\lambda_{is}}^*$	${Q_{is}}^*$	${r_{is}}^*$	ETC_{is}	Source	${\lambda_{is}}^*$	${Q_{is}}^*$	${r_{is}}^*$	ETC_{is}	Source
1	0.5778	32.036	5.796	0.463	1	0.0675	36.084	2.899	0.359	1
2	0.8342	26.699	5.633	0.932	1	0.1685	30.419	2.890	0.693	1
3	0.6074	26.812	5.638	1.254	1	0.0219	30.666	2.883	0.940	1
Min TC				**2.649**					**1.992**	

6. CONCLUSION

The optimal order quantity Q^* and the optimal reorder point r^* for the i^{th} item and s^{th} source are introduced after researching the multi-item multi-source constrained probabilistic inventory model with decreasing holding cost utilizing the multipliers of Lagrange method. Then, when the demand for lead time follows the Weibull and Lindley-Weibull distributions, the minimum expected total cost min E(TC) is calculated for each item and source. We can also choose the best source for each item. Eventually, we concluded that the findings of the Lindley-Weibull distribution are superior to those of the Weibull distribution at the lowest value of the holding cost.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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