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EQUATION OF GEODESIC FOR A $(\alpha,\ \beta)$ −METRIC IN A TWO-DIMENSIONAL FINSLER SPACE

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Abstract: In the present paper we have found out the equation of geodesic for a more general $(\alpha,\ \beta)$ −metric as compared to Randers, Kropina and Matsumoto mertric under the same conditions as for the Randers, Kropina and Matsumoto metric, the geodesic of the two-dimensional space with following metrics are the same as that of Matsumoto metric

$$L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$$

$$L = \frac{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}{\alpha + \beta}$$

We have also deal the geodesic of two-dimensional Finsler space with metric,

$$L = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$$

All the above three metrics are special form of the general metric,

$$L = \frac{k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2}{a_1\alpha + a_2\beta}$$

where$a$'s and k's are constants.

Keywords: $(\alpha,\ \beta)$ −metric, geodesic, two-dimensional Finsler space

2000 AMS Subject Classifications: 53B20, 53B40

0. Introduction

In the year 1997 Matsumoto and Park [1] obtained the equation of geodesic in two-dimensional Finsler spaces with the Randers metric $(L = \alpha + \beta)$ and the Kropina metric

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$(L = \alpha^2/\beta)$, whereas in 1998, they have [2] obtained the equation of geodesic in two-dimensional Finsler space with the slope metrics, i.e. Matsumoto metric given by $L = \alpha^2/(\alpha - \beta)$, by considering β as an infinitesimal of degree one and neglecting infinitesimal of degree more than two they obtained the equations of geodesic of two-dimensional Finsler space in the form $y'' = f(x,\ y,\ y')$, where $(x,\ y)$ are the co-ordinate of two-dimensional Finsler space.

In the present paper we have shown that under the same conditions as for the Matsumoto metric, the equations of geodesic of the two-dimensional space with following metrics are the same as that of Matsumoto metric

$$L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$$

$$L = \frac{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}{\alpha + \beta}$$

We have also deal the geodesic of two-dimensional Finsler space with metric,

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All the above three metrics are special form of the general metric,

$$L = \frac{k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2}{a_1\alpha + a_2\beta}$$

where$a$'s and k's are constants.

1. Preliminaries

We consider a two-dimensional Finsler space $F^2 = (M^2, L(x,y))$ with the $(\alpha, \beta)-$metric ([3], [4], [5], [7]) where $\alpha = \sqrt{a_{ij}(x)\dot{x}^i\dot{x}^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form on M^2. The space $R^2 = (M^2, \alpha)$ is said to be a Riemannian space associated to F^2.

M. Matsumoto constructed the problem in his paper [2] as follows:

(I). The underlying manifold M^2 is thought of as a surface S of the ordinary 3-space with an orthonormal co-ordinate system (X^α), $\alpha = 1,2,3$, which by the parametric equation $(X^\alpha) = X^\alpha(x^1,\ x^2)$. Then S is equipped with the induced Riemannian metric α. Thus two tangent vector field B_i, $i = 1,2$, are given with the components $B_i^\alpha = \frac{\partial X^\alpha}{\partial x^i}$ and then $a_{ij} = \sum_\alpha B_i^\alpha B_j^\alpha$. Let $N = (N^\alpha)$ be the unit normal vector to S.

In S an isothermal co-ordinate system $x^i = (x,\ y)$ may be referred in which α is of the form $\alpha = aE$, where $a = a(x,\ y)$ is a positive-valued function and $E = \sqrt{\dot{x}^2 + \dot{y}^2}$. Then the Christoffel symbols $\gamma_{jk}^i(x,\ y)$ of S in (x^i) are given by $(\gamma_{11}^1,\ \gamma_{12}^1,\ \gamma_{22}^1;\ \gamma_{11}^2,\ \gamma_{12}^2,\ \gamma_{22}^2) = \left(a_x,\ a_y,\ -a_x,\ -a_y,\ a_x,\ a_y\right)/a$. We shall denote by (;) the covariant differentiation with respect to Christoffel symbols in R^2.

(II). Let$B = (B^\alpha)$ be a constant vector field in the ambient 3-space, and put,

(1.1) $$B = b^i B_i + b^0 N$$

along S. Then the tangential component of B gives to the linear form,

(1.2) $$\beta = b_i \dot{x}^i, \qquad b_i = a_{ij} b^j$$

The Gauss-Weingarten derivation formulae lead from (1.1) to,

$$B_{;j} = \left(b_{;j}^i B_i + b^i H_{ij} N\right) + (b_{;j}^0 N - b^0 H_j^i B_i)$$

where, H_{ij} is the second fundamental tensor of S and $H_{ij} = a_{ik} H_j^k$. From, $B_{;j} = 0$, we get $b_{;j}^i = b^0 H_j^i$, i.e.,

(1.3) $$b_{i;j} = b^0 H_{ij}.$$

Consequently, we have, $b_{i;j} = b_{j;i}$ i.e. $b_{1y} = b_{2x}$ and hence b_i is a gradient vector field in S.

(III). The linear form β was originally to be induced one in S by the earth's gravity [4]. Hence it is here assumed that the constant vector field B is parallel to the X^3 −axis, i.e. $B^\alpha = (0,\ 0, -G), G = \text{const.} > 0$. Then from (1.1) we have $G^2 = a_{ij} b^i b^j + (b^0)^2$. Since $(a_{11},\ a_{12},\ a_{22}) = (a^2,\ 0,\ a^2)$, then

$$\left(\frac{G}{a}\right)^2 = (b^1)^2 + (b^2)^2 + \left(\frac{b^0}{a}\right)^2.$$

We shall regard the quantity $\frac{G}{a}$ as an infinitesimal of degree one, and neglect the infinitesimal of degree more than two. Then it is natural from the above that b^1, b^2 and $\frac{b^0}{a}$ are also those of degree one. Further (1.3) shows that $\beta_{;0}/a = \left(b_{i;j}\dot{x}^i\dot{x}^j\right)/a$ may be regarded as an infinitesimal of degree one. Consequently,

(1.4) $$\lambda = \beta/a^2\,, \qquad \mu = \gamma/a^2\,, \qquad \nu = \beta_{;0}/a$$

are infinitesimals of degree one, whereas $\gamma = b_1\dot{y} - b_2\dot{x}$. Thus,

(I) $\alpha$is the induced Riemannian metric in a surface S and, in particular $\alpha = aE$.

(II) $\beta$is the linear form in $(\dot{x}^i)$ induced from a constant vector field $(0,\ 0, -G)$ by (1.1) and (1.2).

(III) λ, μ, and ν of (1.4) are regarded as infinitesimals of degree one, and infinitesimals of degree more than two are neglected.

2. Special $(\alpha,\ \beta)$ −metric

Here we shall consider the special $(\alpha,\ \beta)$ −metric

(2.1) $$\boldsymbol{L = \frac{k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2}{a_1\alpha + a_2\beta}},$$

where$a$'s and k's are constants. It is obvious that by homothetic change of $\alpha$and β. This kind of metric may be classified as follows:

(I)$a_1 \neq 0,\ a_2 = 0$, we have the Randers metric $L = \alpha + \beta$,

(2.2) $$\boldsymbol{L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}}$$

(II)$a_1 = 0,\ a_2 \neq 0$, we have the Randers metric $L = \alpha + \beta$,

(2.3) $$\boldsymbol{L = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}}$$

(III)$a_1, a_2 \neq 0$, we have,

(2.4) $$\boldsymbol{L = \frac{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}{\alpha + \beta}}$$

Remark: If, $c_1 = 1,\ c_2 = 1$, then (2.2) reduced to $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ which is the Matsumoto metric of second kind. If,$c_1 = 1$and $c_2 = 0$, then (2.2) reduced to $L = \alpha + \frac{\beta^2}{\alpha}$. This metric is also a very special metric introduced by Matsumoto [6]. If $k_1 = 1,\ k_2 = k_3 = 0$and $a_1 = 1 = -a_2$, then the metric (2.1) is reduced to Matsumoto metric, $L = \alpha^2/(\alpha - \beta)$.

Now, we study the geodesic in two-dimensional Finsler space with above metrics.

3. Geodesics of the special $(\boldsymbol{\alpha},\ \boldsymbol{\beta})$ −metric

M. Matsumoto in his paper [1] found out the differential equation of the geodesic in an isothermal co-ordinate system $(x^i) = (x,\ y)$ for the $(\alpha,\ \beta)$ −metric is

(3.1) $$(L_\alpha + aEw\gamma^2)Ri(C) - \beta_{;0}a^2w\gamma - L_\beta(b_{1y} - b_{2x}) = 0$$

where we put, $w = \frac{L_{\alpha\alpha}}{\beta^2} = -\frac{L_{\alpha\beta}}{\alpha\beta} = \frac{L_{\beta\beta}}{\alpha^2}$

$$Ri(C) = a(\dot{x}\ddot{y} - \dot{y}\ddot{x})/E^3 + (a_x\dot{y} - a_y\dot{x})/E$$

It is remarked that the equation $Ri(C) = 0$ gives the geodesic of the associated Riemannian space.

Now according to the above contribution (3.1) may be written for the metric (2.2) in the form

$$\boldsymbol{\left(1 - \frac{a^2\lambda^2}{c_1E^2} + \frac{2a^2\mu^2}{c_1E^3}\right)Ri(C) = \frac{2a^2\mu\nu}{c_1E^3}}$$

Let us neglect the infinitesimals of degree more than two. Then we have

(3.2) $$\boldsymbol{Ri(C) = \frac{2a^2\mu\nu}{c_1E^3}}$$

Remark: If we take $c_1 = 1$ then equation (3.2) reduced to $Ri(C) = \frac{2a^2\mu\nu}{E^3}$ that is the result is reduced as for the Matsumoto metric [2].

Therefore on our construction, we obtain the approximate equation of geodesics in the form

(3.3) $$\boldsymbol{y'' = \frac{2}{c_1a^2}\beta^*_{;0}\gamma^* - \frac{1}{a}(E^*)^2(a_xy' - a_y)}$$

where,

(3.3)' $$\begin{cases} y' = dy/dx, \quad E^* = \sqrt{1 + (y')^2}, \quad \gamma^* = b_1y' - b_2 \\ \beta^*_{;0} = b_{1;1} + (b_{1;2} + b_{2;1})y' + b_{2;2}(y')^2 \end{cases}$$

Next if we take the metric (2.3) then the differential equation (3.1) of geodesic is written as

$$\boldsymbol{(c_1a\lambda^3 + 2E\lambda^2 + 2E\mu^2)Ri(C) = 2\mu\nu}$$

Let us neglect the infinitesimals of degree more than two. Then we have

(3.4) $$\boldsymbol{Ri(C)} = \frac{\mu\nu}{E(\lambda^2+\mu^2)}$$

Therefore on our construction, we obtain the approximate equation of geodesics in the form

(3.5) $$\boldsymbol{y''} = \frac{1}{b_1^2+b_2^2}\boldsymbol{\beta^*_{;0}\gamma^*} - \frac{1}{a}(\boldsymbol{E^*})^2(\boldsymbol{a_x y' - a_y})$$

where $\beta^*_{;0}$, γ^*, E^* and y' are given in (3.3)'.

Again if we take the metric (2.4) then the differential equation of (3.1) of geodesic is written as

$$\left(\boldsymbol{c_1 E + 2c_1 a\lambda} + \frac{a^2\lambda^2(c_2-c_3)}{E} + \frac{2a^2\mu^2(c_1-c_2+c_3)}{(E+a\lambda)}\right)\boldsymbol{Ri(C)} = \frac{2a^2\mu\nu(c_1-c_2+c_3)}{E(E+a\lambda)}$$

Let us neglect the infinitesimals of degree more than two. Then we have

$$\frac{\mu^2}{(E+a\lambda)} = \frac{\mu^2}{E}, \qquad \frac{\mu\nu}{E(E+a\lambda)} = \frac{\mu\nu}{E^2}$$

Thus the equation is reduced to

(3.6) $$\boldsymbol{Ri(C)} = \frac{2a^2\mu\nu(c_1-c_2+c_3)}{E^3}$$

Remark: If we take $c_1 = 0$and $c_2 = c_3 = 0$, then equation (3.6) is reduced to, $Ri(C) = \frac{2a^2\mu\nu}{E^3}$ that is the result for Matsumoto metric [2].

Therefore on our construction, we obtain the approximate equation of geodesics in the form

(3.7) $$\boldsymbol{y''} = \frac{2(c_1-c_2+c_3)}{c_1 a^2}\boldsymbol{\beta^*_{;0}\gamma^*} - \frac{1}{a}(\boldsymbol{E^*})^2(\boldsymbol{a_x y' - a_y})$$

where $\beta^*_{;0}$, γ^*, E^* and y' are given in (3.3)'.

4. Some Examples

In the following we shall use the notation as follows:

$$(X^\alpha) = (X,\ Y,\ Z), \qquad (x^i) = (x,\ y)$$

Example 1 We consider the circular cylinder S: $X^2 + Z^2 = 1,\ \ Y = y$, which is also written as

$$\text{S: } X = \cos x, \qquad Y = y, \qquad Z = \sin x$$

Then we get

$B_1 = (-\sin x,\ 0,\ \cos x), \qquad B_2 = (0,\ 1,\ 0), \qquad N = (\cos x\ \ 0,\ \sin x)$

$(a_{11},\ a_{12},\ a_{22}) = (1,\ 0,\ 1), \qquad (b^1,\ b^2,\ b^0) = (G\cos x,\ 0,\ -G\sin x)$

Consequently we have

$$\alpha^2 = dx^2 + dy^2, \qquad \beta = -G\cos x\ dx$$

Therefore (3.3) gives the approximate differential equation of geodesic is

(4.1) $$\boldsymbol{y'' + \frac{G^2}{c_1}(\sin 2x)y' = 0}$$

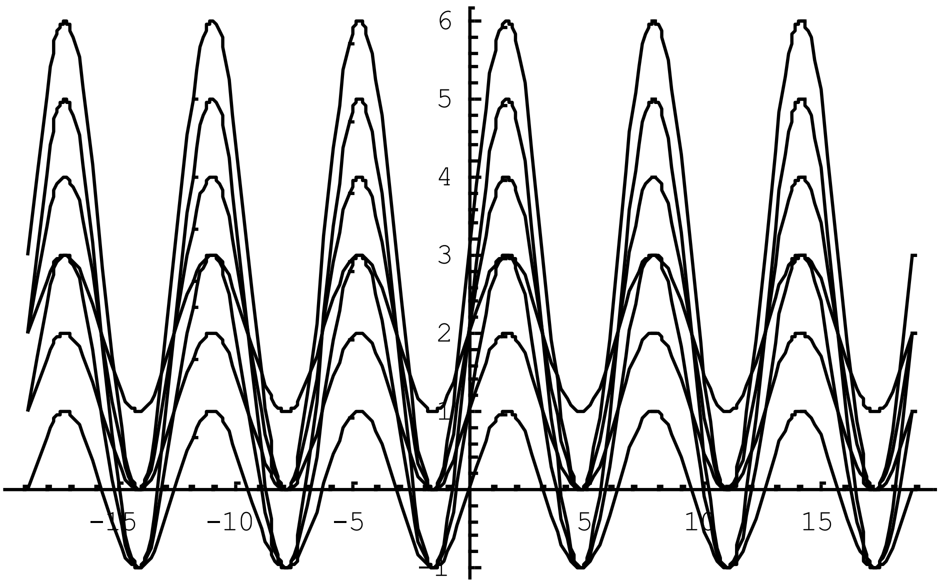
Next (3.5) gives the differential equation of geodesic is as

$$\boldsymbol{y'' + (\tan x)\ y' = 0}$$

which has the solution

(4.2) $$\boldsymbol{y = a' \sin x + b'}$$

where a' and b' are constants. The above equation shows sine curve which shown for different values of a' and b' are given below:



<u>Fig. 1</u> The solution of equation of geodesic for thecircular cylinder S: $\boldsymbol{X^2 + Z^2 = 1}$, $\boldsymbol{Y = y}$, behaves like Sine curves

Again (3.7) gives the approximate differential equation of geodesic is

(4.3) $$\boldsymbol{y'' + (c_1 - c_2 + c_3)G^2(\sin 2x)y' = 0}.$$

Next we are interested in revolution surfaces the axis of which is parallel to the constant vector field B. Such a surface S is given by

$$X = g(u)\cos y\,, \qquad Y = g(u)\sin y\,, \qquad Z = f(u)$$

Denoting (u, y) by (x^i) we have

$B_1 = (g'\cos y\,,\ g'\sin y\,,\ f'),$ $\qquad B_2 = (-g\sin y\,,\ g\cos y\,,\ 0)$

$N = (-f'\cos y\,,\ -f'\sin y,\ \ g')/F\,,$ $\qquad F = \sqrt{(f')^2 + (g')^2}$

$(a_{11},\ a_{12},\ a_{22}) = (F^2,\ 0,\ g^2),$ $\quad (b^1,\ b^2,\ b^0) = \left(-\frac{Gf'}{F},\ 0,\ -\frac{Gg'}{F}\right),$

$(b_1,\ b_2) = (Gf',\ 0)$

Consequently we get

$$\alpha^2 = F^2du^2 + g^2dy^2, \qquad \beta = -Gf'\,du.$$

We need an isothermal co-ordinate system if we take

(4.4) $$x = \int \frac{F}{g}\,du$$

Then we obtain

(4.5) $$\alpha^2 = g(u)^2(dx^2 + dy^2), \qquad \beta = -G\,\frac{f'g}{F}\,dx$$

Example 2 We shall deal with the sphere, surface of constant curvature +1: $g(u) = \cos u$and $f(u) = \sin u$. Then $F = 1$ and (4.4) gives,

$$x = \int \frac{1}{\cos u}\,du = \frac{1}{2}\log\frac{1+\sin u}{1-\sin u}$$

Then $\frac{1+\sin u}{1-\sin u} = e^{2x}$ implies, $\frac{1}{\cos u} = \cosh u$, and hence $du = \frac{dx}{\cosh x}$. Consequently (4.5) leads to

$$\alpha^2 = \frac{1}{cosh^2x}(dx^2 + dy^2), \qquad \beta = -\frac{G}{cosh^2x}dx$$

Therefore (3.3) gives the approximate differential equation of geodesics in the form

(4.6) $$\boldsymbol{y'' = \tanh x\left(1 - \frac{2G^2}{c_1 cosh^2x}\right)\{y' + (y')^3\}}$$

Again (3.5) gives the approximate differential equation of geodesics in the form

$$\boldsymbol{y'' = 0}$$

which has the solution

(4.7) $$\boldsymbol{y = a'\,x + b'}$$

where a' and b' are constants. The above equation shows sin curve which shown for different values of a' and b' are given below

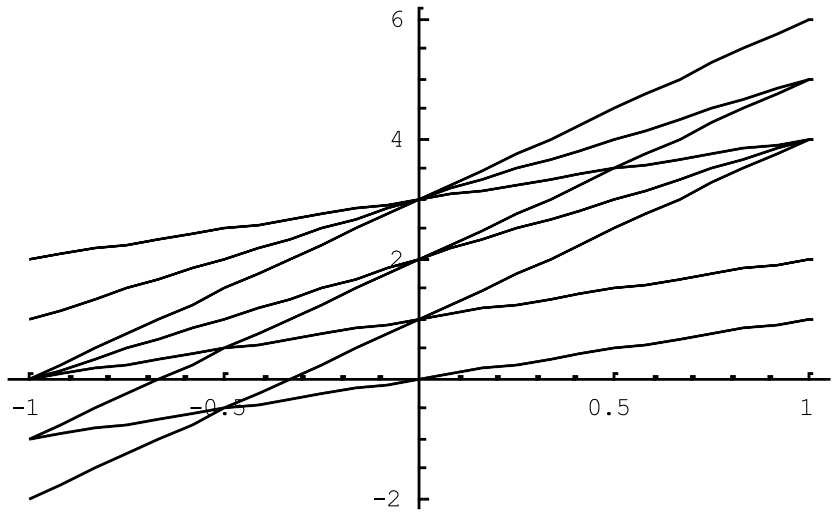


Fig. 2 **The solution of equation of geodesic for the sphere, surface of constant curvature +1: $g(u) = \cos u$ and $f(u) = \sin u$, behaves like Straight line**

Again (3.7) gives the approximate differential equation of geodesics in the form

(4.8) $$y'' = \tanh x\left(1 - \frac{2(c_1 - c_2 + c_3)G^2}{cosh^2 x}\right)\{y' + (y')^3\}$$

Example 3 We shall treat of the pseudo-sphere, surface of constant curvature -1: $g(u) = \cos u$and $f(u) = \log\tan\left(\frac{u}{2} + \frac{\pi}{4}\right) - \sin u$. We have, $f' = \frac{sin^2 u}{\cos u}$, $F = \frac{\sin u}{\cos u}$, and (4.4) gives $x = \frac{1}{\cos u}$. Therefore (4.5) leads to

$$\alpha^2 = \frac{1}{x^2}(dx^2 + dy^2), \quad \beta = -G\frac{\sqrt{x^2-1}}{x^2}dx$$

We shall exchange x and y as usual:

$$\alpha^2 = \frac{1}{y^2}(dx^2 + dy^2), \quad \beta = -G\frac{\sqrt{y^2-1}}{y^2}dy.$$

Then (3.3) yields the approximate differential equation of geodesic as

(4.9) $$y'' = -\frac{(y')^2 + 1}{y} - \frac{2G^2}{c_1 y^3}\{1 - y^2 + (y')^2\}$$

Again (3.5) gives the approximate differential equation of geodesics in the form

$$y'' + \frac{y(y')^2}{y^2 - 1} = 0$$

which, has the solution

$$\frac{1}{2}y\sqrt{y^2-1}-\frac{1}{2}\log\left(y+\sqrt{y^2-1}\right)=a'\,x+b' \tag{4.10}$$

where, a' and b' are constants.

Again (3.7) gives the approximate differential equation of geodesics in the form,

$$y''=-\frac{(y')^2+1}{y}-\frac{2(c_1-c_2+c_3)G^2}{y^3}\{1-y^2+(y')^2\}. \tag{4.11}$$

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