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SOME SUBORDINATION PROPERTIES OF MULTIVALENT FUNCTIONS DEFINED BY CERTAIN LINEAR OPERATORS

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ABSTRACT: In this paper, we investigate some interesting properties among certain subclasses of analytic and p-valent functions, which are defined by a new generalized differential operator $I^m_{p,\alpha,\beta}$ and a new generalized integral operator $J^m_{p,\alpha,\beta}$, using the techniques of the first order differential subordination.

Key words: Analytic functions, Differential subordination, Differential operator, Integral operator.

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1. Introduction

Let A_p denote the class of functions of the form

$$(1.1) \qquad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in N = \{1,2,3...\}),$$

which are analytic and p-valent in the unit disc $U = \{z \in C : |z| < 1\}$, and we set $A_1 = A$, a well-known class of normalized analytic functions in U. For $f \in A_p$ given by (1.1) and $g \in A_p$

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defined by $g(z)=z^p+\sum_{k=p+1}^{\infty}b_k z^k$,the Hadamard (or convolution) product of f and g is given by $(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_k b_k z^k=(g*f)(z).$ If f and g are analytic in U, we say that the function f is subordinate to g , or the function g is superordinate to f , if there exists a Schwarz function w, analytic in U , with $w(0)=0$ and $|w(z)|<1$, for all $z\in U$, such that $f(z)=g(w(z))$, for $z\in U.$ In such a case we write $f\prec g.$ In particular, if the function g is univalent in U , then we have the following equivalence(See [8,16]:

$$f(z)\prec g(z)\ \ (z\in U)\ \text{if and only if}\ \ f(0)=g(0)\ \ \text{and}\ \ f(U)\subset g(U).$$

For $f\in A_p$, the author [23, 24] defined a new differential operator $I^m_{p,\alpha,\beta}$ by the following infinite series

$$(1.2)\qquad I^m_{p,\alpha,\beta}f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{\alpha+k\beta}{\alpha+p\beta}\right)^m a_k z^k, z\in U,$$

where $p\in N,\ m\in N_0=N\bigcup\{0\}$, $\beta\geq 0$ and α a real number with $\alpha+p\beta>0$.

Remark 1.1 If $f\in A_p$ and the differential operator $I^m_{p,\alpha,\beta}$ is given by (1.2), then

$$(1.3)\qquad (\alpha+p\beta)I^{m+1}_{p,\alpha,\beta}f(z)=\alpha I^m_{p,\alpha,\beta}f(z)+\beta z(I^m_{p,\alpha,\beta}f(z))', \beta>0.$$

We note that

- $I^m_{1,\alpha,\beta}f(z)=I^m_{\alpha,\beta}f(z)$ (See [22]).
- $I^m_{p,\alpha,1}f(z)=I^m_p(\alpha)f(z),\alpha>-p$ (See [1]).
- $I^m_{p,l+p-p\beta,\beta}f(z)=I^m_p(\beta,l)f(z), l>-p,\beta\geq 0$ (See Catas [9]).
- $I^m_{p,0,\beta}f(z)=D^m_p f(z)$ (See [4]).

Remark 1.2 i) $I_p^m(\alpha)f(z)$ was considered in [1], for $\alpha \ge 0$ and $I_p^m(\beta,l)f(z)$ was defined in [9] for $l \ge 0, \beta \ge 0$, ii) $I_p^m(l)f(z) = I_p^m(1,l)f(z), l > -p$, iii) $I_p^m(\beta,0)f(z) = D_p^m(\beta)f(z)$, $\beta \ge 0$, was mentioned in Aouf et.al. [5], iv) $D_1^m(\beta), \beta \ge 0$, was introduced by Al-Oboudi [2], v) $D_1^m(1)f(z) = D^m f(z)$ was defined by Salagean [20] and was considered for $m \ge 0$ in [7] , vi) $I_1^m(\alpha)f(z), \alpha \ge 0$, was investigated in [10] and [11] and vii) $I_1^m(1)$ was due to Uralegaddi and Somanatha[27].

In [26], the author defined a new integral operator $J_{p,\alpha,\beta}^m$ and is as follows:

Definition 1.3 For $f \in A_p$, we define an integral operator $J_{p,\alpha,\beta}^m f(z)$ by

$$J_{p,\alpha,\beta}^0 f(z) = f(z),$$

$$J_{p,\alpha,\beta}^1 f(z) = J_{p,\alpha,\beta} f(z) = \left(\frac{\alpha + p\beta}{\beta}\right) z^{p-\left(\frac{\alpha+p\beta}{\beta}\right)} \int_0^z t^{\left(\frac{\alpha+p\beta}{\beta}\right)-p-1} f(t)dt, z \in U,$$

$$J_{p,\alpha,\beta}^2 f(z) = \left(\frac{\alpha + p\beta}{\beta}\right) z^{p-\left(\frac{\alpha+p\beta}{\beta}\right)} \int_0^z t^{\left(\frac{\alpha+p\beta}{\beta}\right)-p-1} J_{p,\alpha,\beta}^1 f(t)dt, z \in U,$$

$$\ldots$$

$$J_{p,\alpha,\beta}^m f(z) = \left(\frac{\alpha + p\beta}{\beta}\right) z^{p-\left(\frac{\alpha+p\beta}{\beta}\right)} \int_0^z t^{\left(\frac{\alpha+p\beta}{\beta}\right)-p-1} J_{p,\alpha,\beta}^{m-1} f(t)dt$$

$$= J_{p,\alpha,\beta}^1\left(\frac{z^p}{1-z}\right) * J_{p,\alpha,\beta}^1\left(\frac{z^p}{1-z}\right) * \ldots * J_{p,\alpha,\beta}^1\left(\frac{z^p}{1-z}\right) * f(z)$$

← -------------------- m – times --------------------→

where $p \in N$, $m \in N_0 = N \cup \{0\}$, $\beta > 0$ and α a real number with $\alpha + p\beta > 0$.

We see that for $f(z) \in A_p$, we have

(1.4) $$J^{m}_{p,\alpha,\beta} f(z) = z^{p} + \sum_{k=p+1}^{\infty} \left(\frac{\alpha + p\beta}{\alpha + k\beta} \right)^{m} a_k z^k, z \in U,$$

where $p \in N$, $m \in N_0 = N \bigcup \{0\}$, $\beta > 0$ and α a real number with $\alpha + p\beta > 0$.

From (1.4), it is easy to verify that

(1.5) $$(\alpha + p\beta) J^{m}_{p,\alpha,\beta} f(z) = \alpha J^{m+1}_{p,\alpha,\beta} f(z) + \beta z (J^{m+1}_{p,\alpha,\beta} f(z))'.$$

We also note that for $f(z) \in A$, we have

$$J^{m}_{1,\alpha,\beta} f(z) = J^{m}_{\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + \beta}{\alpha + k\beta} \right)^{m} a_k z^k, z \in U,$$

where $m \in N_0 = N \bigcup \{0\}$, $\beta > 0$ and α a real number with $\alpha + \beta > 0$.

Remark 1.4 i) $J^{m}_{1,\alpha,\beta} f(z) = J^{m}_{\alpha,\beta} f(z)$ [See 25],ii) $J^{m}_{p,l+p-p\beta,\beta} f(z) = J^{m}_{p}(\beta,l) f(z), l > -p, \beta > 0$ (See [6(considered for $l \geq 0$)], iii) $J^{m}_{p,\alpha,1} f(z) = J^{m}_{p}(\alpha) f(z), \alpha > -p$ (See [6(considered for $\alpha \geq 0$)], iv) $J^{m}_{p,0,\beta} f(z) = J^{m}_{p} f(z)$ (See[6]),v) $J^{m}_{p,p-p\beta,\beta} f(z) = L^{m}_{p}(\beta) f(z), \beta > 0$ (See[6]),vi) $J^{m}_{p,1,1} f(z) = L^{m}_{p} f(z)$ (See [17, 21]), vii) $J^{m}_{1,1,1} f(z) = L^{m}_{1} f(z) = L^{m} f(z)$ (See [12, 14]) and viii) $J^{m}_{1,1-\beta,\beta} f(z) = L^{m}(\beta) f(z)$ (See [19]).

Remark 1.3 we observe that $I^{m}_{p,\alpha,\beta}$ and $J^{m}_{p,\alpha,\beta}$ are linear operators and for $f \in A_p$, we have $J^{m}_{p,\alpha,\beta}(I^{m}_{p,\alpha,\beta} f(z)) = I^{m}_{p,\alpha,\beta}(J^{m}_{p,\alpha,\beta} f(z)) = f(z).$

For $f(z) \in A_p$, the function $F_\delta(z)$ is defined by

(1.6) $$F_\delta(z) = \frac{\delta + p}{z^\delta} \int_0^z t^{\delta-1} f(t) dt, z \in U.$$

where $\delta > -p$. Clearly $F_\delta(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{\delta + p}{\delta + k} \right) a_k z^k$ and it is easy to verify that

(1.7) $$(\delta+p)I^{m}_{p,\alpha,\beta}f(z)=\delta I^{m}_{p,\alpha,\beta}F_{\delta}(z)+z(I^{m}_{p,\alpha,\beta}F_{\delta}(z))',$$

and from Remark 1.3, we have

(1.8) $$(\delta+p)J^{m}_{p,\alpha,\beta}f(z)=\delta J^{m}_{p,\alpha,\beta}F_{\delta}(z)+z(J^{m}_{p,\alpha,\beta}F_{\delta}(z))'.$$

In this paper we will determine some subordination properties of multivalent functions defined using a new generalized differential operator or a new generalized integral operator

2. Preliminaries

The following lemmas will be required in our investigation.

Lemma 2.1[13] Let $\gamma\in C,\gamma\neq 0,\mathrm{Re}(\gamma)>0,h(z)$ be a convex (univalent) in U, with $h(0)=1$ and let $p(z)=1+p_1z+p_2z^2+...,$ be analytic in U. If $p(z)+\dfrac{zp'(z)}{\gamma}\prec h(z),z\in U,$ then

$p(z)\prec q(z)=\dfrac{\gamma}{z^{\gamma}}\displaystyle\int_0^z t^{\gamma-1}h(t)dt\prec h(z),z\in U,$ and $q(z)$ is the best dominant.

For any complex numbers $a,b,c(c\notin Z_0^-=\{0,-1,-2,...\}),$ the Gauss hypergeometric function is defined by

$${}_2F_1(a,b;c;z)=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+...\ .$$

The above series converges absolutely for all $z\in U,$ and hence represents an analytic function in the unit disc U (See, for details, [28]).

The each of the identities asserted by lemma below is well-known

Lemma 2.2[28] For any complex parameters a,b and c $(c\notin Z_0^-)$, $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, we have

(2.1) $$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z);$$

(2.2) $${}_2F_1(a,b;c;z)={}_2F_1(b,a;c;z);$$

(2.3) $${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(b,c-b;c;\frac{z}{z-1}\right).$$

3. Main Results

Unless otherwise mentioned, we shall assume in the remainder of this paper that $z\in U$, the powers are understood as principle values and the parameters $p,m,A,B,\delta,\lambda,\mu,\alpha,$ and β are constrained as follows:

$$p\in N, m\in N_0, -1\le B<A\le 1, \delta>-p, \lambda>0, \mu>0, \beta>0, \alpha\in R \text{ such that } \alpha+p\beta>0.$$

Theorem 3.1 If the function $f\in A_p$, satisfy the following subordination condition

(3.1) $$(1-\lambda)\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}\left(\frac{I^{m+1}_{p,\alpha,\beta}f(z)}{I^m_{p,\alpha,\beta}f(z)}\right)\prec\frac{1+Az}{1+Bz},$$

then

$$\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}\prec q(z)\prec\frac{1+Az}{1+Bz},$$

where

(3.2) $$q(z)=\begin{cases}\dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{\mu(\alpha+p\beta)}{\lambda\beta}+1;\dfrac{Bz}{Bz+1}\right), & B\neq 0\\[2ex] 1+\dfrac{\mu(\alpha+p\beta)}{\mu(\alpha+p\beta)+\lambda\beta}Az, & B=0\end{cases}$$

and $q(z)$ is the best dominant. Furthermore,

$$\operatorname{Re}\left(\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}\right)>M(p,A,B,\mu,\lambda,\alpha,\beta)$$

where

$$M(p,A,B,\mu,\lambda,\alpha,\beta)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{\mu(\alpha+p\beta)}{\lambda\beta}+1;\frac{B}{B-1}\right), & B\neq 0\\ 1-\frac{\mu(\alpha+p\beta)}{\mu(\alpha+p\beta)+\lambda\beta}A, & B=0.\end{cases} \tag{3.3}$$

This result is sharp.

Proof. Let

$$p(z)=\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu} \tag{3.4}$$

then $p(z)$ is analytic in U with $p(0)=1$. Using (1.3), (3.1) and (3.4), we obtain

$$p(z)+\frac{\lambda\beta}{\mu(\alpha+p\beta)}zp'(z)\prec\frac{1+Az}{1+Bz}.$$

Thus, by Lemma 2.1 for $\gamma=\frac{\mu(\alpha+p\beta)}{\lambda\beta}$, we deduce that

$$\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}\prec\left(\frac{\mu(\alpha+p\beta)}{\lambda\beta}\right)z^{-\left(\frac{\mu(\alpha+p\beta)}{\lambda\beta}\right)}\int_0^z t^{\left(\frac{\mu(\alpha+p\beta}{\lambda\beta}\right)-1}\left(\frac{1+At}{1+Bt}\right)dt=q(z),$$

where $q(z)$ is given by (3.2) and is obtained by change of variables followed by the use of identities (2.1),(2.2) and (2.3) from Lemma 2.2.Following the same lines as in Theorem 4[18], we can prove that $\inf_{z\in U}\left(\mathrm{Re}(q(z)\right)=q(-1)$.The proof of Theorem 3.1 is thus completed.

In a manner similar to that of Theorem 3.1, we can easily prove the following theorem, using the identity (1.5).

Theorem 3.2 Let $f\in A_p$, satisfies

$$(1-\lambda)\left(\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}\left(\frac{J^{m}_{p,\alpha,\beta}f(z)}{J^{m+1}_{p,\alpha,\beta}f(z)}\right)\prec\frac{1+Az}{1+Bz},$$

then

$$\left(\frac{J_{p,\alpha,\beta}^{m+1} f(z)}{z^p}\right)^{\mu} \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $q(z)$ is given by (3.2) and $q(z)$ is the best dominant. Furthermore,

$$\mathrm{Re}\left(\left(\frac{J_{p,\alpha,\beta}^{m+1} f(z)}{z^p}\right)^{\mu}\right) > M(p, A, B, \mu, \lambda, \alpha, \beta),$$

where $M(p, A, B, \mu, \lambda, \alpha, \beta)$ is given by (3.3) and this result is sharp.

Remark 3.3 For $p=1, \mu=1,$ and $\alpha=1-\beta,$ Theorem 3.1 and Theorem 3.2 agree with Theorem 3.1 and Theorem 3.2, respectively, of Al-Oboudi and Al-Qahtani [3]. For $\beta=1$ in Theorem 3.1 and Theorem 3.2, our results for operators $I_p^m(\alpha)$ and $J_p^m(\alpha)$ hold true for $\alpha > -p.$ Similarly, results obtained for operators $I_p^m(\beta, l)$ and $J_p^m(\beta, l)$ from Theorem 3.1 and Theorem 3.2, by putting $\alpha = l + p - p\beta,$ hold true for $l > -p.$

Now we prove the following.

Theorem 3.4 If the function $f \in A_p$, satisfy the following subordination condition

$$(1-\lambda)\left(\frac{I_{p,\alpha,\beta}^{m} F_{\delta}(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{I_{p,\alpha,\beta}^{m} F_{\delta}(z)}{z^p}\right)^{\mu-1}\left(\frac{I_{p,\alpha,\beta}^{m} f(z)}{z^p}\right) \prec \frac{1+Az}{1+Bz},$$

then

$$\left(\frac{I_{p,\alpha,\beta}^{m} F_{\delta}(z)}{z^p}\right)^{\mu} \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $F_{\delta}(z)$ is defined by (1.6) and $q(z)$ is given by

(3.5) $$q(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\frac{\mu(\delta+p)}{\lambda}+1;\frac{Bz}{Bz+1}\right), & B\neq 0\\ 1+\frac{\mu(\delta+p)}{\mu(\delta+p)+\lambda}Az, & B=0\end{cases}$$

and $q(z)$ is the best dominant. Furthermore

$$\mathrm{Re}\left(\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu\right)>M_1(p,A,B,\delta,\lambda,\mu)$$

where

(3.6) $$M_1(p,A,B,\delta,\lambda,\mu)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{\mu(\delta+p)}{\lambda}+1;\frac{B}{B-1}\right), & B\neq 0\\ 1-\frac{\mu(\delta+p)}{\mu(\delta+p)+\lambda}A, & B=0.\end{cases}$$

This result is sharp.

Proof. Setting

(3.7) $$p(z)=\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu,$$

we note that $p(z)$ is analytic in U and $p(z)=1+p_1z+p_2z^2+...$ .Carrying out logarithmic differentiation of (3.7) and using the identity (1.7) , one obtains

$$p(z)+\frac{\lambda}{\mu(\delta+p)}zp'(z)\prec\frac{1+Az}{1+Bz}.$$

Using Lemma 2.1 for $\gamma=\frac{\mu(\delta+p)}{\lambda}$, we get

$$p(z)\prec\left(\frac{\mu(\delta+p)}{\lambda}\right)z^{-\left(\frac{\mu(\delta+p)}{\lambda}\right)}\int_0^z t^{\left(\frac{\mu(\delta+p)}{\lambda}\right)-1}\left(\frac{1+At}{1+Bt}\right)dt=q(z),$$

where $q(z)$ is given by (3.5) and the remaining part of the proof is similar to Theorem 3.1.

In the following theorem we prove the corresponding result, using the identity (1.8), for the defined new integral operator, the proof of which is similar to that of Theorem 3.4.

Theorem 3.5 Let $f \in A_p$, satisfies

$$(1-\lambda)\left(\frac{J^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu + \lambda\left(\frac{J^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^{\mu-1}\left(\frac{J^m_{p,\alpha,\beta}f(z)}{z^p}\right) \prec \frac{1+Az}{1+Bz},$$

then

$$\left(\frac{J^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $F_\delta(z)$ is defined by (1.6) , $q(z)$ is given by (3.5) and $q(z)$ is the best dominant. Furthermore

$$\mathrm{Re}\left(\left(\frac{J^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu\right) > M_1(p,A,B,\delta,\lambda,\mu)$$

where $M_1(p,A,B,\delta,\lambda,\mu)$ is given by (3.6) and this result is sharp.

Remark 3.6 For $p=1, \lambda=1, \mu=1,$ and $\alpha=1-\beta,$ Theorem 3.4 and Theorem 3.5 agree with Theorem 3.3 and Theorem 3.4, respectively, of Al-Oboudi and Al-Qahtani [3]. For $\beta=1$ in Theorem 3.4 and Theorem 3.5, our results for operators $I^m_p(\alpha)$ and $J^m_p(\alpha)$ hold true for $\alpha > -p.$ Similarly, results obtained for operators $I^m_p(\beta,l)$ and $J^m_p(\beta,l)$ from Theorem 3.4 and Theorem 3.5, by putting $\alpha = l+p-p\beta,$ hold true for $l > -p.$

Now we prove the partial converse of Theorem 3.4 and Theorem 3.5, for $A=1-2\rho,$ $0 \le \rho < 1$ and $B=-1.$

Theorem 3.7 Let $f \in A_p$, satisfies

(3.8) $$\operatorname{Re}\left(\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu\right) > \rho, 0 \le \rho < 1,$$

then

$$\operatorname{Re}\left((1-\lambda)\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu + \lambda\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^{\mu-1}\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right)\right) > \rho, |z| < R_1,$$

where

(3.9) $$R_1 = \frac{\sqrt{\lambda^2 + (\mu(\delta+P))^2} - \lambda}{\mu(\delta+p)}.$$

The bound R_1 is the best possible.

Proof. From (3.8), we have

(3.10) $$\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu = \rho + (1-\rho)p(z).$$

We see that $p(z) = 1 + p_1 z + p_2 z^2 + ...$ is analytic and $\operatorname{Re}(p(z)) > 0, z \in U$. Differentiating both sides of (3.10) and making use of (1.7), we obtain

(3.11)
$$\operatorname{Re}\left((1-\lambda)\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu + \lambda\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^{\mu-1}\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right) - \rho\right) = (1-\rho)\operatorname{Re}\left(p(z) + \frac{\lambda z p'(z)}{\mu(\delta+p)}\right)$$

$$\ge (1-\rho)\left(\operatorname{Re}(p(z) - \frac{\lambda|zp'(z)|}{\mu(\delta+p)}\right).$$

By making use of the well-known estimate (See [15]), $\frac{|zp'(z)|}{\mathrm{Re}(p(z))} \le \frac{2r}{1-r^2}, (|z| = r < 1)$, in (3.11), we obtain

$$\mathrm{Re}\left((1-\lambda)\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu + \lambda\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^{\mu-1}\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right) - \rho\right) \ge (1-\rho)\,\mathrm{Re}\,p(z)\left(1 - \frac{2\lambda r}{\mu(\delta+p)(1-r^2)}\right),$$

which is positive if $r < R_1$, where R_1 is given by (3.9).

To show that the bound R_1 is the best possible, we consider the function $f \in A_p$ defined by

$$\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu = \rho + (1-\rho)\frac{1+z}{1-z},$$

where $F_\delta(z)$ is defined by (1.6). By noting that

$$\mathrm{Re}\left((1-\lambda)\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu + \lambda\left(\frac{I^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^{\mu-1}\left(\frac{I^m_{p,\alpha,\beta}f(z)}{z^p}\right) - \rho\right) = (1-\rho)\,\mathrm{Re}\left(\frac{1+z}{1-z} + \frac{2\lambda}{\mu(\delta+p)}\frac{z}{(1-z)^2}\right)$$

$$= 0$$

for $z = R_1$, we conclude that the bound is best possible. Theorem 3.7 is thus proved.

By applying the technique of proof of Theorem 3.7, we easily get the following result.

Theorem 3.8 Let $f \in A_p$, satisfies $\mathrm{Re}\left(\left(\frac{J^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu\right) > \rho, 0 \le \rho < 1,$ then

$$\mathrm{Re}\left((1-\lambda)\left(\frac{J^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^\mu + \lambda\left(\frac{J^m_{p,\alpha,\beta}F_\delta(z)}{z^p}\right)^{\mu-1}\left(\frac{J^m_{p,\alpha,\beta}f(z)}{z^p}\right)\right) > \rho, |z| < R_1,$$

where R_1 is given by (3.9). The bound R_1 is the best possible.

Remark 3.9 For $p=1, \mu=1,$ and $\alpha=1-\beta,$ Theorem 3.7 and Theorem 3.8 agree with Theorem 3.5 and Theorem 3.6, respectively, of Al-Oboudi and Al-Qahtani [3]. For $\beta=1$ in Theorem 3.7 and Theorem 3.8, our results for operators $I_p^m(\alpha)$ and $J_p^m(\alpha)$ hold true for $\alpha>-p.$ Likewise, results obtained for operators $I_p^m(\beta,l)$ and $J_p^m(\beta,l)$ from Theorem 3.7 and Theorem 3.8, by putting $\alpha=l+p-p\beta,$ hold true for $l>-p.$

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