Available online at http://scik.org
J. Math. Comput. Sci. 3 (2013), No. 2, 456-465
ISSN: 1927-5307

QUINTIC C^1-SPLINE COLLOCATION METHODS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

H.M. EL-HAWARY* AND K.A. EL-SHAMI

Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt

Abstract. An efficient numerical method based on a quintic spline collocation is proposed for the numerical solution of neutral delay differential equations (NDDEs). The convergence analysis is given and the method is shown to have fifth-order convergence. Finally, numerical results for two nonlinear examples are given to illustrate the efficiency of our method.

Keywords: Neutral delay differential equations; Quintic spline; Collocation methods.

2010 AMS Subject Classification: 65L05; 41A15; 65D07

1. Introduction

Neutral delay differential equations (NDDEs), arise widely in scientific fields, such as control theory, bioscience, physics; we refer reader for many examples to the monograph [8]. This class of equations play an important role in modeling phenomena of the real world. So it is valuable to investigate the properties of the solutions of these equations. Since most of these equations cannot be solved exactly, it is necessary to study efficient numerical methods to solve these equations.

*Corresponding author

Received January 22, 2013

In this paper we will be concerned with the numerical solution for the initial value problems (IVPs) of NDDEs:

$$\begin{aligned} & y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)), \qquad t \in [t_0, t_f], \\ & y(t) = \phi(t), \qquad t \le t_0, \end{aligned} \tag{1.1}$$

where $f \in C^5([t_0, t_f] \times R \times R \times R)$ is Lipschitz continuous with respect to y, τ is a positive constant, ϕ is given continuously differentiable function.

Spline methods for solving delay differential equations (DDEs) and NDDEs are considered in [1, 3]. C^3-spline collocation methods with four points for solving DDEs and second order NDDEs were presented in [4, 5]. C^1-spline collocation methods for solving stiff delay and DDEs were studied in [6, 7]. More detailed analysis for both the convergence and absolute stability was also given. The convergence of numerical methods for NDDEs have previously been considered by several authors (see, e.g., [2, 10, 13]). Analysis of convergence and absolute stability of quintic C^2-spline integrating method for solving second-order ordinary initial value problems were studied in [9].

The outline of this paper is as follows: Section 2 contains an investigation of the existence, uniqueness, and the precise definition of spline collocation methods. Analysis of convergence for the method has been discussed in Section 3. In Section 4, numerical results are included to demonstrate the efficiency of the method. The last section is conclusion.

2. Description of the methods

Consider the initial value problem for the NDDEs (1.1). For a given positive integer n, the interval $[t_0, t_f]$ is partitioned into n equal subintervals $I_i = [t_{i-1}, t_i], i = 1(1)n$ with $t_i = t_{i-1} + h, n = (t_f - t_0)/h, h$ is the stepsize. The basic idea is to generate a quintic spline collocation methods $S \in C^1[t_0, t_f]$ at the Chebyshev points

$$c_\ell = \frac{t_i + t_{i-1}}{2} + \frac{t_i - t_{i-1}}{2}(-cos\frac{\ell\pi}{4}), \quad \ell = 0(1)4, i = 1(1)n. \tag{2.1}$$

Let $S^{(1)}_{n,5} = \{S(t) : S \in C^1[t_0, t_f], S \in \Pi_5,$ for $t \in I_i, i = 1(1)n\}$, where Π_5 denotes the collection of all polynomials of degree ≤ 5. Using the notations

$$S'_{i-1} = S'(t_{i-1}), \quad S'_{i-1+c_1} = S'(t_{i-1+c_1}), \quad S'_{i-1+c_2} = S'(t_{i-1+c_2}),$$
$$S'_{i-1+c_3} = S'(t_{i-1+c_3}), \quad S'_i = S'(t_i),\ i = 1(1)n,$$

a quintic spline functions $S \in S^{(1)}_{n,5}$ can be represented on each I_i by

$$\begin{aligned} S_i(t) =& S_{i-1} + hA(\xi)S'_{i-1} + hB(\xi)S'_{i-1+c_1} + hC(\xi)S'_{i-1+c_2} \\ &+ hD(\xi)S'_{i-1+c_3} + hE(\xi)S'_i, \end{aligned} \tag{2.2}$$

where $A(\xi), ..., E(\xi)$ are given in the Appendix of this paper, $t = t_{i-1} + \xi h,\ \xi \in [0, 1]$. Since $S \in S^{(1)}_{n,5}$, then the approximate spline solution $S(t)$ to the exact solution $y(t)$ of Eq. (1.1) will be constructing as follows: for $i = 1(1)n$

$$\underline{S}_i = M_0 S_{i-1} + hM_1 S'_{i-1} + hM_2 \underline{S}'_i \tag{2.3}$$

where $\underline{S}_i = (S_{i-1+c_1}, S_{i-1+c_2}, S_{i-1+c_3}, S_i)^T$, $\underline{S}'_i = (S'_{i-1+c_1}, S'_{i-1+c_2}, S'_{i-1+c_3}, S'_i)^T$,

$$S'_{i-1+c_\ell} = f\Big(t_{i-1+c_\ell}, S(t_{i-1+c_\ell}), S(t_{i-1+c_\ell} - \tau), S'(t_{i-1+c_\ell} - \tau)\Big), \tag{2.4}$$

c_ℓ be given in Eq. (2.1), $M_0 = (1, 1, 1, 1)^T$, M_1 and M_2 are also given in the Appendix of this paper.

It is easy to define that $S(t_{i-1+c_\ell} - \tau) = \phi(t_{i-1+c_\ell} - \tau)$ when $(t_{i-1+c_\ell} - \tau) \leq t_0$, and if $(t_{i-1+c_\ell} - \tau) \in [t_{k-1}, t_k], k = 1(1)i$, then $S(t_{i-1+c_\ell} - \tau)$ can be calculated by

$$\begin{aligned} S(t_{i-1+c_\ell} - \tau) =& S_{k-1} + hA(\zeta)S'_{k-1} + hB(\zeta)S'_{k-1+c_1} + hC(\zeta)S'_{k-1+c_2} \\ &+ hD(\zeta)S'_{k-1+c_3} + hE(\zeta)S'_k, \end{aligned} \tag{2.5}$$

where

$$\zeta = \frac{(t_{i-1+c_\ell} - \tau) - t_{k-1}}{h} \ \in [0, 1].$$

and $A(\zeta), ..., E(\zeta)$ be given in Eq. (5.1).

Also $S'(t_{i-1+c_\ell}-\tau)=\phi'(t_{i-1+c_\ell}-\tau)$ when $(t_{i-1+c_\ell}-\tau)\le t_0$, and if $(t_{i-1+c_\ell}-\tau)\in[t_{m-1},t_m], m=1(1)i$, then $S'(t_{i-1+c_\ell}-\tau)$ can be calculated by

$$\begin{aligned} S'(t_{i-1+c_\ell}-\tau)=&A'(\zeta)S'_{m-1}+B'(\zeta)S'_{m-1+c_1}+C'(\zeta)S'_{m-1+c_2}\\ &+D'(\zeta)S'_{m-1+c_3}+E'(\zeta)S'_m, \end{aligned} \tag{2.6}$$

with

$$\zeta=\frac{(t_{i-1+c_\ell}-\tau)-t_{m-1}}{h}\ \in[0,1],$$

and $A'(\zeta),...,E'(\zeta)$ can be calculated from Eq. (5.1).

From Equations (2.4)-(2.6), system (2.3) can be solved for $S_{i-1+c_1},S_{i-1+c_2},S_{i-1+c_3},S_i$.

Theorem 2.1. *If f satisfies Lipschitz condition, and if*

$$h<\frac{1}{1.02427L}, \tag{2.7}$$

then there exists a unique spline approximation solution of Eq. (1.1) given by system (2.3).

Proof. It is sufficient to prove that $\underline{S}_i=(S_{i-1+c_1},S_{i-1+c_2},S_{i-1+c_3},S_i)^T$ can be uniquely determined for an arbitrary given S_{i-1}. Since, we can write system (2.3) as follows:

$$\underline{S}_i=M_0S_{i-1}+hM_1f_{i-1}+hM_2\underline{f}_i \tag{2.8}$$

where $\underline{f}_i=(f_{i-1+c_1},f_{i-1+c_2},f_{i-1+c_3},f_i)^T$, from Eq. (2.8), we have

$$\begin{aligned} \underline{S}_{i,1}&=M_0S_{i-1}+hM_1f_{i-1,1}+hM_2\underline{f}_{i,1}\\ \underline{S}_{i,2}&=M_0S_{i-1}+hM_1f_{i-1,2}+hM_2\underline{f}_{i,2}. \end{aligned}$$

Thus, $\underline{S}_{i,1}$ and $\underline{S}_{i,2}$ can be written in the form

$$\begin{aligned} \underline{S}_{i,1}&=\underline{Q}_{i,1}(S_{i-1+c_1,1},S_{i-1+c_2,1},S_{i-1+c_3,1},S_{i,1}),\\ \underline{S}_{i,2}&=\underline{Q}_{i,2}(S_{i-1+c_1,2},S_{i-1+c_2,2},S_{i-1+c_3,2},S_{i,2}). \end{aligned}$$

Applying $\|.\|_1$, Lipschitz condition, we get

$$\begin{aligned}
\|\underline{Q}_{i,1} - \underline{Q}_{i,2}\| = \|&(M_0 S_{i-1} + hM_1 f_{i-1,1} + hM_2 \underline{f}_{i,1}) \\
&- (M_0 S_{i-1} + hM_1 f_{i-1,2} + hM_2 \underline{f}_{i,2})\| \\
\leq&\Big\{ \|M_1\| h |f_{i-1,1} - f_{i-1,2}| + \|M_2\| h \Big(|f_{i-1+c_1,1} - f_{i-1+c_1,2}| \\
&+ |f_{i-1+c_2,1} - f_{i-1+c_2,2}| + |f_{i-1+c_3,1} - f_{i-1+c_3,2}| + |f_{i,1} - f_{i,2}| \Big) \Big\} \\
<&\Big\{ \frac{7}{48} h L_1 |S_{i-1,1} - S_{i-1,2}| + 1.02427 h \Big(L_2 |S_{i-1+c_1,1} - S_{i-1+c_1,2}| \\
&+ L_3 |S_{i-1+c_2,1} - S_{i-1+c_2,2}| + L_4 |S_{i-1+c_3,1} - S_{i-1+c_3,2}| + L_5 |S_{i,1} - S_{i,2}| \Big) \Big\} \\
<&1.02427 hL \Big\{ |S_{i-1,1} - S_{i-1,2}| + |S_{i-1+c_1,1} - S_{i-1+c_1,2}| \\
&+ |S_{i-1+c_2,1} - S_{i-1+c_2,2}| + |S_{i-1+c_3,1} - S_{i-1+c_3,2}| + |S_{i,1} - S_{i,2}| \Big\}
\end{aligned}$$

where

$$L = max(L_1, L_2, L_3, L_4, L_5).$$

Thus, the function $\underline{Q}_i$ defines a contraction mapping, if $(1.02427)hL < 1$, which satisfies Eq. (2.7). Hence, there exists a unique $\underline{S}_i$ that satisfies

$$\underline{S}_i = \underline{Q}_i(S_{i-1+c_1}, S_{i-1+c_2}, S_{i-1+c_3}, S_i)$$

which may be found by iteration

$$\underline{S}_i^{p+1} = \underline{Q}_i(\underline{S}_i^p), \quad p = 0, 1, 2, ...$$

The proof of Theorem 2.1 is now complete.

3. Convergence of the method

In this section the emphasis is on conditions for convergence of the proposed method. It is shown that the method is a continuous extension of a multi-step method, and its derivative reproduces the values given by the well-known closed four-panel Newton-Cotes formula at the mesh points. A priori error estimates in $L_\infty-$norm shows that the method is a fifth order as well as its first derivatives, according to the following:

Lemma 3.1. *Let $f \in C^6([t_0, t_f] \times R \times R \times R)$, then*

$$e_i = O(h^5), \quad i = 0(1)n, \tag{3.1}$$

where $e_i = S_i - y_i$, with $y_i = y(t_i)$.

Proof. Since

$$S_i = S_{i-1} + \frac{h}{30}\Big(S'_{i-1} + 8S'_{i-1+c_1} + 12S'_{i-1+c_2} + 8S'_{i-1+c_3} + S'_i\Big),$$

which is the well-known closed four-panel Newton-Cotes formula, applied to $y'(t)$, if $y \in C^6[t_0, t_f]$, then it follows that

$$e'_i = O(h^5).$$

But

$$e_i = e_{i-1} + \delta_i.$$

where

$$\delta_i = \frac{h}{30}\Big(e'_{i-1} + 8e'_{i-1+c_1} + 12e'_{i-1+c_2} + 8e'_{i-1+c_3} + e'_i\Big) + O(h^6), \quad e_0 = 0,$$

thus

$$e_i = \sum_{j=1}^{i} \delta_j$$

or

$$e_i = O(h^5). \tag{3.2}$$

The proof of Lemma 3.1 is now completed.

We now turn to prove the following main theorem, which provides estimation for the global error for $S(t) - y(t)$ and its first derivative.

Theorem 3.1. *Let $f \in C^6([t_0, t_f] \times R \times R \times R)$, then for all $t \in [t_0, t_f]$, we have*

$$|S^{(k)}(t) - y^{(k)}(t)| < C_k h^5, \ k = 0, 1, \tag{3.3}$$

where C_k denote generic constants independent of h, but dependent on the order of the various derivatives.

Proof. On $[t_{i-1}, t_i]$, we have

$$e'_i(t) = S'(t) - u'(t) + u'(t) - y'(t),$$

where $u'(t)$ is the quartic interpolant of $y'(t)$ at $t_{i-1}, t_{i-1+c_1}, t_{i-1+c_2}, t_{i-1+c_3}$ and t_i. It can be easily verified that

$$u'(t) = y'_{i-1}A'(\xi) + y'_{i-1+c_1}B'(\xi) + y'_{i-1+c_2}C'(\xi) + y'_{i-1+c_3}D'(\xi) + y'_iE'(\xi),$$

with $A'(\xi), ..., E'(\xi)$ can be calculated from Eq. (5.1).

But

$$S'(t) - u'(t) = e'_{i-1}A'(\xi) + e'_{i-1+c_1}B'(\xi) + e'_{i-1+c_2}C'(\xi) + e'_{i-1+c_3}D'(\xi) + e'_iE'(\xi).$$

Therefore,

$$\begin{aligned}|S'(t) - u'(t)| &\leq |e'_{i-1}||A'(\xi)| + |e'_{i-1+c_1}||B'(\xi)| + |e'_{i-1+c_2}||C'(\xi)| \\ &+ |e'_{i-1+c_3}||D'(\xi)| + |e'_i||E'(\xi)| \\ &\leq |e'_{i-1}| + |e'_{i-1+c_1}| + |e'_{i-1+c_2}| + |e'_{i-1+c_3}| + |e'_i|,\end{aligned}$$

and using Lemma 3.1, it follows that

$$|S'(t) - u'(t)| = O(h^5).$$

Also from the construction of $u'(t)$, it follows that $|u'(t) - y'(t)| = O(h^5)$, provided $f \in C^6([t_0, t_f] \times R \times R \times R)$. Hence, $|e'(t)| \leq C_1h^5$.

On $[t_{i-1}, t_i]$, we have

$$e(t) = \int_{t_{i-1}}^{t} e'(\xi)d\xi + e_{i-1}$$

or, using Lemma 3.1, we get

$$|e(t)| \leq C_0h^5.$$

This completes the proof of Theorem 3.1.

4. Numerical results

To demonstrate the applicability of our presented method for the approximate solution of the NDDEs computationally, two nonlinear examples are considered. All calculations are implemented by MATLAB 7.

Example 4.1. [11] *Consider the following nonlinear NDDEs:*

$$\begin{aligned} y'(t) =& r\,y(t) + a\,cos\Big(y(t-\tau) + y'(t-\tau)\Big) + cos(t)exp(-t) \\ & - a\,cos\Big(cos(t-\tau)exp(-(t-\tau))\Big), \quad t \geq 0, \\ y(t) =& exp(-t)sin(t), \qquad -1 \leq t \leq 0, \end{aligned}$$

where $\tau = 1, r = -1, a = 0.9$*. The exact solution is given by*

$$y(t) = exp(-t)sin(t), \quad t \geq -1.$$

In Table 1, we give the absolute errors between the exact solution and the numerical results by the present method.

TABLE 1. Absolute errors for the solution of Example 4.1

t	$h = 0.2$	$h = 0.1$
2	5.807605E-11	9.045403E-13
4	1.200563E-11	1.875131E-13
6	4.509175E-13	6.898019E-15
8	1.705967E-13	2.669847E-15
10	2.450184E-14	3.962657E-16

TABLE 2. Errors $|y_i^n - y_{2i}^{2n}|$ of Example 4.2 for $n = 50$.

t	$\|y_i^n - y_{2i}^{2n}\|$
2	2.018320E-10
4	3.718039E-10
6	6.768029E-11
8	2.129114E-12
10	1.285190E-13

Example 4.2. [12] *Consider the following nonlinear NDDEs:*

$$y'(t) = -20y(t) + 0.25cos(y(t-1))sin(y'(t-1)), \quad t \in [0, 10].$$

In Table 2, the absolute errors for this example are calculated, with initial function $\phi(t) = t$, using the double mesh principle $|y_i^n - y_{2i}^{2n}|$ (because the exact solution for this example is not available).

5. Conclusion

In this paper, a numerical method based on a quintic C^1-spline collocation is proposed for the numerical solution of NDDEs. Our present methods have convergence of order five. Numerical results are presented in tables. It can be observed from the tables that the results of the present method are very encouraging.

References

[1] S.M. Abo-Hasha, Numerical solution of delay differential and neutral differential equations using spline methods. Ph. D Thesis, Assuit University, 1992.

[2] A. Bellen and M. Zennaro, Numerical Methods for Delay Differential Equations. Oxford University Press, Oxford, 2003.

[3] F. Calio, E. Marchetti, and R. Pavani, About the defficient spline collocation method for particular differential and integral equations with delay. Rend. Sem. Mat. Univ. Pol. Torino, 61 (2003) 287-300.

[4] H.M. El-Hawary and S.M. Mahmoud, Spline collocation methods for solving delay-differential equations. Appl. Math. Comput. 146 (2003) 359-372.

[5] H.M. El-Hawary and K.A. El-Shami, Spline collocation methods for solving second order neutral delay differential equations. Int. J. Open Problems Compt. Math. 2 (2009) 536-545.

[6] H.M. El-Hawary and K.A. El-Shami, Quintic C^1-spline collocation methods for stiff delay differential equations. Int. J. Open Problems Compt. Math. 5 (2012) 58-72.

[7] H.M. El-Hawary and K.A. El-Shami, Sextic C^1-spline collocation methods for solving delay differential equations. Int. J. Comput. Math. 89 (2012) 679-690.

[8] V. Kolmanovskii and A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations. Kluwer Academic Publishers, Dordrecht, 1999.

[9] S. Sallam and M.N. Anwar, Quintic C^2-spline integration methods for solving second-order ordinary initial value problems. J. Comput. Appl. Math. 115 (2000) 495-502.

[10] W. Wang and S.F. Li, Convergence of waveform relaxation methods for neutral delay differential equations. Math. Comput. Model. 48 (2008) 1875-1887.

[11] W. Wang, S.F. Li, and K. Su, Nonlinear stability of Runge-Kutta methods for neutral delay differential equations. J. Comput. Appl. Math. 214 (2008) 175-185.

[12] W. Wang, Y. Zhang, and S.F. Li, Stability of continuous Runge-Kutta-type methods for nonlinear neutral delay-differential equations. Appl. Math. Model. 33 (2009) 3319-3329.

[13] S. Wu and C. Huang, Convergence analysis of waveform relaxation methods for neutral differential-functional systems. J. Comput. Appl. Math. 223 (2009) 263-277.

Appendices

A. Appendix In this Appendix, we give the $A(\xi), ...E(\xi)$.

$$\begin{aligned}
A(\xi) &= \xi - \frac{11}{2}\xi^2 + \frac{34}{3}\xi^3 - 10\xi^4 + \frac{16}{5}\xi^5,\\
B(\xi) &= 6.82842712474616\xi^2 - 18.99018758282566\xi^3 + 18.82842712474616\xi^4 - \frac{32}{5}\xi^5,\\
C(\xi) &= -2\xi^2 + 12\xi^3 - 16\xi^4 + 6.4\xi^5,\\
D(\xi) &= 1.17157287525384\xi^2 - 7.67647908384101\xi^3 + 13.17157287525384\xi^4 - 6.4\xi^5,\\
E(\xi) &= -0.5\xi^2 + \frac{10}{3}\xi^3 - 6\xi^4 + \frac{16}{5}\xi^5.
\end{aligned} \tag{5.1}$$

B. Appendix In this Appendix, we give the M_1 and the matrix M_2.

$$M_1 = (0.05970177968644, 1/60, 0.03613155364689, 1/30)^T,$$

$$M_2 = \begin{bmatrix} 0.09503171601907 & -0.01213203435596 & 0.00664336837074 & 0.00279822031356 \\ 0.31011002862997 & 0.2 & -0.04344336196330 & 0.01666666666667 \\ 0.26002329829592 & 0.41213203435597 & 0.17163495064760 & -0.02636844635311 \\ 4/15 & 2/5 & 4/15 & 1/30 \end{bmatrix}.$$