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FINITE ITERATIVE ALGORITHM FOR SOLVING THE GENERALIZED COUPLED SYLVESTER – CONJUGATE MATRIX EQUATIONS

$A_1V + B_1W = E_1\overline{V}F_1 + C_1$ AND $A_2V + B_2W = E_2\overline{V}F_2 + C_2$

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Abstract. In this paper, we consider an iterative algorithm for solving a generalized coupled Sylvester– conjugate matrix equation. With the iterative algorithm, the existence of a comman solution of these two matrix equation can be determined automatically. When these two matrix equations are consistent, for any initial matrices V_1, W_1 the solutions can be obtained by iterative algorithm within finite iterative steps in the absence of round off errors. Some lemmas and theorems are stated and proved where the iterative solutions are obtained. A numerical example is given to illustrate the effectiveness of the proposed method and to support the theoretical results of this paper.

Keywords: coupled Sylvester–conjugate matrix equation; iterative algorithm; inner product; frobenius norm.

1. Introduction

Consider the generalized coupled Sylvester – conjugate matrix equation

$$\begin{aligned} A_1V + B_1W &= E_1\overline{V}F_1 + C_1, \\ A_2V + B_2W &= E_2\overline{V}F_2 + C_2, \end{aligned} \tag{1}$$

where $A_1, E_1, A_2, E_2 \in \mathbb{C}^{n\times n}$, $B_1, B_2 \in \mathbb{C}^{n\times r}$, $F_1, F_2 \in \mathbb{C}^{p\times p}$ and $C_1, C_2 \in \mathbb{C}^{n\times p}$ are given matrices, while $V \in \mathbb{C}^{n\times p}$ and $W \in \mathbb{C}^{r\times p}$ are matrices to be determined. Matrix equations are often encountered in many areas of computational mathematics, control and system theory. Research on solving linear matrix equations has been actively engaged in for many years. For example, Navarra et al.

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studied a representation of the general common solution of the matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$ [1]; van der Woude obtained the existence of a common solution X for matrix equations $A_iXB_j = C_{ij}$ [2]; Bhimasankaram considered the linear matrix equation $AX = C, XB = D$ and $FXG = H$ [3]. Mitra has provided conditions for the existence of a solution and a representation of the general common solution of the matrix equations $AX = C, XB = D$ and the matrix equation $A_1XB_1 = C_1, A_2XB_2 = C_2$ [4, 5]. Ramadan et al. [6] introduced a complete, general and explicit solution to the Yakubovich matrix equation $V - AVF = BW$, the matrix equation $(AXB, GXH) = (C, D)$ have some important results have been developed. In [7], necessary and sufficient conditions for its solvability and the expression of the solution were derived by means of generalized inverse. Moreover, in [7] the least-squares solution was also obtained by using the generalized singular value decomposition. While in [8], when this matrix equation is consistent, the minimum-norm solution was given by the use of the canonical correlation decomposition. In [9], based on the projection theorem in Hilbert space, an analytical expression of the least-squares solution was given for the matrix equations $(AXB, GXH) = (C, D)$ by making use of the generalized singular value decomposition and the canonical correlation decomposition. In [10], by using the matrix rank method a necessary and sufficient condition was derived for the matrix equations $AX_1B = C$ and $GX_2H = D$ to have a common least square solution. In the aforementioned methods, the coefficient matrices of the considered equations are required to be firstly transformed into some canonical forms. Recently, an iterative algorithm was presented in [11] to solve the matrix equation $(AXB, CXD) = (E, F)$. Different from the above mentioned methods, this algorithm can be implemented by initial coefficient matrices, and can provide a solution within finite iteration steps for any initial values.

Based on the iterative solutions of matrix equations, Ding and Chen presented the hierarchical gradient iterative algorithms for general matrix equations [12,13] and hierarchical least squares iterative algorithms for generalized coupled Sylvester matrix equations and general coupled matrix equations [14,15]. The hierarchical gradient iterative algorithms [12,13] and hierarchical least squares iterative algorithms [12,15,16] for solving general (coupled) matrix equations are innovational and computationally efficient numerical ones and were proposed based on the hierarchical identification principle [14,17] which regards the unknown matrix as the system parameter matrix to be identified. The generalized Sylvester matrix equations (1) have very wide application in many problems such as pole/eigenstructure assignment design [18, 19], observer design [20].

This paper is organized as follows: First, in section 2, we introduce some notations, a lemma and a theorem that will be needed to develop this work. In section 3, we propose iterative methods to obtain numerical solution to the generalized coupled Sylvester–conjugate matrix equation $A_1V + B_1W = E_1\overline{V}F_1 + C_1$ and $A_2V + B_2W = E_2\overline{V}F_2 + C_2$ using iterative method. In section 4, numerical example is given to explore the simplicity and the neatness of the presented methods.

2. Preliminaries

The following notations, definitions, lemmas and theorems will be used to develop the proposed work. We use $A^T, \overline{A}, A^H$ and $tr(A)$ to denote the transpose, conjugate, conjugate transpose and the trace of a matrix A respectively. We denote the set of all $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$, $\mathrm{Re}(a)$ denote the real part of number a

Definition 1 Inner product [21]

A real inner product space is a vector space V over the real field $\mathbb{R}$ together with an inner product that is with a map

$$\langle .,. \rangle : V \times V \to \mathbb{R}$$

Satisfying the following three axioms for all vectors $x, y, z \in V$ and all scalars $a \in \mathbb{R}$

(1) Symmetry: $\langle x, y\rangle = \langle y, x\rangle$.

(2) Linearity in the first argument:

$$\langle ax, y\rangle = a\langle x, y\rangle, \quad \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle.$$

(3) Positive definiteness: $\langle x, x\rangle > 0$ for all $x \neq 0$.

The following theorem defines a real inner product on space $\mathbb{C}^{m\times n}$ over the field $\mathbb{R}$

Theorem 1 [22]

In the space $\mathbb{C}^{m\times n}$ over the field $\mathbb{R}$, an inner product can be defined as

$$\langle A, B\rangle = \mathrm{Re}[tr(A^H B)] \tag{2}$$

Proof

(1) For $A, B \in \mathbb{C}^{m\times n}$, according to the properties of trace of a matrix one has

$$\langle A, B\rangle = \mathrm{Re}[tr(A^H B)] = \mathrm{Re}[tr(B^T \overline{A})] = \mathrm{Re}[\overline{tr(B^T \overline{A})}]$$

$$= \mathrm{Re}[tr(B^H A)] = \langle B, A\rangle.$$

(2) For a real number a, and $A, B, C \in \mathbb{C}^{m\times n}$, one has

$$\langle aA, B\rangle = \mathrm{Re}[tr((aA)^H B)] = \mathrm{Re}[tr(aA^H B)] = \mathrm{Re}[a\, tr(A^H B)]$$

$$= a\ \mathrm{Re}[tr(A^H B)] = a\langle A, B\rangle.$$

$$\langle A+B, C\rangle = \mathrm{Re}[tr((A+B)^H C)] = \mathrm{Re}[tr(A^H + B^H)C]$$

$$= \mathrm{Re}[tr(A^H C)] + \mathrm{Re}[tr(B^H C)] = \langle A, C\rangle + \langle B, C\rangle.$$

(3) It is well-known that $tr(A^H A) > 0$ for all $x \neq 0$. Thus, $\langle A, A\rangle = \mathrm{Re}[tr(A^H A)] > 0$ for all $x \neq 0$.

According to definition 1, all the above argument reveals that the space $\mathbb{C}^{m\times n}$ over field $\mathbb{R}$ with the inner product defined by (2) is an inner product space. The Frobenius norm of A is denoted by $\|A\|$, that is $\|A\| = \sqrt{tr(A^H A)}$

3. Main results

In this section, we propose an iterative solution to the generalized coupled Sylvester – conjugate matrix equation

$$\begin{aligned} A_1V + B_1W &= E_1\overline{V}F_1 + C_1, \\ A_2V + B_2W &= E_2\overline{V}F_2 + C_2, \end{aligned} \tag{1}$$

where $A_1, E_1, A_2, E_2 \in \mathbb{C}^{n\times n}$, $B_1, B_2 \in \mathbb{C}^{n\times r}$, $F_1, F_2 \in \mathbb{C}^{p\times p}$ and $C_1, C_2 \in \mathbb{C}^{n\times p}$ are given matrices, while $V \in \mathbb{C}^{n\times p}$ and $W \in \mathbb{C}^{r\times p}$ are matrices to be determined.

Let $f(V,W) = A_1V + B_1W - E_1\overline{V}F_1$,

and $g(V,W) = A_2V + B_2W - E_2\overline{V}F_2$.

We introduce the following finite iterative algorithm to solve the generalized coupled Sylvester – conjugate matrix equation (1)

Algorithm I

1. Input $A_1, E_1, A_2, E_2, B_1, B_2, C_1, C_2$;
2. Chosen arbitrary matrices $V_1 \in \mathbb{R}^{n\times p}$ and $W_1 \in \mathbb{R}^{r\times p}$;
3. set

$$R_1 = diag(C_1 - f(V_1, W_1)\ ,\ C_2 - g(V_1, W_1));$$

$$S_1 = A_1^H(C_1 - f(V_1, W_1)) - \overline{E}_1^H \overline{(C_1 - f(V_1, W_1))} \overline{F}_1^H$$

$$+ A_2^H(C_2 - g(V_1, W_1)) - \overline{E}_2^H \overline{(C_2 - g(V_1, W_1))} \overline{F}_2^H;$$

$$T_1 = B_1^H(C_1 - f(V_1, W_1)) + B_2^H(C_2 - g(V_1, W_1));$$

$$k := 1;$$

4. If $R_k = 0$, then stop and V_k , W_k are the solution ; else let $k := k + 1$ go to STEP 5
5. compute

$$V_{k+1} = V_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k;$$

$$W_{k+1} = W_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} T_k;$$

$$R_{k+1} = diag(C_1 - f(V_{k+1}, W_{k+1})\ , C_2 - g(V_{k+1}, W_{k+1}))$$

$$= R_k - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2}\ diag(f(S_k, T_k)\ ,\ g(S_k, T_k));$$

$$S_{k+1} = A_1^H(C_1 - f(V_{k+1}, W_{k+1})) - \overline{E}_1^H \overline{(C_1 - f(V_{k+1}, W_{k+1}))} \overline{F}_1^H$$

$$+ A_2^H(C_2 - g(V_{k+1}, W_{k+1})) - \overline{E}_2^H \overline{(C_2 - g(V_{k+1}, W_{k+1}))} \overline{F}_2^H + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} S_k;$$

$$T_{k+1} = B_1^H(C_1 - f(V_{k+1}, W_{k+1})) + B_2^H(C_2 - g(V_{k+1}, W_{k+1})) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} T_k;$$

6. If $R_{k+1} = 0$, then stop; else let $k = k + 1$ go to STEP 5.

To prove the convergence property of Algorithm I, we first establish the following basic properties

Lemma 1.

Suppose that the system of matrix equations (1) is consistent and let V^*, W^* be its arbitrary solutions. Then for any initial matrices V_1 and W_1, we have

$$tr[S_i^H(V^* - V_i) + T_i^H(W^* - W_i)] + \overline{tr[S_i^H(V^* - V_i) + T_i^H(W^* - W_i)]} = 2\|R_i\|^2 \tag{3}$$

Or, equivalently

$$\mathrm{Re}\{tr[S_i^H(V^* - V_i) + T_i^H(W^* - W_i)]\} = \|R_i\|^2,$$

where the sequences $\{V_i\},\{S_i\},\{W_i\},\{T_i\}$ and $\{R_i\}$ are generated by Algorithm I for $i=1,2,\ldots$

Proof

We apply mathematical induction

For $i=1$, from Algorithm I one has

$$\begin{aligned}
tr[S_1^H(V^*-V_1)+T_1^H(W^*-W_1)] &= tr[(A_1^H(C_1-f(V_1,W_1))-\overline{E}_1^H\overline{(C_1-f(V_1,W_1))}\overline{F}_1^H \\
&\quad + A_2^H(C_2-g(V_1,W_1))-\overline{E}_2^H\overline{(C_2-g(V_1,W_1))}\overline{F}_2^H)^H(V^*-V_1) \\
&\quad + (B_1^H(C_1-f(V_1,W_1))+B_2^H(C_2-g(V_1,W_1)))^H(W^*-W_1)] \\
&= tr[(C_1-f(V_1,W_1))^H(A_1(V^*-V_1)+B_1(W^*-W_1)) \\
&\quad + (C_2-g(V_1,W_1))^H(A_2(V^*-V_1)+B_2(W^*-W_1)) \\
&\quad - \overline{(C_1-f(V_1,W_1))}^H(\overline{E}_1(V^*-V_1)\overline{F}_1)-\overline{(C_2-g(V_1,W_1))}^H(\overline{E}_2(V^*-V_1)\overline{F}_2)]
\end{aligned}$$

In view that V^*,W^* are solutions of the generalized coupled Sylvester – conjugate matrix equation (1), it is easy one can obtain from above relation

$$\begin{aligned}
& tr[S_1^H(V^*-V_1)+T_1^H(W^*-W_1)]+\overline{tr[S_1^H(V^*-V_1)+T_1^H(W^*-W_1)]} = tr[(C_1-f(V_1,W_1))^H \\
&\quad (A_1(V^*-V_1)+B_1(W^*-W_1))+(C_2-g(V_1,W_1))^H(A_2(V^*-V_1)+B_2(W^*-W_1)) \\
&\quad -\overline{(C_1-f(V_1,W_1))}^H(\overline{E}_1(V^*-V_1)\overline{F}_1)-\overline{(C_2-g(V_1,W_1))}^H(\overline{E}_2(V^*-V_1)\overline{F}_2)] \\
&\quad +\overline{tr[(C_1-f(V_1,W_1))^H(A_1(V^*-V_1)+B_1(W^*-W_1))} \\
&\quad +\overline{(C_2-g(V_1,W_1))^H(A_2(V^*-V_1)+B_2(W^*-W_1))} \\
&\quad -\overline{\overline{(C_1-f(V_1,W_1))}^H(\overline{E}_1(V^*-V_1)\overline{F}_1)-\overline{(C_2-g(V_1,W_1))}^H(\overline{E}_2(V^*-V_1)\overline{F}_2)]} \\
&= tr[(C_1-f(V_1,W_1))^H(A_1V^*+B_1W^*-E_1\overline{V^*}F_1-A_1V_1-B_1W_1+E_1\overline{V}_1F_1) \\
&\quad +(C_2-g(V_1,W_1))^H(A_2V^*+B_2W^*-E_2\overline{V^*}F_2-A_2V_1-B_2W_1+E_2\overline{V}_1F_2)] \\
&\quad +\overline{tr[(C_1-f(V_1,W_1))^H(A_1V^*+B_1W^*-E_1\overline{V^*}F_1-A_1V_1-B_1W_1+E_1\overline{V}_1F_1)} \\
&\quad +\overline{(C_2-g(V_1,W_1))^H(A_2V^*+B_2W^*-E_2\overline{V^*}F_2-A_2V_1-B_2W_1+E_2\overline{V}_1F_2)]} \\
&= tr[(C_1-f(V_1,W_1))^H(C_1-f(V_1,W_1))+(C_2-g(V_1,W_1))^H(C_2-g(V_1,W_1))] \\
&\quad +\overline{tr[(C_1-f(V_1,W_1))^H(C_1-f(V_1,W_1))+(C_2-g(V_1,W_1))^H(C_2-g(V_1,W_1))]}
\end{aligned}$$

$$= tr[\begin{bmatrix} C_1 - f(V_1,W_1) & 0 \\ 0 & C_2 - g(V_1,W_1) \end{bmatrix}^H \begin{bmatrix} C_1 - f(V_1,W_1) & 0 \\ 0 & C_2 - g(V_1,W_1) \end{bmatrix}]$$

$$+ \overline{tr[\begin{bmatrix} C_1 - f(V_1,W_1) & 0 \\ 0 & C_2 - g(V_1,W_1) \end{bmatrix}^H \begin{bmatrix} C_1 - f(V_1,W_1) & 0 \\ 0 & C_2 - g(V_1,W_1) \end{bmatrix}]}$$

$$= tr(R_1^H R_1) + tr(\overline{R}_1^H \overline{R}_1) = 2\|R_1\|^2$$

This implies that (3) holds for $i = 1$.

Now assume that (3) holds for $i = k$. That is,

$$tr[S_k^H (V^* - V_k) + T_k^H (W^* - W_k)] + \overline{tr[S_k^H (V^* - V_k) + T_k^H (W^* - W_k)]} = 2\|R_k\|^2$$

Then we have to prove that the conclusion holds for $i = k+1$. It follows from Algorithm I that

$$tr[S_{k+1}^H (V^* - V_{k+1}) + T_{k+1}^H (W^* - W_{k+1})] = tr[(A_1^H (C_1 - f(V_{k+1}, W_{k+1})) - \overline{E}_1^H \overline{(C_1 - f(V_{k+1}, W_{k+1}))} \overline{F}_1^H$$

$$+ A_2^H (C_2 - g(V_{k+1}, W_{k+1})) - \overline{E}_2^H \overline{(C_2 - g(V_{k+1}, W_{k+1}))} \overline{F}_2^H + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} S_k)^H (V^* - V_{k+1})$$

$$+ (B_1^H (C_1 - f(V_{k+1}, W_{k+1})) + B_2^H (C_2 - g(V_{k+1}, W_{k+1})) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} T_k)^H (W^* - W_{k+1})]$$

$$= tr[(C_1 - f(V_{k+1}, W_{k+1}))^H (A_1 (V^* - V_{k+1}) + B_1 (W^* - W_{k+1})) + (C_2 - g(V_{k+1}, W_{k+1}))^H (A_2 (V^* - V_{k+1})$$

$$+ B_2 (W^* - W_{k+1})) - \overline{(C_1 - f(V_{k+1}, W_{k+1}))}^H (\overline{E}_1 (V^* - V_{k+1}) \overline{F}_1) - \overline{(C_2 - g(V_{k+1}, W_{k+1}))}^H$$

$$(\overline{E}_2 (V^* - V_{k+1}) \overline{F}_2)] + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} tr[S_k^H (V^* - V_{k+1}) + T_k^H (W^* - W_{k+1})] \tag{4}$$

In view that V^*, W^* are solutions of the generalized coupled Sylvester – conjugate matrix equation (1), with relation (4) one has

$$tr[S_{k+1}^H (V^* - V_{k+1}) + T_{k+1}^H (W^* - W_{k+1})] + \overline{tr[S_{k+1}^H (V^* - V_{k+1}) + T_{k+1}^H (W^* - W_{k+1})]} = tr[(C_1 - f(V_{k+1}, W_{k+1}))^H$$

$$(A_1 (V^* - V_{k+1}) + B_1 (W^* - W_{k+1})) + (C_2 - g(V_{k+1}, W_{k+1}))^H (A_2 (V^* - V_{k+1}) + B_2 (W^* - W_{k+1}))$$

$$- \overline{(C_1 - f(V_{k+1}, W_{k+1}))}^H (\overline{E}_1 (V^* - V_{k+1}) \overline{F}_1) - \overline{(C_2 - g(V_{k+1}, W_{k+1}))}^H (\overline{E}_2 (V^* - V_{k+1}) \overline{F}_2)]$$

$$+ \overline{tr[(C_1 - f(V_{k+1}, W_{k+1}))^H (A_1 (V^* - V_{k+1}) + B_1 (W^* - W_{k+1}))}$$

$$\overline{+ (C_2 - g(V_{k+1}, W_{k+1}))^H (A_2 (V^* - V_{k+1}) + B_2 (W^* - W_{k+1}))}$$

$$\overline{- \overline{(C_1 - f(V_{k+1}, W_{k+1}))}^H (\overline{E}_1 (V^* - V_{k+1}) \overline{F}_1) - \overline{(C_2 - g(V_{k+1}, W_{k+1}))}^H (\overline{E}_2 (V^* - V_{k+1}) \overline{F}_2)]}$$

$$+ \frac{\|R_{k+1}\|^2}{\|R_k\|^2} tr[S_k^H (V^* - V_{k+1}) + T_k^H (W^* - W_{k+1}) + \overline{S_k^H (V^* - V_{k+1}) + T_k^H (W^* - W_{k+1})}]$$

$$= tr[(C_1 - f(V_{k+1},W_{k+1}))^H (A_1V^* + B_1W^* - E_1\overline{V^*}F_1 - A_1V_{k+1} - B_1W_{k+1} + E_1\overline{V_{k+1}}F_1)$$
$$+ (C_2 - g(V_{k+1},W_{k+1}))^H (A_2V^* + B_2W^* - E_2\overline{V^*}F_2 - A_2V_{k+1} - B_2W_{k+1} + E_2\overline{V_{k+1}}F_2)$$
$$+ \overline{tr[(C_1 - f(V_{k+1},W_{k+1}))^H (A_1V^* + B_1W^* - E_1\overline{V^*}F_1 - A_1V_{k+1} - B_1W_{k+1} + E_1\overline{V_{k+1}}F_1)}$$
$$\overline{+ (C_2 - g(V_{k+1},W_{k+1}))^H (A_2V^* + B_2W^* - E_2\overline{V^*}F_2 - A_2V_{k+1} - B_2W_{k+1} + E_2\overline{V_{k+1}}F_2)}]$$
$$+ \frac{\|R_{k+1}\|^2}{\|R_k\|^2} tr[S_k^H (V^* - V_k - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k) + T_k^H (W^* - W_k - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} T_k)]$$
$$+ \overline{\frac{\|R_{k+1}\|^2}{\|R_k\|^2} tr[S_k^H (V^* - V_k - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k) + T_k^H (W^* - W_k - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} T_k)]}$$
$$= tr[(C_1 - f(V_{k+1},W_{k+1}))^H (C_1 - f(V_{k+1},W_{k+1})) + (C_2 - g(V_{k+1},W_{k+1}))^H (C_2 - g(V_{k+1},W_{k+1}))]$$
$$+ \overline{tr[(C_1 - f(V_{k+1},W_{k+1}))^H (C_1 - f(V_{k+1},W_{k+1})) + (C_2 - g(V_{k+1},W_{k+1}))^H (C_2 - g(V_{k+1},W_{k+1}))]}$$
$$+ \frac{\|R_{k+1}\|^2}{\|R_k\|^2} tr\{[S_k^H (V^* - V_k) + T_k^H (W^* - W_k) + \overline{S_k^H (V^* - V_k) + T_k^H (W^* - W_k)}] -$$
$$- [\frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} ((S_k^H S_k + T_k^H T_k) + \overline{(S_k^H S_k + T_k^H T_k)})]\}$$
$$= tr[\begin{bmatrix} C_1 - f(V_{k+1},W_{k+1}) & 0 \\ 0 & C_2 - g(V_{k+1},W_{k+1}) \end{bmatrix}^H \begin{bmatrix} C_1 - f(V_{k+1},W_{k+1}) & 0 \\ 0 & C_2 - g(V_{k+1},W_{k+1}) \end{bmatrix}]$$
$$+ \overline{tr[\begin{bmatrix} C_1 - f(V_{k+1},W_{k+1}) & 0 \\ 0 & C_2 - g(V_{k+1},W_{k+1}) \end{bmatrix}^H \begin{bmatrix} C_1 - f(V_{k+1},W_{k+1}) & 0 \\ 0 & C_2 - g(V_{k+1},W_{k+1}) \end{bmatrix}]}$$
$$+ \frac{\|R_{k+1}\|^2}{\|R_k\|^2} \{2\|R_k\|^2 - [\frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} (2(\|S_k\|^2 + \|T_k\|^2))]\}$$
$$= tr(R_{k+1}^H R_{k+1}) + \overline{tr(R_{k+1}^H R_{k+1})} + 2\|R_{k+1}\|^2 - 2\|R_{k+1}\|^2 = 2\|R_{k+1}\|^2$$

This implies that (3) holds for $i = k+1$. Hence relation (3) holds by principle of induction.

Lemma 2.

Suppose that system of matrix equations (1) is consistent and the sequences $\{R_i\}$, $\{S_i\}$ and $\{T_i\}$ are generated by Algorithm I with any initial matrices V_1, W_1 , such that $R_i \neq 0$ for all $i = 1,2,\ldots,k$, then

$$\operatorname{Re}\{trace(R_j^H R_i)\} = 0 \qquad (5)$$

and $\mathrm{Re}\{trace(S_j^H S_i + T_j^H T_i)\} = 0$, for $i, j = 1,2,\ldots,k, \quad i \neq j$. (6)

Proof

We apply mathematical induction

Step 1: We prove

$$tr(R_{i+1}^H R_i) = 0 \tag{7}$$

and

$$tr(S_{i+1}^H S_i + T_{i+1}^H T_i) = 0 \tag{8}$$

for $i = 1,2,\ldots,k$.

First from Algorithm I we have

$$\begin{aligned}
R_{k+1} &= diag(C_1 - f(V_{k+1}, W_{k+1}) \,, C_2 - g(V_{k+1}, W_{k+1})) \\
&= diag(C_1 - A_1 V_{k+1} - B_1 W_{k+1} + E_1 \overline{V_{k+1}} F_1, C_2 - A_2 V_{k+1} - B_2 W_{k+1} + E_2 \overline{V_{k+1}} F_2) \\
&= diag(C_1 - A_1 (V_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k) - B_1 (W_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} T_k) + E_1 \overline{(V_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k)} F_1 \\
&\quad , C_2 - A_2 (V_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k) - B_2 (W_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} T_k) + E_2 \overline{(V_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k)} F_2) \\
&= diag(C_1 - A_1 V_k - B_1 W_k + E_1 \overline{V_k} F_1, C_2 - A_2 V_k - B_2 W_k + E_2 \overline{V_k} F_2) \\
&\quad - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} diag(A_1 S_k + B_1 T_k - E_1 \overline{S_k} F_1, A_2 S_k + B_2 T_k - E_2 \overline{S_k} F_2) \\
&= diag(C_1 - f(V_k, W_k) \,, C_2 - g(V_k, W_k)) - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} \; diag(f(S_k, T_k) \,, g(S_k, T_k)) \\
&= R_k - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} \; diag(f(S_k, T_k) \,, g(S_k, T_k)) \; .
\end{aligned} \tag{9}$$

For $i = 1$, it follows from (9) that

$$\begin{aligned}
tr(R_2^H R_1) &= tr[(R_1 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} \begin{bmatrix} f(S_1, T_1) & 0 \\ 0 & g(S_1, T_1) \end{bmatrix})^H R_1] \\
&= tr(R_1^H R_1) - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} tr[\begin{bmatrix} A_1 S_1 + B_1 T_1 - E_1 \overline{S_1} F_1 & 0 \\ 0 & A_2 S_1 + B_2 T_1 - E_2 \overline{S_1} F_2 \end{bmatrix}^H \\
&\qquad \cdot \begin{bmatrix} C_1 - f(V_1, W_1) & 0 \\ 0 & C_2 - g(V_1, W_1) \end{bmatrix}]
\end{aligned}$$

$$= \|R_1\|^2 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} tr[(A_1 S_1 + B_1 T_1 - E_1 \overline{S_1} F_1)^H (C_1 - f(V_1, W_1))$$

$$+ (A_2 S_1 + B_2 T_1 - E_2 \overline{S_1} F_2)^H (C_2 - g(V_1, W_1))]$$

$$= \|R_1\|^2 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} tr[S_1^H A_1^H (C_1 - f(V_1, W_1)) + T_1^H B_1^H (C_1 - f(V_1, W_1))$$

$$- \overline{S_1}^H E_1^H (C_1 - f(V_1, W_1)) F_1^H + S_1^H A_2^H (C_2 - g(V_1, W_1)) + T_1^H B_2^H (C_2 - g(V_1, W_1))$$

$$- \overline{S_1}^H E_2^H (C_2 - g(V_1, W_1)) F_2^H]$$

$$= \|R_1\|^2 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} tr[S_1^H (A_1^H (C_1 - f(V_1, W_1)) + A_2^H (C_2 - g(V_1, W_1))) + T_1^H (B_1^H (C_1 - f(V_1, W_1))$$

$$+ B_2^H (C_2 - g(V_1, W_1))) - \overline{S_1}^H (E_1^H (C_1 - f(V_1, W_1)) F_1^H + E_2^H (C_2 - g(V_1, W_1)) F_2^H)]$$

From this last relation one has

$$tr(R_2^H R_1) + \overline{tr(R_2^H R_1)} = 2\|R_1\|^2 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} tr[S_1^H (A_1^H (C_1 - f(V_1, W_1)) + A_2^H (C_2 - g(V_1, W_1)))$$

$$+ T_1^H (B_1^H (C_1 - f(V_1, W_1)) + B_2^H (C_2 - g(V_1, W_1))) - \overline{S_1}^H (E_1^H (C_1 - f(V_1, W_1)) F_1^H$$

$$+ E_2^H (C_2 - g(V_1, W_1)) F_2^H) + \overline{S_1^H (A_1^H (C_1 - f(V_1, W_1)) + A_2^H (C_2 - g(V_1, W_1)))}$$

$$+ \overline{T_1^H (B_1^H (C_1 - f(V_1, W_1)) + B_2^H (C_2 - g(V_1, W_1)))}$$

$$- \overline{\overline{S_1}^H (E_1^H (C_1 - f(V_1, W_1)) F_1^H + E_2^H (C_2 - g(V_1, W_1)) F_2^H)}]$$

$$= 2\|R_1\|^2 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} tr[S_1^H (A_1^H (C_1 - f(V_1, W_1)) - \overline{E}_1^H \overline{(C_1 - f(V_1, W_1))} \overline{F}_1^H + A_2^H (C_2 - g(V_1, W_1))$$

$$- \overline{E}_2^H \overline{(C_2 - g(V_1, W_1))} \overline{F}_2^H) + T_1^H (B_1^H (C_1 - f(V_1, W_1)) + B_2^H (C_2 - g(V_1, W_1)))$$

$$+ \overline{T_1^H (B_1^H (C_1 - f(V_1, W_1)) + B_2^H (C_2 - g(V_1, W_1)))} +$$

$$\overline{S_1^H (A_1^H (C_1 - f(V_1, W_1)) - \overline{E}_1^H \overline{(C_1 - f(V_1, W_1))} \overline{F}_1^H + A_2^H (C_2 - g(V_1, W_1)) - \overline{E}_2^H \overline{(C_2 - g(V_1, W_1))} \overline{F}_2^H)}]$$

$$= 2\|R_1\|^2 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} tr[S_1^H S_1 + \overline{S_1^H S_1} + T_1^H T_1 + \overline{T_1^H T_1}]$$

$$= 2\|R_1\|^2 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} [2\|S_1\|^2 + 2\|T_1\|^2] = 0$$

This implies that (7) is satisfied for $i = 1$.

From Algorithm I we also have

$$tr(S_2^H S_1 + T_2^H T_1) = tr[(A_1^H (C_1 - f(V_2,W_2)) - \overline{E}_1^H \overline{(C_1 - f(V_2,W_2))}\overline{F}_1^H + A_2^H (C_2 - g(V_2,W_2))$$
$$- \overline{E}_2^H \overline{(C_2 - g(V_2,W_2))}\overline{F}_2^H + \frac{\|R_2\|^2}{\|R_1\|^2} S_1)^H S_1 + (B_1^H (C_1 - f(V_2,W_2)) + B_2^H (C_2 - g(V_2,W_2))$$
$$+ \frac{\|R_2\|^2}{\|R_1\|^2} T_1)^H T_1]$$
$$= tr[(C_1 - f(V_2,W_2))^H (A_1 S_1 + B_1 T_1) - \overline{(C_1 - f(V_2,W_2))}^H (\overline{E_1} S_1 \overline{F_1}) + (C_2 - g(V_2,W_2))^H$$
$$(A_2 S_1 + B_2 T_1) - \overline{(C_2 - g(V_2,W_2))}^H (\overline{E_2} S_1 \overline{F_2})] + \frac{\|R_2\|^2}{\|R_1\|^2} tr(S_1^H S_1 + T_1^H T_1)$$

It follows from this relation that

$$tr(S_2^H S_1 + T_2^H T_1) + \overline{tr(S_2^H S_1 + T_2^H T_1)} = tr[(C_1 - f(V_2,W_2))^H (A_1 S_1 + B_1 T_1) - \overline{(C_1 - f(V_2,W_2))}^H (\overline{E_1} S_1 \overline{F_1})$$
$$+ (C_2 - g(V_2,W_2))^H (A_2 S_1 + B_2 T_1) - \overline{(C_2 - g(V_2,W_2))}^H (\overline{E_2} S_1 \overline{F_2})] + \frac{\|R_2\|^2}{\|R_1\|^2}[tr(S_1^H S_1 + T_1^H T_1)$$
$$+ \overline{tr(S_1^H S_1 + T_1^H T_1)}] + \overline{tr[(C_1 - f(V_2,W_2))^H (A_1 S_1 + B_1 T_1) - \overline{(C_1 - f(V_2,W_2))}^H (\overline{E_1} S_1 \overline{F_1})}$$
$$\overline{+ (C_2 - g(V_2,W_2))^H (A_2 S_1 + B_2 T_1) - \overline{(C_2 - g(V_2,W_2))}^H (\overline{E_2} S_1 \overline{F_2})]}$$
$$= tr[(C_1 - f(V_2,W_2))^H (A_1 S_1 + B_1 T_1 - E_1 \overline{S_1} F_1) + (C_2 - g(V_2,W_2))^H (A_2 S_1 + B_2 T_1 - E_2 \overline{S_1} F_2)]$$
$$+ \overline{tr[(C_1 - f(V_2,W_2))^H (A_1 S_1 + B_1 T_1 - E_1 \overline{S_1} F_1) + (C_2 - g(V_2,W_2))^H (A_2 S_1 + B_2 T_1 - E_2 \overline{S_1} F_2)]}$$
$$+ \frac{\|R_2\|^2}{\|R_1\|^2}[tr(S_1^H S_1 + T_1^H T_1) + \overline{tr(S_1^H S_1 + T_1^H T_1)}]$$
$$= tr[\begin{bmatrix} C_1 - f(V_2,W_2) & 0 \\ 0 & C_2 - g(V_2,W_2) \end{bmatrix}^H \begin{bmatrix} A_1 S_1 + B_1 T_1 - E_1 \overline{S_1} F_1 & 0 \\ 0 & A_2 S_1 + B_2 T_1 - E_2 \overline{S_1} F_2 \end{bmatrix}$$
$$+ \overline{\begin{bmatrix} C_1 - f(V_2,W_2) & 0 \\ 0 & C_2 - g(V_2,W_2) \end{bmatrix}^H \begin{bmatrix} A_1 S_1 + B_1 T_1 - E_1 \overline{S_1} F_1 & 0 \\ 0 & A_2 S_1 + B_2 T_1 - E_2 \overline{S_1} F_2 \end{bmatrix}}]$$
$$+ 2\frac{\|R_2\|^2}{\|R_1\|^2}(\|S_1\|^2 + \|T_1\|^2)$$

$$=\frac{\|S_1\|^2+\|T_1\|^2}{\|R_1\|^2}[tr(R_2^H(R_1-R_2))+\overline{tr(R_2^H(R_1-R_2))}]+2\frac{\|R_2\|^2}{\|R_1\|^2}(\|S_1\|^2+\|T_1\|^2)$$

$$=-\frac{\|S_1\|^2+\|T_1\|^2}{\|R_1\|^2}[2\|R_2\|^2]+2\frac{\|R_2\|^2}{\|R_1\|^2}(\|S_1\|^2+\|T_1\|^2)=0$$

Thus, (8) satisfied for $i=1$

Now, assume (7) and (8) hold for $i=k-1$. From (9) and applying mathematical assumption, from Algorithm I one has

$$tr(R_{k+1}^H R_k)=tr[(R_k-\frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}\; diag(f(S_k,T_k)\;,\;g(S_k,T_k)))^H R_k]$$

$$=tr[R_k{}^H R_k-\frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}\begin{bmatrix}A_1S_k+B_1T_k-E_1\overline{S_k}F_1 & 0\\ 0 & A_2S_k+B_2T_k-E_2\overline{S_k}F_2\end{bmatrix}^H$$

$$\begin{bmatrix}C_1-f(V_k,W_k) & 0\\ 0 & C_2-g(V_k,W_k)\end{bmatrix}]$$

$$=\|R_k\|^2-\frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}tr[(A_1S_k+B_1T_k-E_1\overline{S_k}F_1)^H(C_1-f(V_k,W_k))$$

$$+(A_2S_k+B_2T_k-E_2\overline{S_k}F_2)^H(C_2-g(V_k,W_k)]$$

$$=\|R_k\|^2-\frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}tr[S_k^H(A_1^H(C_1-f(V_k,W_k))+A_2^H(C_2-g(V_k,W_k)))+T_k^H(B_1^H(C_1-f(V_k,W_k))$$

$$+B_2^H(C_2-g(V_k,W_k)))-\overline{S_k}^H(E_1^H(C_1-f(V_k,W_k))F_1^H+E_2^H(C_2-g(V_k,W_k))F_2^H)]$$

It follows from this relation that

$$tr(R_{k+1}^H R_k)+\overline{tr(R_{k+1}^H R_k)}=2\|R_k\|^2-\frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}tr[S_k^H(A_1^H(C_1-f(V_k,W_k))+A_2^H(C_2-g(V_k,W_k)))$$

$$+T_k^H(B_1^H(C_1-f(V_k,W_k))+B_2^H(C_2-g(V_k,W_k)))-\overline{S_k}^H(E_1^H(C_1-f(V_k,W_k))F_1^H$$

$$+E_2^H(C_2-g(V_k,W_k))F_2^H]-\frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}tr[\overline{S_k^H(A_1^H(C_1-f(V_k,W_k))+A_2^H(C_2-g(V_k,W_k)))}$$

$$+\overline{T_k^H(B_1^H(C_1-f(V_k,W_k))+B_2^H(C_2-g(V_k,W_k)))}$$

$$-\overline{\overline{S_k}^H(E_1^H(C_1-f(V_k,W_k))F_1^H+E_2^H(C_2-g(V_k,W_k))F_2^H}]$$

$$= 2\|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} tr[S_k^H (A_1^H (C_1 - f(V_k, W_k)) - \overline{E}_1^H \overline{(C_1 - f(V_k, W_k))} \overline{F}_1^H + A_2^H (C_2 - g(V_k, W_k))$$

$$- \overline{E}_2^H \overline{(C_2 - g(V_k, W_k))} \overline{F}_2^H) + T_k^H (B_1^H (C_1 - f(V_k, W_k)) + B_2^H (C_2 - g(V_k, W_k)))$$

$$+ \overline{S_k^H (A_1^H (C_1 - f(V_k, W_k)) - \overline{E}_1^H \overline{(C_1 - f(V_k, W_k))} \overline{F}_1^H} + \overline{A_2^H (C_2 - g(V_k, W_k)) - \overline{E}_2^H \overline{(C_2 - g(V_k, W_k))} \overline{F}_2^H)}$$

$$+ \overline{T_k^H (B_1^H (C_1 - f(V_k, W_k)) + B_2^H (C_2 - g(V_k, W_k)))}]$$

$$= 2\|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} \{tr[S_k^H (S_k - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} S_{k-1}) + T_k^H (T_k - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} T_{k-1})]$$

$$+ \overline{tr[S_k^H (S_k - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} S_{k-1}) + T_k^H (T_k - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} T_{k-1})]}]\}$$

$$= 2\|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} [2(\|S_k\|^2 + \|T_k\|^2) - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} (tr(S_k^H S_{k-1} + T_k^H T_{k-1}) + \overline{tr(S_k^H S_{k-1} + T_k^H T_{k-1})})] = 0$$

Thus, (7) holds for $i = k$.

Also, from Algorithm I one also has

$$tr(S_{k+1}^H S_k + T_{k+1}^H T_k) = tr[(A_1^H (C_1 - f(V_{k+1}, W_{k+1})) - \overline{E}_1^H \overline{(C_1 - f(V_{k+1}, W_{k+1}))} \overline{F}_1^H$$

$$+ A_2^H (C_2 - g(V_{k+1}, W_{k+1})) - \overline{E}_2^H \overline{(C_2 - g(V_{k+1}, W_{k+1}))} \overline{F}_2^H + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} S_k)^H S_k$$

$$+ (B_1^H (C_1 - f(V_{k+1}, W_{k+1})) + B_2^H (C_2 - g(V_{k+1}, W_{k+1})) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} T_k)^H T_k]$$

$$= tr[(C_1 - f(V_{k+1}, W_{k+1}))^H (A_1 S_k + B_1 T_k) - \overline{(C_1 - f(V_{k+1}, W_{k+1}))}^H (\overline{E_1 S_k F_1}) - \overline{(C_2 - g(V_{k+1}, W_{k+1}))}^H$$

$$(\overline{E_2 S_k F_2}) + (C_2 - g(V_{k+1}, W_{k+1}))^H (A_2 S_k + B_2 T_k)] + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} tr(S_k^H S_k + T_k^H T_k)$$

Thus, from above relation one has

$$tr(S_{k+1}^H S_k + T_{k+1}^H T_k) + \overline{tr(S_{k+1}^H S_k + T_{k+1}^H T_k)} = tr[(C_1 - f(V_{k+1}, W_{k+1}))^H (A_1 S_k + B_1 T_k)$$

$$- \overline{(C_1 - f(V_{k+1}, W_{k+1}))}^H (\overline{E_1} S_k \overline{F_1}) - \overline{(C_2 - g(V_{k+1}, W_{k+1}))}^H (\overline{E_2} S_k \overline{F_2})$$

$$+ (C_2 - g(V_{k+1}, W_{k+1}))^H (A_2 S_k + B_2 T_k)] + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} tr(S_k^H S_k + \overline{S_k^H S_k} + T_k^H T_k + \overline{T_k^H T_k})$$

$$+ \overline{tr[(C_1 - f(V_{k+1}, W_{k+1}))^H (A_1 S_k + B_1 T_k) - \overline{(C_1 - f(V_{k+1}, W_{k+1}))}^H (\overline{E_1} S_k \overline{F_1})}$$

$$\overline{- \overline{(C_2 - g(V_{k+1}, W_{k+1}))}^H (\overline{E_2} S_k \overline{F_2}) + (C_2 - g(V_{k+1}, W_{k+1}))^H (A_2 S_k + B_2 T_k)]}$$

$$= tr[(C_1 - f(V_{k+1}, W_{k+1}))^H (A_1 S_k + B_1 T_k - E_1 \overline{S_k} F_1) + (C_2 - g(V_{k+1}, W_{k+1}))^H (A_2 S_k + B_2 T_k - E_2 \overline{S_k} F_2)$$

$$+ \overline{(C_1 - f(V_{k+1}, W_{k+1}))^H (A_1 S_k + B_1 T_k - E_1 \overline{S_k} F_1) + (C_2 - g(V_{k+1}, W_{k+1}))^H (A_2 S_k + B_2 T_k - E_2 \overline{S_k} F_2)}$$

$$+ 2 \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2)$$

$$= tr[\begin{bmatrix} C_1 - f(V_{k+1}, W_{k+1}) & 0 \\ 0 & C_2 - g(V_{k+1}, W_{k+1}) \end{bmatrix}^H \begin{bmatrix} A_1 S_k + B_1 T_k - E_1 \overline{S_k} F_1 & 0 \\ 0 & A_2 S_k + B_2 T_k - E_2 \overline{S_k} F_2 \end{bmatrix}$$

$$+ \overline{\begin{bmatrix} C_1 - f(V_{k+1}, W_{k+1}) & 0 \\ 0 & C_2 - g(V_{k+1}, W_{k+1}) \end{bmatrix}^H \begin{bmatrix} A_1 S_k + B_1 T_k - E_1 \overline{S_k} F_1 & 0 \\ 0 & A_2 S_k + B_2 T_k - E_2 \overline{S_k} F_2 \end{bmatrix}}]$$

$$+ 2 \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2)$$

$$= tr(R_{k+1}^H (\frac{\|S_k\|^2 + \|T_k\|^2}{\|R_k\|^2} (R_k - R_{k+1}))) + \overline{tr(R_{k+1}^H (\frac{\|S_k\|^2 + \|T_k\|^2}{\|R_k\|^2} (R_k - R_{k+1})))} + 2 \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2)$$

$$= \frac{\|S_k\|^2 + \|T_k\|^2}{\|R_k\|^2} [tr(R_{k+1}^H R_k) + \overline{tr(R_{k+1}^H R_k)} - 2\|R_{k+1}\|^2] + 2 \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2)$$

$$= \frac{\|S_k\|^2 + \|T_k\|^2}{\|R_k\|^2} (-2\|R_{k+1}\|^2) + 2 \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2) = 0$$

This implies that (7) and (8) hold for $i = k$.

Hence, relation (7) and (8) hold for all $1 \le i \le k$

Step2: we want to show that

$$\mathrm{Re}(tr(R_{i+l}^H R_i)) = 0 \tag{10}$$

and

$$\mathrm{Re}(tr(S_{i+l}^H S_i + T_{i+l}^H T_i)) = 0 \tag{11}$$

hold for integer $l \geq 1$. We will prove this conclusion by induction. The case of $l = 1$ has been proven in Step 1. Now we assume that (10) and (11) hold for $l \leq q, q \geq 1$ the aim is to show

$$\mathrm{Re}(tr(R_{i+q+1}^H R_i)) = 0 \tag{12}$$

and

$$\mathrm{Re}(tr(S_{i+q+1}^H S_i + T_{i+q+1}^H T_i)) = 0 \tag{13}$$

First we prove the following

$$\mathrm{Re}(tr(R_{q+1}^H R_0)) = 0 \tag{14}$$

and

$$\mathrm{Re}(tr(S_{q+1}^H S_0 + T_{q+1}^H T_0)) = 0$$

(15)

according Algorithm I, from (9) and induction assumption one has

$$tr(R_{q+1}^H R_0) = tr[(R_q - \frac{\|R_q\|^2}{\|S_q\|^2 + \|T_q\|^2} \; diag(f(S_q, T_q) \;,\; g(S_q, T_q)))^H R_0]$$

$$= tr(R_q^H R_0) - \frac{\|R_q\|^2}{\|S_q\|^2 + \|T_q\|^2} tr(\begin{bmatrix} A_1 S_q + B_1 T_q - E_1 \overline{S_q} F_1 & 0 \\ 0 & A_2 S_q + B_2 T_q - E_2 \overline{S_q} F_2 \end{bmatrix}^H$$

$$\begin{bmatrix} C_1 - f(V_0, W_0) & 0 \\ 0 & C_2 - g(V_0, W_0) \end{bmatrix})$$

$$= tr(R_q^H R_0) - \frac{\|R_q\|^2}{\|S_q\|^2 + \|T_q\|^2} [tr((A_1 S_q + B_1 T_q - E_1 \overline{S_q} F_1)^H (C_1 - f(V_0, W_0))$$

$$+ (A_2 S_q + B_2 T_q - E_2 \overline{S_q} F_2)^H (C_2 - g(V_0, W_0))]$$

$$= tr(R_q^H R_0) - \frac{\|R_q\|^2}{\|S_q\|^2 + \|T_q\|^2} [tr(S_q^H (A_1^H (C_1 - f(V_0, W_0)) + A_2^H (C_2 - g(V_0, W_0)))$$

$$+ T_q^H (B_1^H (C_1 - f(V_0, W_0)) + B_2^H (C_2 - g(V_0, W_0)))$$

$$- \overline{S_q}^H (E_1^H (C_1 - f(V_0, W_0)) F_1^H + E_2^H (C_2 - g(V_0, W_0)) F_2^H)]$$

Thus, from above relation one has

$$tr(R_{q+1}^H R_0)+\overline{tr(R_{q+1}^H R_0)}=tr(R_q^H R_0)+\overline{tr(R_q^H R_0)}-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}[tr(S_q^H(A_1^H(C_1-f(V_0,W_0))+A_2^H(C_2-g(V_0,W_0)))$$

$$+T_q^H(B_1^H(C_1-f(V_0,W_0))+B_2^H(C_2-g(V_0,W_0)))-\overline{S_q}^H(E_1^H(C_1-f(V_0,W_0))F_1^H$$

$$+E_2^H(C_2-g(V_0,W_0))F_2^H)]-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}[\overline{tr(S_q^H(A_1^H(C_1-f(V_0,W_0))+A_2^H(C_2-g(V_0,W_0)))}$$

$$+\overline{T_q^H(B_1^H(C_1-f(V_0,W_0))+B_2^H(C_2-g(V_0,W_0)))}$$

$$-\overline{\overline{S_q}^H(E_1^H(C_1-f(V_0,W_0))F_1^H+E_2^H(C_2-g(V_0,W_0))F_2^H)}]$$

$$=-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}[tr(S_q^H(A_1^H(C_1-f(V_0,W_0))+A_2^H(C_2-g(V_0,W_0))-\overline{E}_1^H\,\overline{(C_1-f(V_0,W_0))}\,\overline{F}_1^H$$

$$-\overline{E}_2^H\,\overline{(C_2-g(V_0,W_0))}\,\overline{F}_2^H)+T_q^H(B_1^H(C_1-f(V_0,W_0))+B_2^H(C_2-g(V_0,W_0)))$$

$$+\overline{T_q^H(B_1^H(C_1-f(V_0,W_0))+B_2^H(C_2-g(V_0,W_0)))}+tr(\overline{S}_q^H(\overline{A}_1^H\,\overline{(C_1-f(V_0,W_0))}$$

$$+\overline{A_2^H(C_2-g(V_0,W_0))}-\overline{\overline{E}_1^H\,\overline{(C_1-f(V_0,W_0))}\,\overline{F}_1^H-\overline{E}_2^H\,\overline{(C_2-g(V_0,W_0))}\,\overline{F}_2^H}))]$$

$$=-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}[tr(S_q^H S_0+T_q^H T_0)+\overline{tr(S_q^H S_0+T_q^H T_0)}]=0$$

And

$$tr(S_{q+1}^H S_0+T_{q+1}^H T_0)=tr[(A_1^H(C_1-f(V_{q+1},W_{q+1}))-\overline{E}_1^H\,\overline{(C_1-f(V_{q+1},W_{q+1}))}\,\overline{F}_1^H$$

$$+A_2^H(C_2-g(V_{q+1},W_{q+1}))-\overline{E}_2^H\,\overline{(C_2-g(V_{q+1},W_{q+1}))}\,\overline{F}_2^H+\frac{\|R_{q+1}\|^2}{\|R_q\|^2}S_q)^H S_0+(B_1^H(C_1-f(V_{q+1},W_{q+1}))$$

$$+B_2^H(C_2-g(V_{q+1},W_{q+1}))+\frac{\|R_{q+1}\|^2}{\|R_q\|^2}T_q)^H T_0]$$

$$=tr[(C_1-f(V_{q+1},W_{q+1}))^H(A_1S_0+B_1T_0)+(C_2-g(V_{q+1},W_{q+1}))^H(A_2S_0+B_2T_0)$$

$$-\overline{(C_2-f(V_{q+1},W_{q+1}))}^H(\overline{E}_1S_0\overline{F}_1)-\overline{(C_2-g(V_{q+1},W_{q+1}))}^H(\overline{E}_2S_0\overline{F}_2)]+\frac{\|R_{q+1}\|^2}{\|R_q\|^2}tr(S_q^H S_0+T_q^H T_0)$$

Thus, from above relation one has

$$
\begin{aligned}
& tr(S_{q+1}^H S_0 + T_{q+1}^H T_0) + \overline{tr(S_{q+1}^H S_0 + T_{q+1}^H T_0)} = tr((C_1 - f(V_{q+1}, W_{q+1}))^H (A_1 S_0 + B_1 T_0) \\
&\qquad + (C_2 - g(V_{q+1}, W_{q+1}))^H (A_2 S_0 + B_2 T_0) - \overline{(C_2 - f(V_{q+1}, W_{q+1}))}^H (\overline{E}_1 S_0 \overline{F}_1) \\
&\qquad - \overline{(C_2 - g(V_{q+1}, W_{q+1}))}^H (\overline{E}_2 S_0 \overline{F}_2)) \\
&\qquad + \overline{tr((C_1 - f(V_{q+1}, W_{q+1}))^H (A_1 S_0 + B_1 T_0) + (C_2 - g(V_{q+1}, W_{q+1}))^H (A_2 S_0 + B_2 T_0)} \\
&\qquad - \overline{\overline{(C_2 - f(V_{q+1}, W_{q+1}))}^H (\overline{E}_1 S_0 \overline{F}_1) - \overline{(C_2 - g(V_{q+1}, W_{q+1}))}^H (\overline{E}_2 S_0 \overline{F}_2))} \\
&\qquad + \frac{\|R_{q+1}\|^2}{\|R_q\|^2} tr[(S_q^H S_0 + T_q^H T_0) + \overline{(S_q^H S_0 + T_q^H T_0)}] \\
&= tr[(C_1 - f(V_{q+1}, W_{q+1}))^H (A_1 S_0 + B_1 T_0 - E_1 \overline{S_0} F_1) + (C_2 - g(V_{q+1}, W_{q+1}))^H (A_2 S_0 + B_2 T_0 - E_2 \overline{S_0} F_2) \\
&\quad + \overline{(C_1 - f(V_{q+1}, W_{q+1}))^H (A_1 S_0 + B_1 T_0 - E_1 \overline{S_0} F_1)} + \overline{(C_2 - g(V_{q+1}, W_{q+1}))^H (A_2 S_0 + B_2 T_0 - E_2 \overline{S_0} F_2)}] \\
&\quad + \frac{\|R_{q+1}\|^2}{\|R_q\|^2} tr[(S_q^H S_0 + T_q^H T_0) + \overline{(S_q^H S_0 + T_q^H T_0)}] \\
&= tr(\begin{bmatrix} C_1 - f(V_{q+1}, W_{q+1}) & 0 \\ 0 & C_2 - g(V_{q+1}, W_{q+1}) \end{bmatrix}^H \begin{bmatrix} A_1 S_0 + B_1 T_0 - E_1 \overline{S_0} F_1 & 0 \\ 0 & A_2 S_0 + B_2 T_0 - E_2 \overline{S_0} F_2 \end{bmatrix} \\
&\quad + \overline{\begin{bmatrix} C_1 - f(V_{q+1}, W_{q+1}) & 0 \\ 0 & C_2 - g(V_{q+1}, W_{q+1}) \end{bmatrix}^H \begin{bmatrix} A_1 S_0 + B_1 T_0 - E_1 \overline{S_0} F_1 & 0 \\ 0 & A_2 S_0 + B_2 T_0 - E_2 \overline{S_0} F_2 \end{bmatrix}}) \\
&= tr(R_{q+1}^H (\frac{\|S_0\|^2 + \|T_0\|^2}{\|R_0\|^2} (R_0 - R_1)) + \overline{tr(R_{q+1}^H (\frac{\|S_0\|^2 + \|T_0\|^2}{\|R_0\|^2} (R_0 - R_1))} = 0
\end{aligned}
$$

Then (14) and (15) hold

From Algorithm I and (9) , induction assumption one has

$$
\begin{aligned}
&tr(S^H_{i+q+1}S_i + T^H_{i+q+1}T_i) + \overline{tr(S^H_{i+q+1}S_i + T^H_{i+q+1}T_i)} = tr[(A_1^H(C_1 - f(V_{i+q+1},W_{i+q+1}))\\
&\quad - \overline{E}_1^H\,\overline{(C_1 - f(V_{i+q+1},W_{i+q+1}))}\,\overline{F}_1^H + A_2^H(C_2 - g(V_{i+q+1},W_{i+q+1})) - \overline{E}_2^H\,\overline{(C_2 - g(V_{i+q+1},W_{i+q+1}))}\,\overline{F}_2^H\\
&\quad + \frac{\|R_{i+q+1}\|^2}{\|R_{i+q}\|^2} S_{i+q})^H S_i + (B_1^H(C_1 - f(V_{i+q+1},W_{i+q+1})) + B_2^H(C_2 - g(V_{i+q+1},W_{i+q+1}))\\
&\quad + \frac{\|R_{i+q+1}\|^2}{\|R_{i+q}\|^2} T_{i+q})^H T_i] + \overline{tr[(A_1^H(C_1 - f(V_{i+q+1},W_{i+q+1})) - \overline{E}_1^H\,\overline{(C_1 - f(V_{i+q+1},W_{i+q+1}))}\,\overline{F}_1^H}\\
&\quad \overline{+ A_2^H(C_2 - g(V_{i+q+1},W_{i+q+1})) - \overline{E}_2^H\,\overline{(C_2 - g(V_{i+q+1},W_{i+q+1}))}\,\overline{F}_2^H + \frac{\|R_{i+q+1}\|^2}{\|R_{i+q}\|^2} S_{i+q})^H S_i}\\
&\quad \overline{+ (B_1^H(C_1 - f(V_{i+q+1},W_{i+q+1})) + B_2^H(C_2 - g(V_{i+q+1},W_{i+q+1})) + \frac{\|R_{i+q+1}\|^2}{\|R_{i+q}\|^2} T_{i+q})^H T_i]}\\
&= tr[(C_1 - f(V_{i+q+1},W_{i+q+1}))^H(A_1S_i + B_1T_i) - \overline{(C_1 - f(V_{i+q+1},W_{i+q+1}))}^H(\overline{E}_1 S_i \overline{F}_1)\\
&\quad - \overline{(C_2 - g(V_{i+q+1},W_{i+q+1}))}^H(\overline{E}_2 S_i \overline{F}_2) + (C_2 - g(V_{i+q+1},W_{i+q+1}))^H(A_2S_i + B_2T_i)]\\
&\quad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} tr[S^H_{i+q}S_i + \overline{S^H_{i+q}S_i} + T^H_{i+q}T_i + \overline{T^H_{i+q}T_i}]\\
&\quad \overline{+ tr[(C_1 - f(V_{i+q+1},W_{i+q+1}))^H(A_1S_i + B_1T_i) - \overline{(C_1 - f(V_{i+q+1},W_{i+q+1}))}^H(\overline{E}_1 S_i \overline{F}_1)}\\
&\quad \overline{- \overline{(C_2 - g(V_{i+q+1},W_{i+q+1}))}^H(\overline{E}_2 S_i \overline{F}_2) + (C_2 - g(V_{i+q+1},W_{i+q+1}))^H(A_2S_i + B_2T_i)]}\\
&= tr[(C_1 - f(V_{i+q+1},W_{i+q+1}))^H(A_1S_i + B_1T_i - E_1\overline{S}_iF_1) + (C_2 - g(V_{i+q+1},W_{i+q+1}))^H(A_2S_i + B_2T_i - E_2\overline{S}_iF_2)\\
&\quad \overline{+ (C_1 - f(V_{i+q+1},W_{i+q+1}))^H(A_1S_i + B_1T_i - E_1\overline{S}_iF_1) + (C_2 - g(V_{i+q+1},W_{i+q+1}))^H(A_2S_i + B_2T_i - E_2\overline{S}_iF_2)}]\\
&= tr(\begin{bmatrix} C_1 - f(V_{i+q+1},W_{i+q+1}) & 0 \\ 0 & C_2 - g(V_{i+q+1},W_{i+q+1}) \end{bmatrix}^H \begin{bmatrix} A_1S_i + B_1T_i - E_1\overline{S}_iF_1 & 0 \\ 0 & A_2S_i + B_2T_i - E_2\overline{S}_iF_2 \end{bmatrix}\\
&\quad + \overline{\begin{bmatrix} C_1 - f(V_{i+q+1},W_{i+q+1}) & 0 \\ 0 & C_2 - g(V_{i+q+1},W_{i+q+1}) \end{bmatrix}^H \begin{bmatrix} A_1S_i + B_1T_i - E_1\overline{S}_iF_1 & 0 \\ 0 & A_2S_i + B_2T_i - E_2\overline{S}_iF_2 \end{bmatrix}})\\
&= tr(R^H_{i+q+1}(\frac{\|S_i\|^2 + \|T_i\|^2}{\|R_i\|^2}(R_i - R_{i+1})) + \overline{tr(R^H_{i+q+1}(\frac{\|S_i\|^2 + \|T_i\|^2}{\|R_i\|^2}(R_i - R_{i+1}))}\\
&= \frac{\|S_i\|^2 + \|T_i\|^2}{\|R_i\|^2}[tr(R^H_{i+q+1}R_i) + \overline{tr(R^H_{i+q+1}R_i)}] \qquad (16)
\end{aligned}
$$

in addition, from (9) it can be shown that

$$tr(R_{i+q+1}^H R_i)+\overline{tr(R_{i+q+1}^H R_i)}=tr[(R_{i+q}-\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}\ diag(f(S_{i+q},T_{i+q})\ ,\ g(S_{i+q},T_{i+q})))^H R_i]$$

$$+\overline{tr[(R_{i+q}-\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}\ diag(f(S_{i+q},T_{i+q})\ ,\ g(S_{i+q},T_{i+q})))^H R_i]}$$

$$=tr(R_{i+q}^H R_i)+\overline{tr(R_{i+q}^H R_i)}-\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}[$$

$$tr(\begin{bmatrix} A_1S_{i+q}+B_1T_{i+q}-E_1\overline{S_{i+q}}F_1 & 0 \\ 0 & A_2S_{i+q}+B_2T_{i+q}-E_2\overline{S_{i+q}}F_2 \end{bmatrix}^H \begin{bmatrix} C_1-f(V_i,W_i) & 0 \\ 0 & C_2-g(V_i,W_i) \end{bmatrix})$$

$$+\overline{tr(\begin{bmatrix} A_1S_{i+q}+B_1T_{i+q}-E_1\overline{S_{i+q}}F_1 & 0 \\ 0 & A_2S_{i+q}+B_2T_{i+q}-E_2\overline{S_{i+q}}F_2 \end{bmatrix}^H \begin{bmatrix} C_1-f(V_i,W_i) & 0 \\ 0 & C_2-g(V_i,W_i) \end{bmatrix})}]$$

$$=tr(R_{i+q}R_i)+\overline{tr(R_{i+q}R_i)}-\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}[tr((A_1S_{i+q}+B_1T_{i+q}-E_1\overline{S_{i+q}}F_1)^H(C_1-f(V_i,W_i))$$

$$+(A_2S_{i+q}+B_2T_{i+q}-E_2\overline{S_{i+q}}F_2)^H(C_2-g(V_i,W_i)))+\overline{tr((A_1S_{i+q}+B_1T_{i+q}-E_1\overline{S_{i+q}}F_1)^H(C_1-f(V_i,W_i))}$$

$$\overline{+(A_2S_{i+q}+B_2T_{i+q}-E_2\overline{S_{i+q}}F_2)^H(C_2-g(V_i,W_i)))}]$$

$$=tr(R_{i+q}^H R_i)+\overline{tr(R_{i+q}^H R_i)}-\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}tr[S_{i+q}^H(A_1^H(C_1-f(V_i,W_i))+A_2^H(C_2-g(V_i,W_i)))$$

$$+T_{i+q}^H(B_1^H(C_1-f(V_i,W_i))+B_2^H(C_2-g(V_i,W_i)))-\overline{S}_{i+q}^H(E_1^H(C_1-f(V_i,W_i))F_1^H$$

$$+E_2^H(C_2-g(V_i,W_i))F_2^H)]-\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}(tr[\overline{S_{i+q}^H(A_1^H(C_1-f(V_i,W_i))+A_2^H(C_2-g(V_i,W_i)))}$$

$$+\overline{T_{i+q}^H(B_1^H(C_1-f(V_i,W_i))+B_2^H(C_2-g(V_i,W_i)))}$$

$$-\overline{\overline{S}_{i+q}^H(E_1^H(C_1-f(V_i,W_i))F_1^H+E_2^H(C_2-g(V_i,W_i))F_2^H)}])$$

$$=-\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}[tr[S_{i+q}^H(A_1^H(C_1-f(V_i,W_i))+A_2^H(C_2-g(V_i,W_i))-\overline{E}_1^H\overline{(C_1-f(V_i,W_i))}\overline{F}_1^H$$

$$-\overline{E}_2^H\overline{(C_2-g(V_i,W_i))}\overline{F}_2^H)+T_{i+q}^H(B_1^H(C_1-f(V_i,W_i))+B_2^H(C_2-g(V_i,W_i)))$$

$$-\overline{S}_{i+q}^H\overline{(A_1^H(C_1-f(V_i,W_i))+A_2^H(C_2-g(V_i,W_i))-\overline{E}_1^H\overline{(C_1-f(V_i,W_i))}\overline{F}_1^H}$$

$$\overline{-\overline{E}_2^H\overline{(C_2-g(V_i,W_i))}\overline{F}_2^H)+T_{i+q}^H(B_1^H(C_1-f(V_i,W_i))+B_2^H(C_2-g(V_i,W_i)))}]$$

$$=-\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}[tr(S_{i+q}^H(S_i-\frac{\|R_i\|^2}{\|R_{i-1}\|^2}S_{i-1})+T_{i+q}^H(T_i-\frac{\|R_i\|^2}{\|R_{i-1}\|^2}T_{i-1}))$$

$$+\overline{tr(S_{i+q}^H(S_i-\frac{\|R_i\|^2}{\|R_{i-1}\|^2}S_{i-1})+T_{i+q}^H(T_i-\frac{\|R_i\|^2}{\|R_{i-1}\|^2}T_{i-1}))}]$$

$$=\frac{\|R_{i+s}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2}\frac{\|R_i\|^2}{\|R_{i-1}\|^2}[tr(S_{i+q}^HS_{i-1}+T_{i+q}^HT_{i-1})+\overline{tr(S_{i+q}^HS_{i-1}+T_{i+q}^HT_{i-1})}] \tag{17}$$

Repeating (16) and (17), one can easily obtain for certain α and β

$$tr(S_{i+q+1}^HS_i+T_{i+q+1}^HT_i)+\overline{tr(S_{i+q+1}^HS_i+T_{i+q+1}^HT_i)}=\alpha[tr(S_{q+1}^HS_1+S_{q+1}^HS_1)+\overline{tr(S_{q+1}^HS_1+S_{q+1}^HS_1)}]$$

And

$$tr(R_{i+q+1}^HR_i)+\overline{tr(R_{i+q+1}^HR_i)}=\beta[tr(R_{q+1}^HR_1)+\overline{tr(R_{q+1}^HR_1)}]$$

Combining these two relations with (14) and (15) implies that (10) and (11) holds for $l=s+1$.

From step (1) and (2) the conclusion holds by the principle of induction.

With the above two lemmas, one has the following theorem

Remark

Lemma1 implies that if there exist a positive number i such that $P_i=0$ and $Q_i=0$ but $R_j\neq 0$, then the system of matrix equation (1) is inconsistent .

Theorem 2.

If the system of matrix equation (1) is consistent, then a solution can be obtained within finite iteration steps by using Algorithm I for any initial matrices V_1, W_1 .

Proof.

Suppose that $R_i \neq 0$ for $i = 1,2,3,\ldots,2np$ we get $P_i \neq 0$ or $Q_i \neq 0$ from the previous lemma and remark.

Then we can compute $V_{2np+1}, W_{2np+1}, R_{2np+1}$ by Algorithm I. Also, from Lemma2 we have

$trace(R_{2np+1}^T R_i) = 0$ and $trace(R_i^T R_j) = 0$ for $i = 1,2,3,\ldots,2np, i \neq j$

So the set of $R_1, R_2, \ldots, R_{2np}$ is an orthogonal basis of the linear space Ω of dimension $2np$ where $\Omega = \{\, U \mid U = diag(K_1, K_2) \quad where \quad K_1, K_2 \in \mathbb{C}^{n\times p} \,\}$

Which implies that

$R_{2np+1} = 0$ that is V_{2np+1}, W_{2np+1} Is the solution of system of matrix equation (1).

4. Numerical example

In this section, a numerical example is given to illustrate the application of our proposed algorithm.

Consider the system of matrix equation $A_1V + B_1W = E_1\overline{V}F_1 + C_1 \quad , A_2V + B_2W = E_2\overline{V}F_2 + C_2$

Where

$$A_1 = \begin{bmatrix} 1-3i & 2i & -3i \\ 1 & 2+3i & 4i \\ 1-2i & 0 & 2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 3-i & 1+i & -1 \\ 4+i & -i & 4i \\ 0 & 1-i & 2+2i \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & i \\ 1+i & 0 \\ -1-i & 3i \end{bmatrix}, \quad C_1 = \begin{bmatrix} -16+27i & -45-52i \\ -50+9i & 21-79i \\ -8+7i & -3-28i \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 1 \\ -3i & 4+i \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 1 & i \\ -i & 2+3i \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2+3i & -i & 1+i \\ 5 & 1+2i & -3i \\ 0 & 1-i & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 3+i & -1-i & -3i \\ 0 & 2-i & i \\ -1+3i & 2 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 1-3i & 4+i \\ 2i & -3i \end{bmatrix}, \quad C_2 = \begin{bmatrix} 6i & 21-22i \\ 5-15i & 8-30i \\ -3-18i & 19+23i \end{bmatrix}.$$

Taking

$V_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $, W_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ We apply Algorithm I to compute V_k, W_k

After iterating 33 steps we obtain

$$V_{33} = \begin{bmatrix} 1-i & 2-3i \\ 3-i & 1+i \\ 1 & 2-i \end{bmatrix}, \quad W_{33} = \begin{bmatrix} 1-2i & i \\ 1-2i & -2i \end{bmatrix}$$

which satisfy the system of matrix equation

$$A_1V + B_1W = E_1\overline{V}F_1 + C_1, \qquad A_2V + B_2W = E_2\overline{V}F_2 + C_2$$

With the corresponding residual

$$\|R_{33}\| = \|diag(C_1 - f(V_{33}, W_{33}),\, C_2 - g(V_{33}, W_{33}))\| = 1.8151 \times 10^{-10}$$

The obtained results are presented in figure 1, where

$r_k = \|R_k\|$ (Residual)

$\delta_k = \dfrac{\|[V_k, W_k] - [V, W]\|}{\|[V, W]\|}$ (Relative error)

From Fig. 1, it is clear that the error δ_k is becoming smaller and approaches zero as iteration number k increases. This indicates that the proposed algorithm is effective and convergent.

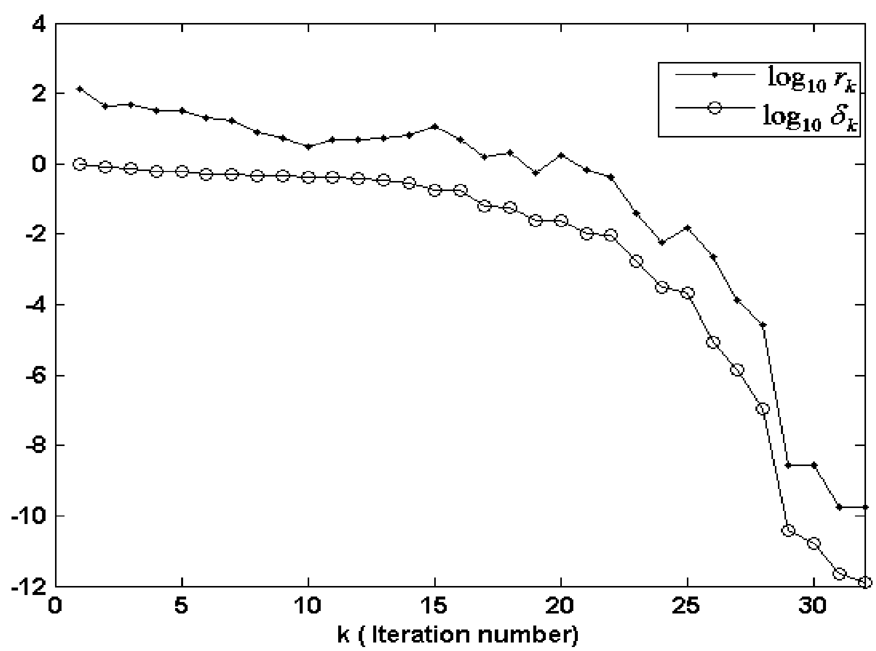

Fig. 1. The residual and the relative error versus k (iteration number)

5. Conclusions

An iterative algorithm for solving the generalized coupled Sylvester – conjugate Matrix Equation $A_1V + B_1W = E_1\overline{V}F_1 + C_1$ and $A_2V + B_2W = E_2\overline{V}F_2 + C_2$ is presented. We have proven that the iterative algorithms always converge to the solution for any initial matrices. We stated and proved some lemmas and theorems where the solutions are obtained. The obtained results show that the methods are very neat and efficient. The proposed methods are illustrated by numerical example. Example we tested using MATLAB to verify our theoretical results.

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